应用数学和力学，第13卷第1期（1992年1月）
Applied Mathematics and Mechanics
应用数学和力学编委会编
重庆出版社出版

多自由度非线性振动系统的内共振

刘世龄　　徐　兆
（香港理工学院，香港）（中山大学，广州）

（钱伟长推荐．1991年3月28日收到）

摘　要

本文研究多自由度非线性自治系统的周期解问题，我们推广了KBM[1]法，本方法可以求得极限环的相图，振幅，周期及其稳定性．

关键词　非线性振动　锁相周期解　相图

一、引　言

我们研究下列多自由度非线性自治系统

$$\frac{d^2x_i}{dt^2}+\omega_i^2x_i=\varepsilon F_i\left(x_1,\cdots,x_N,\ \frac{dx_1}{dt},\cdots,\frac{dx_N}{dt}\right)\qquad(i=1,2,\cdots,N)\tag{1.1}$$

其中　F_i为其变量的解析函数，ε为小参数，$0<\varepsilon\ll1$，ω_i满足共振条件：

$$\omega_1:\omega_2:\cdots:\omega_N\approx k_1:k_2:\cdots:k_N\tag{1.2}$$

这里k_1，$k_2,\cdots,k_N$是没有公约数的正整数．

对于两自由度系统，已经有一些作者用不同的方法做了研究，例如多尺度法[2]，推广的L-P法[3]和平均法[4]等．而多自由度系统则研究得较少，缺乏简单有效的一般分析方法．本文提出一种新的渐近解法，我们把新的渐近方法[5]和关于多自由度非共振情形的KBM法[1]结合起来，本方法可求出系统(1.1)的近似解，把锁相周期解问题化为求解$2N-1$个函数方程的解．

二、渐近方法

由(1.2)可设

$$\omega_i^2=c^2k_i^2+\varepsilon\sigma_i\qquad(i=1,2,\cdots,N)\tag{2.1}$$

其中　σ_i为解谐参数，常数$c>0$，不失一般性，可设整数k_i为奇数．引入新的时间变量$\tau=ct$，方程组(1.1)可化为

$$\frac{d^2x_i}{d\tau^2}+k_i^2x_i=\varepsilon f_i\left(x_1,\cdots,x_N,\ \frac{dx_1}{d\tau},\cdots,\frac{dx_N}{d\tau}\right)\tag{2.2}$$

其中 $f_i\left(x_1,\cdots,x_N,\ \frac{dx_1}{d\tau},\cdots,\frac{dx_N}{d\tau}\right)$

$$=\frac{1}{c^2}\left[F_i\left(x_1,\ \cdots,\ x_N,\ c\frac{dx_1}{d\tau},\cdots,c\frac{dx_N}{d\tau}\right)-\sigma_i x_i\right] \tag{2.3}$$

类似KBM法，我们把方程组(2.2)的解写成如下形式

$$x_1=a_1\cos k_1\varphi+\varepsilon x_{11}(a_1,\cdots,a_N,\ \theta_2,\cdots,\ \theta_N)+\varepsilon^2 x_{12}(a_1,\cdots,a_N,\theta_2,\cdots,\theta_N)+\cdots \tag{2.4}$$

$$x_s=a_s\cos(k_s\varphi+\theta_s)+\varepsilon x_{s1}(a_1,\cdots,a_N,\ \theta_2,\cdots,\ \theta_N,\ \varphi)+\varepsilon^2 x_{s2}(a_1,\cdots,a_N,\theta_2,\cdots,\theta_N,\ \varphi)+\cdots \tag{2.5}$$

其中 x_{1k}与φ无关，而x_{sk}是φ的以2π为周期的函数，$a_i\neq 0$ ($s=2,3,\cdots,N$; $k=1,2,\cdots$; $i=1,2,\cdots,N$)。我们假设a_i，θ_s和φ为τ的函数，满足下列方程

$$\frac{da_i}{d\tau}=\varepsilon A_{i1}(a_1,\cdots,a_N,\ \theta_2,\cdots,\theta_N)+\varepsilon^2 A_{i2}(a_1,\cdots,a_N,\theta_2,\cdots,\theta_N)+\cdots \tag{2.6}$$

$$\frac{d\theta_s}{d\tau}=\varepsilon\Theta_{s1}(a_1,\cdots,a_N,\ \theta_2,\cdots,\theta_N)+\varepsilon^2\Theta_{s2}(a_1,\cdots,a_N,\theta_2,\cdots,\theta_N)+\cdots \tag{2.7}$$

$$(s=2,3,\cdots,N)$$

$$\frac{d\varphi}{d\tau}=1+\varepsilon\Phi_1(a_1,\cdots,\ a_N,\theta_2,\cdots,\theta_N,\varphi)+\varepsilon^2\Phi_2(a_1,\cdots,a_N,\theta_2,\cdots,\theta_N,\varphi)+\cdots \tag{2.8}$$

其中 $\Phi_k(k=1,2,\cdots)$是φ的以2π为周期的函数. 本方法与KBM法的重要差别是(2.8)式，即当a_i和θ_s为常数时，φ不再是时间的线性函数，而是时间的非线性函数，这是新的渐近方法[5]的特点，它不仅能够提高近似解的精度而且可应用于强非线性系统。

对x_1和x_s求导得

$$\frac{dx_1}{d\tau}=-a_1k_1\sin k_1\varphi+\varepsilon(A_{11}\cos k_1\varphi-a_1k_1\Phi_1\sin k_1\varphi)+\varepsilon^2\left(A_{12}\cos k_1\varphi-a_1k_1\Phi_2\sin k_1\varphi+\sum_{m=1}^{N}A_{m1}\frac{\partial x_{11}}{\partial a_m}+\sum_{l=2}^{N}\Theta_{l1}\frac{\partial x_{11}}{\partial\theta_l}\right)+O(\varepsilon^3) \tag{2.9}$$

$$\begin{aligned}\frac{dx_s}{d\tau}=&-a_sk_s\sin(k_s\varphi+\theta_s)+\varepsilon\Big[A_{s1}\cos(k_s\varphi+\theta_s)-a_sk_s\Phi_1\sin(k_s\varphi+\theta_s)\\&-a_s\Theta_{s1}\sin(k_s\varphi+\theta_s)+\frac{\partial x_{s1}}{\partial\varphi}\Big]+\varepsilon^2\Big[A_{s2}\cos(k_s\varphi+\theta_s)\\&-a_sk_s\Phi_2\sin(k_s\varphi+\theta_s)-a_s\Theta_{s2}\sin(k_s\varphi+\theta_s)+\sum_{m=1}^{N}A_{m1}\frac{\partial x_{s1}}{\partial a_m}\\&+\sum_{l=2}^{N}\Theta_{l1}\frac{\partial x_{s1}}{\partial\theta_l}+\Phi_1\frac{\partial x_{s1}}{\partial\varphi}+\frac{\partial x_{s2}}{\partial\varphi}\Big]+O(\varepsilon^3)\end{aligned} \tag{2.10}$$

$$\frac{d^2x_1}{d\tau^2}=-a_1k_1^2\cos k_1\varphi-\varepsilon\Big[2k_1A_{11}\sin k_1\varphi+a_1k_1^2\Phi_1\cos k_1\varphi+a_1k_1\frac{\partial}{\partial\varphi}(\Phi_1\sin k_1\varphi)\Big]$$

$$-\varepsilon^2\Big[2k_1A_{12}\sin k_1\varphi+a_1k_1^2\Phi_2\cos k_1\varphi+a_1k_1\frac{\partial}{\partial\varphi}(\Phi_2\sin k_1\varphi)$$

$$+2k_1A_{11}\Phi_1\sin k_1\varphi+a_1k_1\Phi_1\frac{\partial}{\partial\varphi}(\Phi_1\sin k_1\varphi)$$

$$-\Big(\sum_{m=1}^{N}A_{m1}\frac{\partial A_{11}}{\partial a_m}+\sum_{l=2}^{N}\Theta_{l1}\frac{\partial A_{11}}{\partial\theta_l}\Big)\cos k_1\varphi$$

$$+a_1k_1\Big(\sum_{m=1}^{N}A_{m1}\frac{\partial\Phi_1}{\partial a_m}+\sum_{l=2}^{N}\Theta_{l1}\frac{\partial\Phi_1}{\partial\theta_l}\Big)\sin k_1\varphi\Big]+O(\varepsilon^3) \qquad (2.11)$$

$$\frac{d^2x_s}{d\tau^2}=-a_sk_s^2\cos(k_s\varphi+\theta_s)-\varepsilon[2k_sA_{s1}\sin(k_s\varphi+\theta_s)+2k_sa_s\Theta_{s1}\cos(k_s\varphi+\theta_s)$$

$$-\frac{\partial^2x_{s1}}{\partial\varphi^2}+a_sk_s^2\Phi_1\cos(k_s\varphi+\theta_s)+a_sk_s\frac{\partial}{\partial\varphi}(\Phi_1\sin(k_s\varphi+\theta_s))]$$

$$-\varepsilon^2\Big\{2k_sA_{s2}\sin(k_s\varphi+\theta_s)+2k_sa_s\Theta_{s2}\cos(k_s\varphi+\theta_s)-\frac{\partial^2x_{s2}}{\partial\varphi^2}$$

$$+a_sk_s^2\Phi_2\cos(k_s\varphi+\theta_s)+a_sk_s\frac{\partial}{\partial\varphi}(\Phi_2\sin(k_s\varphi+\theta_s))$$

$$+a_sk_s\Phi_1\frac{\partial}{\partial\varphi}(\Phi_1\sin(k_s\varphi+\theta_s))+a_sk_s\sum_{m=1}^{N}A_{m1}\frac{\partial}{\partial a_m}(\Phi_1\sin(k_s\varphi+\theta_s))$$

$$+a_sk_s\sum_{l=2}^{N}\Theta_{l1}\frac{\partial}{\partial\theta_l}(\Phi_1\sin(k_s\varphi+\theta_s))-2\sum_{m=1}^{N}A_{m1}\frac{\partial^2x_{s1}}{\partial\varphi\partial a_m}$$

$$-2\sum_{l=2}^{N}\Theta_{l1}\frac{\partial^2x_{s1}}{\partial\varphi\partial\theta_l}-\Phi_1\frac{\partial^2x_{s1}}{\partial\varphi^2}-\frac{\partial}{\partial\varphi}\Big(\Phi_1\frac{\partial x_{s1}}{\partial\varphi}\Big)$$

$$-\Big[\sum_{m=1}^{N}A_{m1}\frac{\partial A_{s1}}{\partial a_m}+\sum_{l=2}^{N}\Theta_{l1}\frac{\partial A_{s1}}{\partial\theta_l}-a_s\Theta_{s1}(k_1\Phi_1+\Theta_{s1})\Big]\cos(k_s\varphi+\theta_s)$$

$$+\Big[a_s\sum_{m=1}^{N}A_{m1}\frac{\partial\Theta_{s1}}{\partial a_m}+a_s\sum_{l=2}^{N}\Theta_{l1}\frac{\partial\Theta_{s1}}{\partial\theta_l}$$

$$+A_{s1}(k_1\Phi_1+2\Theta_{s1})\Big]\sin(k_s\varphi+\theta_s)\Big\}+O(\varepsilon^3) \qquad (2.12)$$

引入记号

$$\left.\begin{array}{l}x_{10}=a_1\cos k_1\varphi,\ x_{s0}=a_s\cos(k_s\varphi+\theta_s)\\ x'_{10}=-a_1k_1\sin k_1\varphi,\ x'_{s0}=-a_sk_s\sin(k_s\varphi+\theta_s)\end{array}\right\}\quad(s=2,3,\cdots,N) \qquad (2.13)$$

由(2.9)和(2.10)，我们可以展开εf_i为如下形式

$$\varepsilon f_i\Big(x_1,\cdots,x_N,\frac{dx_1}{d\tau},\cdots,\frac{dx_N}{d\tau}\Big)=\varepsilon f_i(x_{10},\cdots,x_{N0},x'_{10},\cdots,x'_{N0})$$

$$+\varepsilon^2\left\{\sum_{m=1}^{N}x_{m1}\frac{\partial f_i}{\partial x_m}(x_{10},\cdots,x_{N0},x'_{10},\cdots,x'_{N0})\right.$$

$$+(A_{11}\cos k_1\varphi-a_1k_1\Phi_1\sin k_1\varphi)\frac{\partial f_i}{\partial x'_1}(x_{10},\cdots,x_{N0},x'_{10},\cdots,x'_{N0})$$

$$+\sum_{l=2}^{N}\Big[A_{l1}\cos(k_l\varphi+\theta_l)-a_lk_l\Phi_1\sin(k_l\varphi+\theta_l)-a_l\Theta_{l1}\sin(k_l\varphi+\theta_l)$$

$$\left.+\frac{\partial x_{l1}}{\partial\varphi}\Big]\frac{\partial f_i}{\partial x'_l}(x_{10},\cdots,x_{N0},x'_{10},\cdots,x'_{N0})\right\}+O(\varepsilon^3)\tag{2.14}$$

其中 $\frac{\partial f_i}{\partial x_m}(x_{10},\cdots,x_{N0},x'_{10},\cdots,x'_{N0})$ 记 $\frac{\partial f_i}{\partial x_m}\left(x_1,\cdots,x_N,\frac{dx_1}{d\tau},\cdots,\frac{dx_N}{d\tau}\right)$ 在 $x_j=x_{j0}$，$x'_j=x'_{j0}(j=,2,\cdots,N)$ 的值。

最后把(2.4)，(2.5)，(2.11)，(2.12)和(2.14)代入方程组(2.2)，令 ε 的同次幂系数相等，得到 ε 阶项为

$$a_1k_1\frac{\partial}{\partial\varphi}(\Phi_1\sin^2k_1\varphi)=-f_1(x_{10},\cdots,x_{N0},x'_{10},\cdots,x'_{N0})\sin k_1\varphi$$
$$-2k_1A_{11}\sin^2k_1\varphi+x_{11}k_1^2\sin k_1\varphi\tag{2.15}$$

$$\frac{\partial^2x_{s1}}{\partial\varphi^2}+k_s^2x_{s1}=f_s(x_{10},\cdots,x_{N0},x'_{10},\cdots,x'_{N0})+2k_sA_{s1}\sin(k_s\varphi+\theta_s)$$
$$+2k_sa_s\Theta_{s1}\cos(k_s\varphi+\theta_s)+2a_sk_s^2\Phi_1\cos(k_s\varphi+\theta_s)$$
$$+a_sk_s\frac{\partial\Phi_1}{\partial\varphi}\sin(k_s\varphi+\theta_s)\qquad(s=2,3,\cdots,N)\tag{2.16}$$

我们先确定 $x_{11}(a_1,\cdots,a_N,\theta_2,\cdots,\theta_N)$，$A_{11}(a_1,\cdots,a_N,\theta_2,\cdots,\theta_N)$ 和 $\Phi_1(a_1,\cdots,a_N,\theta_2,\cdots,\theta_N,\varphi)$。对方程(2.15)积分得

$$a_1k_1\Phi_1\sin^2k_1\varphi=-\int_0^\varphi f_1(x_{10},\cdots,x_{N0},x'_{10},\cdots,x'_{N0})\sin k_1\varphi d\varphi$$
$$-2k_1A_{11}\int_0^\varphi\sin^2k_1\varphi d\varphi+x_{11}k_1^2\int_0^\varphi\sin k_1\varphi d\varphi\tag{2.17}$$

按假设条件 k_1 为奇数，所以 $\int_0^\pi\sin k_1\varphi\,d\varphi=\frac{2}{k_1}$，令 $\varphi=2\pi$，π 分别代入(2.17)得

$$A_{11}=-\frac{1}{2\pi k_1}\int_0^{2\pi}f_1(x_{10},\cdots,x_{N0},x'_{10},\cdots,x'_{N0})\sin k_1\varphi d\varphi\tag{2.18}$$

和
$$x_{11}=\frac{1}{2k_1}\int_0^\pi[f_1(x_{10},\cdots,x_{N0},x'_{10},\cdots,x'_{N0})+2k_1A_{11}\sin k_1\varphi]\sin k_1\varphi d\varphi\tag{2.19}$$

这样一来，由(2.17)便可以确定 Φ_1。

其次，由(2.16)我们可以确定 $x_{s1}(a_1,\cdots,a_N,\theta_2,\cdots,\theta_N,\varphi)$，$A_{s1}(a_1,\cdots,a_N,\theta_2,\cdots,\theta_N)$ 和 $\Theta_{s1}(a_1,\cdots,a_N,\theta_2,\cdots,\theta_N)(s=2,\cdots,N)$，为了使 x_{s1} 是 φ 的周期函数，方程组(2.16)右端必须不含 $\cos k_s\varphi$ 和 $\sin k_s\varphi$ 项。由此得

$$A_{s1}=-\frac{1}{2\pi k_s}\int_0^{2\pi}\bar{f}_s(a,\theta,\varphi)\sin(k_s\varphi+\theta_s)d\varphi\tag{2.20}$$

$$\Theta_{s1}=-\frac{1}{2\pi a_sk_s}\int_0^{2\pi}\bar{f}_s(a,\theta,\varphi)\cos(k_s\varphi+\theta_s)d\varphi\tag{2.21}$$

和
$$x_{s1}=\frac{C_{s0}}{2k_s^2}+\sum_{\substack{n=1\\ n\neq k_s}}^{\infty}\frac{1}{k_s^2-n^2}(C_{sn}\cos n\varphi+D_{sn}\sin\varphi) \tag{2.22}$$

其中
$$\bar{f}_s(a,\theta,\varphi)=f_s(x_{10},\cdots,x_{N0},\ x'_{10},\cdots,x'_{N0})+2a_sk_s^2\Phi_1\cos(k_s\varphi+\theta_s)$$
$$+a_s k_s\frac{\partial\Phi_1}{\partial\varphi}\sin(k_s\varphi+\theta_s) \tag{2.23}$$

$$C_{sn}=\frac{1}{2\pi}\int_0^{2\pi}\bar{f}_s(a,\theta,\varphi)\cos n\varphi d\varphi \tag{2.24}$$

$$D_{sn}=\frac{1}{2\pi}\int_0^{2\pi}\bar{f}_s(a,\theta,\varphi)\sin n\varphi d\varphi \tag{2.25}$$

其中 $s=2,3,\cdots,N$。类似可求高阶近似解。

为使近似解(2.4),(2.5)成为锁相周期解，(2.6)和(2.7)必须有平衡点。我们略去ε^2以上的项，在(2.6)和(2.7)中令 $da_i/d\tau=0$ 和 $d\theta_s/d\tau=0$，于是求锁相周期解的问题便化为求解$2N-1$个函数方程：

$$A_{i1}=0,\ \ \Theta_{s1}=0 \qquad (i=1,2,\cdots,N;\ s=2,3,\cdots,N) \tag{2.26}$$

三、周 期 解 的 稳 定

为了研究锁相周期解的稳定性，我们来考察(2.6)和(2.7)平衡点的稳定性。令

$$a_i=a_i^0+\rho_i \qquad (i=1,2,\cdots,\ N) \tag{3.1}$$
$$\theta_s=\theta_s^0+\psi_s \qquad (s=2,3,\cdots,\ N) \tag{3.2}$$

其中 a_i^0 和 θ_s^0 为(2.26)的一组解（平衡点），ρ_i 和 ψ_s 为平衡点附近的小扰动。把(3.1)和(3.2)代入(2.6)和(2.7)并对ρ_i，ψ_s线性化，得到

$$\frac{du}{d\tau}=\varepsilon Ju \tag{3.3}$$

其中 $u=(\rho_1,\ \rho_2,\cdots,\rho_N,\ \psi_2,\cdots,\psi_N)^T$，

和
$$J=\begin{bmatrix}\frac{\partial A_{11}}{\partial a_1} & \cdots & \frac{\partial A_{11}}{\partial a_N} & \frac{\partial A_{11}}{\partial\theta_2} & \cdots & \frac{\partial A_{11}}{\partial\theta_N}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \frac{\partial A_{N1}}{\partial a_1} & \cdots & \frac{\partial A_{N1}}{\partial a_N} & \frac{\partial A_{N1}}{\partial\theta_2} & \cdots & \frac{\partial A_{N1}}{\partial\theta_N}\\ \frac{\partial\Theta_{21}}{\partial a_1} & \cdots & \frac{\partial\Theta_{21}}{\partial a_N} & \frac{\partial\Theta_{21}}{\partial\theta_2} & \cdots & \frac{\partial\Theta_{21}}{\partial\theta_N}\\ \frac{\partial\Theta_{N1}}{\partial a_1} & \cdots & \frac{\partial\Theta_{N1}}{\partial a_N} & \frac{\partial\Theta_{N1}}{\partial\theta_2} & \cdots & \frac{\partial\Theta_{N1}}{\partial\theta_N}\end{bmatrix}. \tag{3.4}$$

矩阵J在平衡点取值。

当J的特征值均有负实部时，平衡点(a_i^0，θ_s^0)对应的锁相周期解是渐近稳定的；当J的特征值有一个具正实部时，则对应的锁相周期解是不稳定的。

四、例：耦合性van der Pol振子

作为例子，我们来研究耦合 van der Pol 振子

$$\frac{d^2x_1}{dt^2}+x_1=\varepsilon(1-x_1^2)\frac{dx_1}{dt}+\varepsilon\gamma(-x_1+x_2) \tag{4.1}$$

$$\frac{d^2x_2}{dt^2}+x_2=\varepsilon(1-x_2^2)\frac{dx_2}{dt}+\varepsilon(\gamma x_1-\delta x_2) \tag{4.2}$$

文献[6]和[7]曾用两变量展开法研究这个系统的锁相周期运动，因此本文方法的结果和功效可与其比较。

对于本例，$k_1=k_2=1$，$c=1$，

$$f_1(x_1,\ x_2,\ \dot{x}_1,\ \dot{x}_2)=(1-x_1^2)\dot{x}_1+\gamma(-x_1+x_2) \tag{4.3}$$

$$f_2(x_1,\ x_2,\ \dot{x}_1,\ \dot{x}_2)=(1-x_2^2)\dot{x}_2+\gamma x_1-\delta x_2 \tag{4.4}$$

其中对时间t的导数用点表示。由(2.17)～(2.25)，通过简单积分得

$$x_{11}=0,\ \Phi_1=\frac{\gamma}{2a_1}\ (a_1-a_2\cos\theta_2)-\frac{1}{8}a_1^2\sin 2\varphi \tag{4.5}$$

$$A_{11}=\frac{1}{2}\left(a_1-\frac{1}{4}a_1^3+\gamma a_2\sin\theta_2\right) \tag{4.6}$$

$$A_{21}=\frac{1}{2}\left(a_2-\frac{1}{4}a_2^3-\gamma a_1\sin\theta_2\right) \tag{4.7}$$

$$\Theta_{21}=\frac{1}{2}\left[\delta-\gamma-\gamma\left(\frac{a_1}{a_2}-\frac{a_2}{a_1}\right)\cos\theta_2\right] \tag{4.8}$$

$$x_{21}=\frac{1}{32}a_2[a_1^2\sin(3\varphi+\theta_2)-a_2^2\sin(3\varphi+3\theta_2)] \tag{4.9}$$

由(3.4)式表示的矩阵J变成

$$J=\frac{1}{2}\begin{bmatrix} 1-\frac{3}{4}a_1^2 & \gamma\sin\theta_2 & \gamma a_2\cos\theta_2 \\ -\gamma\sin\theta_2 & 1-\frac{3}{4}a_2^2 & -\gamma a_1\cos\theta_2 \\ -\gamma\cos\theta_2\left(\frac{a_2}{a_1^2}+\frac{1}{a_2}\right) & \gamma\cos\theta_2\left(\frac{a_1}{a_2^2}+\frac{1}{a_1}\right) & \gamma\sin\theta_2\left(\frac{a_1}{a_2}-\frac{a_2}{a_1}\right) \end{bmatrix} \tag{4.10}$$

计算行列式$|J-\lambda E|=0$求得特征方程

$$\lambda^3+h_1\lambda^2+h_2\lambda+h_3=0 \tag{4.11}$$

如果h_1，h_2，$h_3>0$且$h_1h_2-h_3>0$，则由$a_i^0(i=1,\ 2)$和θ_2^0描述的锁相周期运动是渐近稳定的。

现在取参数

$$\gamma=0.4,\ \delta=0.56,\ \varepsilon=0.25 \tag{4.12}$$

则由$A_{11}=A_{21}=\Theta_{21}=0$求得两个平衡点：

$$(\text{i})\quad a_1^0=2.1615,\ a_2^0=1.7108,\ \theta_2^0=0.5596; \tag{4.13}$$

$$(\text{ii})\quad a_1^0=1.7108,\ a_2^0=2.1615,\ \theta_2^0=\pi+0.5596 \tag{4.14}$$

这两个平衡点对应的锁相周期运动分别为

$$(\text{i})\quad x_1=a_1^0\cos\varphi+\varepsilon x_{11}=2.1615\cos\varphi \tag{4.15}$$

$$\begin{aligned}\dot{x}_1&=-a_1^0(1+\varepsilon\Phi_1)\sin\varphi\\&=-2.1615(1.0165-0.1460\sin2\varphi)\sin\varphi\end{aligned} \tag{4.16}$$

$$\begin{aligned}x_2&=a_2^0\cos(\varphi+\theta_2^0)+\varepsilon x_{21}\\&=1.7108\cos(\varphi+0.5596)+0.0624\sin(3\varphi+0.5596)\\&\quad-0.0391\sin(3\varphi+1.6788)\end{aligned} \tag{4.17}$$

$$\dot{x}_2=[-1.7108\sin(\varphi+0.5596)+0.1872\cos(3\varphi+0.5596)-0.1173\cos(3\varphi+1.6788)](1.0165-0.1460\sin2\varphi) \tag{4.18}$$

(ii)
$$x_1=1.7108\cos\varphi \tag{4.19}$$

$$\dot{x}_1=-1.7108(1.1035-0.0915\sin2\varphi)\sin\varphi \tag{4.20}$$

$$x_2=-2.1615\cos(\varphi+0.5596)-0.0494\sin(3\varphi+0.5596)+0.0789\sin(3\varphi+1.6788) \tag{4.21}$$

$$\dot{x}_2=[2.1615\sin(\varphi+0.5596)-0.1482\cos(3\varphi+0.5596)+0.2367\cos(3\varphi+1.6788)](1.1035-0.0915\sin2\varphi) \tag{4.22}$$

特征方程(4.11)变为

$$\lambda^3+3.5991\lambda^2+2.6219\lambda+0.6659=0 \tag{4.23}$$

由此可见，两个周期运动(i)和(ii)均是渐近稳定的．让相角φ从0变到2π，就可作出锁相周期运动的相图．由方程(4.15)～(4.22)作出的相图如图1所示，图中还给出了用增量谐波平衡法[8]求解的结果（增量谐波平衡法简称IHB法），该方法可以得到非线性振动系统事先给定的任意精确度的解．两者比较表明，用本文渐近方法得到的近似与用IHB法求得的解是很吻合的．此外，我们求得的近似解与用数值法求得的解[9]也很吻合．

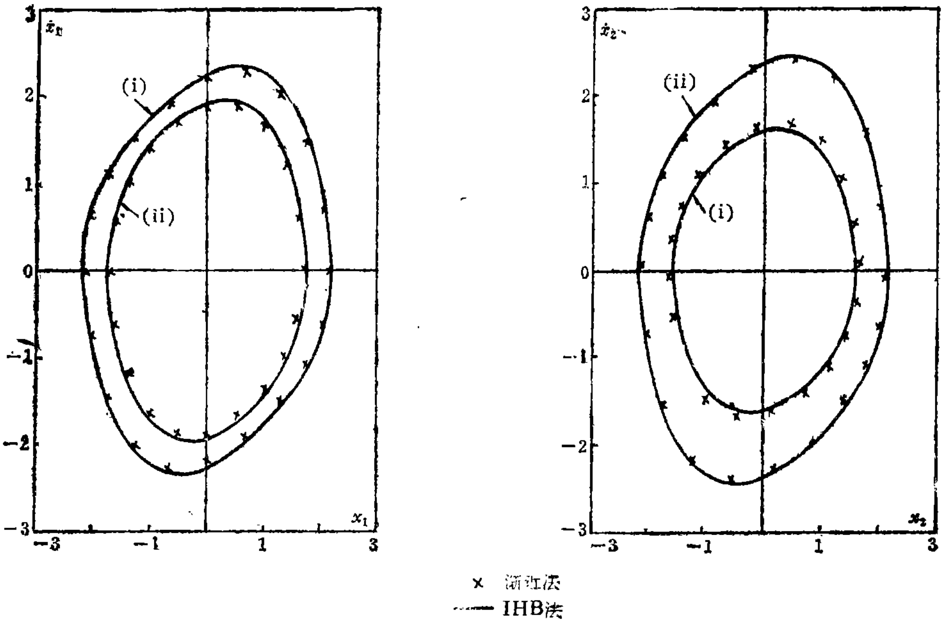

图 1

五、结 论

用本文方法很容易求得多自由度非线性自治系统的周期解．周期解的振幅、相位差和角速度$d\varphi/d\tau$由微分方程(2.6)～(2.8)给出，利用这些方程还可以判别周期解的稳定性．本文方法与著名的KBM法最主要的不同点是对于周期解其角速度$d\varphi/d\tau$不再是常数而是φ的以2π为周期的函数．本文方法的优点是计算方法简单方便并且精度较好．

参 考 文 献

[1] Bogoliubov, N. N. and J. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonliear Vibrations*, Moscow, Nauka (1974).(in Russian)

[2] Nayfeh, A.H., and D.T.Mook, *Nonlinear Oscillations*, Wiley, New York (1979).

[3] Mickens, R. E., A regular perturbation technique for nolinearly coupled oscillators in resonance, *Journal of Sound and Vibration*, 81 (1982), 307—310.

[4] Sprysl, H., Internal resonance of non-linear autonomous vibrating system with two degrees of freedom, *Journal of Sound and Vibration*, 112 (1987), 63—67.

[5] Xu Zhao, and Zhang Ling-qi, Asymptotic method for analysis of non-linear systems with two parameters, *Acta Mathematica Scientia*, 6 (1986), 453—462.

[6] Pavlidis, T., *Biological Oscillators: Their Mathematical Analysis*, Academic Press, New York (1973).

[7] Rand, R.H., and P. J. Holmes, Bifurcation of periodic motions in two weakly coupled van der Pol oscillators, *Int. J. Non-Linear Mechanics*, 15 (1980), 387—399.

[8] Lau, S.L., Y. K. Cheung and S. Y. Wu, A variable parameter incrementation method for dynamic instability of linear and nonlinear elastic systems, *ASME J. Appl. Mech.*, 49 (1982), 849—853.

[9] Xu Jian-xue, R. S. Guttalu and C. S. Hsu, Domains of attraction for multiple limit cycles of coupled van der Pol equations by simple cell mapping, *Int. J. Non-Linear Mech.*, 20 (1985), 507—517.

On Internal Resonance of Nonlinear Vibrating Systems with Many Degrees of Freedom

S. L. Lau

(*Hong Kong Polytechnic, Hong Kong*)

Xu Zhao

(*Zhongshan University, Guangzhou*)

Abstract

The problem of periodic solutions of nonlinear autonomous systems with many degrees of freedom is considered. This is made possible by the development of a modified version of the KBM method[1]. The method can be used to generate limit cycle phase portrait, amplitude, period and to indicate stability of the limit cycle.

Key words nonlinear oscillations, phase-locked solutions, phase portrait