应用数学和力学，第12卷第9期（1991年9月）
Applied Mathematics and Mechanics

应用数学和力学编委会编
重庆出版社出版

广义Airy函数与具有n个转向点的方程*

张 居 铃

（四川轻化工学院，1990年6月20日收到）

摘　要

本文研究二阶线性常微分方程

$$\frac{d^2y}{dx^2}+[\lambda^2q_1(x)+q_2(x)]y=0$$

其中 $q_1(x)=(x-\mu_1)(x-\mu_2)\cdots(x-\mu_n)f(x)$，$f(x)\neq 0$，$\lambda$为大参数，即方程有$n$个转向点。

本文使用匹配法，对三个转向点的情况，进行详细分析研究，得到形式一致有效渐近解，并阐明对n个转向点的情况，方法也是一样的。

关键词　一致有效的　渐近解　转向点

一、引　　言

二阶线性微分方程$\dfrac{d^2y}{dx^2}+p(x)\dfrac{dy}{dx}+q(x)y=0$ 经过变换 $y=w\exp\left[\int^x-\dfrac{1}{2}p(x)dx\right]$，化为

$$\frac{d^2w}{dx^2}-Q(x)w=0,\ \text{其中}Q(x)=\frac{1}{4}p^2(x)+\frac{1}{2}p'(x)-q(x)$$

对含大参数λ的方程 $\dfrac{d^2y}{dx^2}+[\lambda^2q_1(x)+q_2(x)]y=0$ (1.1)

在有限闭区间$[a,b]$上，$q_1(x)$为正则的，$q_2(x)$为连续的。

若$q_1(x)>0$，利用Liouville—Green变换

$$z=\int^x\sqrt{q_1(\tau)}\,d\tau,\quad y=(q_1(x))^{\frac{1}{4}}v$$

(1.1)变为

$$\frac{d^2v}{dz^2}+\lambda^2v=\delta v \tag{1.2}$$

$$\delta=-\frac{q_2}{q_1}-q_1^{-\frac{3}{4}}\frac{d^2q_1^{-\frac{1}{4}}}{dx^2},\quad \delta=O(1)\quad(\text{当}\lambda\longrightarrow+\infty\text{时})$$

得 (1.1) 的首阶近似

* 林宗池推荐.

$$y=\frac{a_1\cos\left(\lambda\int^x\sqrt{q_1(\tau)}\,d\tau\right)+b_1\sin\left(\lambda\int^x\sqrt{q_1(x)}\,d\tau\right)}{\sqrt[4]{q_1(x)}} \tag{1.3}$$

其中 a_1，b_1为任意常数。

若$q_1(x)<0$，同理可得

$$y=\frac{a_1\exp\left(\lambda\int^x\sqrt{-q_1(\tau)}\,d\tau\right)+b_1\exp\left(-\lambda\int^x\sqrt{-q_1(\tau)}\,d\tau\right)}{\sqrt[4]{-q_1(x)}} \tag{1.4}$$

当$q_1(x)$有零点时，(1.3)，(1.4)失效，这是因为分母出现为零的情况，我们把$q_1(x)$的零点称为转向点，把零点的阶数称为转向点的阶数。

对具有转向点的一致有效渐近解问题的文献非常多，尤其是对一个转向点问题的研究，已经得到很好的解决，清华大学蒲富全教授作过重要的工作[1]。

Olver (1959)，Moriguchi (1959)，Lynn和Keller (1970)讨论过两个转向点的问题。

麻省理工学院（MIT）林家翘教授1979年夏在清华大学讲学时曾经提过，研究

$$\frac{d^2w}{dx^2}-Q(x)w=0$$

其中 $Q(x)$为三次多项式在星云结构研究中及其它问题中都很有价值。

本文详细讨论三个转向点的问题，利用广义 Airy 函数，使用匹配法，得到形式一致有效渐近解，所使用的方法可推广到有限个转向点的情况。

Y. Sibuya[5]讨论过$Q(x)$为一n次多项式的情形，但他的结果不同于我们的结果。我们的结果用广义Airy函数较简明，更便于应用。因 Bessel 函数是比较熟悉的函数，对于复数的情况，我们将另文讨论。

二、广义Airy函数

(Ⅰ)
$$\frac{d^2v}{dz^2}+(-1)^nz^nv=0 \quad (n\text{为正整数}) \tag{2.1}$$

叫n阶Airy方程，特殊情况$n=1$，即通常的Airy方程，一般叫广义Airy方程。

若$z>0$，令$t=\frac{2}{n+2}z^{\frac{n+2}{2}}$，$v=\sqrt{z}\,y$

$$\frac{dv}{dz}=\frac{1}{2}z^{-1/2}y+z^{\frac{n+1}{2}}\frac{dy}{dt}$$

$$\frac{d^2v}{dz^2}=-\frac{1}{4}z^{-3/2}y+\frac{n+2}{2}z^{\frac{n-1}{2}}\frac{dy}{dt}+z^{\frac{2n+1}{2}}\frac{d^2y}{dt^2}$$

代入(2.1)得

$$\frac{d^2y}{dt^2}+\frac{1}{t}\frac{dy}{dt}+\left[(-1)^n-\frac{\left(\frac{1}{n+2}\right)^2}{t^2}\right]y=0$$

若 n 为奇数，(2.1)的通解为

$$v=\sqrt{z}\left[AI_{-\frac{1}{n+2}}\left(\frac{2}{n+2}z^{\frac{n+2}{2}}\right)+BI_{\frac{1}{n+2}}\left(\frac{2}{n+2}z^{\frac{n+2}{2}}\right)\right] \tag{2.2}$$

若 n 为偶数

$$v=\sqrt{z}\left[AJ_{-\frac{1}{n+2}}\left(\frac{2}{n+2}z^{\frac{n+2}{2}}\right)+BJ_{\frac{1}{n+2}}\left(\frac{2}{n+2}z^{\frac{n+2}{2}}\right)\right] \tag{2.3}$$

其中 A，B为任意常数．I，J为Bessel函数．

若$z<0$，不论n为奇数为偶数，同理可得（2.1）的通解均为

$$v=\sqrt{-z}\left[\alpha J_{-\frac{1}{n+2}}\left(\frac{2}{n+2}(-z)^{\frac{n+2}{2}}\right)+\beta J_{\frac{1}{n+2}}\left(\frac{2}{n+2}(-z)^{\frac{n+2}{2}}\right)\right] \tag{2.4}$$

其中 α，β为任意常数

i 若n为奇数

$z>0$时，由(2.2)

$$v=A\sum_{m=0}^{\infty}\frac{\left(\frac{1}{n+2}\right)^{2m-\frac{1}{n+2}}z^{m(n+2)}}{m!\,\Gamma\left(m+\frac{n+1}{n+2}\right)}+B\sum_{m=0}^{\infty}\frac{\left(\frac{1}{n+2}\right)^{2m+\frac{1}{n+2}}z^{m(n+2)+1}}{m!\,\Gamma\left(m+\frac{n+3}{n+2}\right)}$$

$$=A\left[\frac{\left(\frac{1}{n+2}\right)^{-\frac{1}{n+2}}}{\Gamma\left(\frac{n+1}{n+2}\right)}+\frac{\left(\frac{1}{n+2}\right)^{2-\frac{1}{n+2}}z^{n+2}}{1!\,\Gamma\left(1+\frac{n+1}{n+2}\right)}+\frac{\left(\frac{1}{n+2}\right)^{4-\frac{1}{n+2}}z^{2(n+2)}}{2!\,\Gamma\left(2+\frac{n+1}{n+2}\right)}\right.$$

$$\left.+\cdots+\frac{\left(\frac{1}{n+2}\right)^{2m-\frac{1}{n+2}}z^{m(n+2)}}{m!\,\Gamma\left(m+\frac{n+1}{n+2}\right)}+\cdots\right]$$

$$+B\left[\frac{\left(\frac{1}{n+2}\right)^{\frac{1}{n+2}}z}{\Gamma\left(\frac{n+3}{n+2}\right)}+\frac{\left(\frac{1}{n+2}\right)^{2+\frac{1}{n+2}}z^{(n+2)+1}}{1!\,\Gamma\left(1+\frac{n+3}{n+2}\right)}+\frac{\left(\frac{1}{n+2}\right)^{4+\frac{1}{n+2}}z^{2(n+2)+1}}{2!\,\Gamma\left(2+\frac{n+3}{n+2}\right)}\right.$$

$$\left.+\cdots+\frac{\left(\frac{1}{n+2}\right)^{2m+\frac{1}{n+2}}z^{m(n+2)+1}}{m!\,\Gamma\left(m+\frac{n+3}{n+2}\right)}+\cdots\right]$$

令 $z\to 0^+$ 知

$$v^{(k)}\Big|_{z\to 0^+}=\begin{cases}A\dfrac{\left(\frac{1}{n+2}\right)^{2m-\frac{1}{n+2}}[m(n+2)]!}{m!\,\Gamma\left(m+\frac{n+1}{n+2}\right)} & (k=m(n+2))\\[2ex] B\dfrac{\left(\frac{1}{n+2}\right)^{2m+\frac{1}{n+2}}[m(n+2)+1]}{m!\,\Gamma\left(m+\frac{n+3}{n+2}\right)} & (k=m(n+2)+1)\\[2ex] 0 & (k=\text{其他})\end{cases}$$

$z<0$时，由(2.4)

$$v=\alpha\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{1}{n+2}\right)^{2m-\frac{1}{n+2}}(-z)^{m(n+2)}}{m!\,\Gamma\left(m+\frac{n+1}{n+2}\right)}$$

$$+\beta\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{1}{n+2}\right)^{2m+\frac{1}{n+2}}(-z)^{m(n+2)+1}}{m!\,\Gamma\left(m+\frac{n+3}{n+2}\right)}$$

$$=\alpha\sum_{m=0}^{\infty}\frac{\left(\frac{1}{n+2}\right)^{2m-\frac{1}{n+2}}z^{m(n+2)}}{m!\,\Gamma\left(m+\frac{n+1}{n+2}\right)}-\beta\sum_{m=0}^{\infty}\frac{\left(\frac{1}{n+2}\right)^{2m+\frac{1}{n+2}}z^{m(n+2)+1}}{m!\,\Gamma\left(m+\frac{n+3}{n+2}\right)}$$

令$z\to0^-$知

$$v^{(k)}\Big|_{z\to0^-}=\begin{cases}\alpha\dfrac{\left(\frac{1}{n+2}\right)^{2m-\frac{1}{n+2}}[m(n+2)]!}{m!\,\Gamma\left(m+\frac{n+1}{n+2}\right)} & (k=m(n+2))\\[2ex] -\beta\dfrac{\left(\frac{1}{n+2}\right)^{2m+\frac{1}{n+2}}[m(n+2)+1]!}{m!\,\Gamma\left(m+\frac{n+3}{n+2}\right)} & (k=m(n+2)+1)\\[2ex] 0 & (k=\text{其他})\end{cases}$$

保持$z=0$处函数及其导数的连续性，比较得知 $A=\alpha$，$B=-\beta$。

故知

$$v=\begin{cases}\sqrt{z}\left[\alpha I_{-\frac{1}{n+2}}\left(\frac{2}{n+2}z^{\frac{n+2}{2}}\right)-\beta I_{\frac{1}{n+2}}\left(\frac{2}{n+2}z^{\frac{n+2}{2}}\right)\right] & (z\geqslant0)\\ \sqrt{-z}\left[\alpha J_{-\frac{1}{n+2}}\left(\frac{2}{n+2}(-z)^{\frac{n+2}{2}}\right)+\beta J_{\frac{1}{n+2}}\left(\frac{2}{n+2}(-z)^{\frac{n+2}{2}}\right)\right] & (z\leqslant0)\end{cases}$$

由于v完全可以用一个z的幂级数（收敛半径为$+\infty$）表示，因而表示一整函数。

若取$\alpha=\beta$得

$$v=\begin{cases}\alpha\sqrt{z}\left[I_{-\frac{1}{n+2}}\left(\frac{2}{n+2}z^{\frac{n+2}{2}}\right)-I_{\frac{1}{n+2}}\left(\frac{2}{n+2}z^{\frac{n+2}{2}}\right)\right] & (z\geqslant0)\\ \alpha\sqrt{-z}\left[J_{-\frac{1}{n+2}}\left(\frac{2}{n+2}(-z)^{\frac{n+2}{2}}\right)+J_{\frac{1}{n+2}}\left(\frac{2}{n+2}(-z)^{\frac{n+2}{2}}\right)\right] & (z<0)\end{cases}$$

由渐近式（$x\to+\infty$时）

$$J_\nu(x)=\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)+O(x^{-3/2})$$ [2]

知$z\to-\infty$时

$$v(z)=a\sqrt{\frac{n+2}{\pi}}\cdot\frac{1}{(-z)^{n/4}}\cdot 2\sin\left(\frac{2}{n+2}(-z)^{\frac{n+2}{2}}+\frac{\pi}{4}\right)\cos\frac{\pi}{2(n+2)}$$
$$+O\left((-z)^{-\frac{3n+4}{4}}\right)$$

取 $$a=\frac{1}{2\sqrt{n+2}\cos\frac{\pi}{2(n+2)}}$$

此时的$v(z)$称为n阶第一类Airy函数，记为

$$\mathrm{Ai}_n(z)=\begin{cases}\dfrac{1}{2\sqrt{n+2}\cos\frac{\pi}{2(n+2)}}\sqrt{z}\left[I_{-\frac{1}{n+2}}\left(\frac{2}{n+2}z^{\frac{n+2}{2}}\right)\right.\\ \qquad\left.-I_{\frac{1}{n+2}}\left(\frac{2}{n+2}z^{\frac{n+2}{2}}\right)\right] \quad (z\geqslant 0)\\ \dfrac{1}{2\sqrt{n+2}\cos\frac{\pi}{2(n+2)}}\sqrt{-z}\left[J_{-\frac{1}{n+2}}\left(\frac{2}{n+2}(-z)^{\frac{n+2}{2}}\right)\right.\\ \qquad\left.+J_{\frac{1}{n+2}}\left(\frac{2}{n+2}(-z)^{\frac{n+2}{2}}\right)\right] \quad (z<0)\end{cases} \tag{2.5}$$

$z\to-\infty$时

$$\mathrm{Ai}_n(z)=\frac{(-z)^{-\frac{n}{4}}}{\sqrt{\pi}}\sin\left(\frac{2}{n+2}(-z)^{\frac{n+2}{2}}+\frac{\pi}{4}\right)+O\left((-z)^{-\frac{3n+4}{4}}\right) \tag{2.6}$$

$\mathrm{Ai}_n(z)$当$n=1$时得，$\mathrm{Ai}_1(z)$即通常的$\mathrm{Ai}(z)$[3]

$z\to-\infty$时，显然

$$\mathrm{Ai}(z)=\frac{(-z)^{-\frac{1}{4}}}{\sqrt{\pi}}\sin\left(\frac{2}{3}(-z)^{3/2}+\frac{\pi}{4}\right)+O\left((-z)^{-\frac{7}{4}}\right) \tag{2.7}$$

又由渐近式($x\to+\infty$时)

$$K_\nu(x)=\frac{\pi}{2\sin\nu\pi}[I_{-\nu}(x)-I_\nu(x)]=\sqrt{\frac{\pi}{2x}}\exp[-x][1+O(x^{-1})]$$ [2]

知$z\to+\infty$时

$$\mathrm{Ai}_n(z)=\frac{\sin\frac{\pi}{2(n+2)}}{\sqrt{\pi}}z^{-\frac{n}{4}}\exp\left[-\frac{2}{n+2}z^{\frac{n+2}{2}}\right]\left(1+O\left(z^{-\frac{n+2}{2}}\right)\right) \tag{2.8}$$

$$\mathrm{Ai}(z)=\frac{z^{-\frac{1}{4}}}{2\sqrt{\pi}}\exp\left[-\frac{2}{3}z^{3/2}\right]\left(1+O\left(z^{-3/2}\right)\right) \tag{2.9}$$

若取$a=-\beta$得

$$v=\begin{cases}a\sqrt{z}\left[I_{-\frac{1}{n+2}}\left(\frac{2}{n+2}z^{\frac{n+2}{2}}\right)+I_{\frac{1}{n+2}}\left(\frac{2}{n+2}z^{\frac{n+2}{2}}\right)\right] & (z\geqslant 0)\\ a\sqrt{-z}\left[J_{-\frac{1}{n+2}}\left(\frac{2}{n+2}(-z)^{\frac{n+2}{2}}\right)-J_{\frac{1}{n+2}}\left(\frac{2}{n+2}(-z)^{\frac{n+2}{2}}\right)\right] & (z<0)\end{cases}$$

$z\to-\infty$时

$$v(z)=a\sqrt{\frac{n+2}{\pi}}\cdot\frac{1}{(-z)^{\frac{n}{4}}}\cdot 2\cos\left(\frac{2}{n+2}(-z)^{\frac{n+2}{2}}+\frac{\pi}{4}\right)\sin\frac{\pi}{2(n+2)}+O\left((-z)^{-\frac{3n+4}{4}}\right)$$

此时取 $a=\dfrac{1}{2\sqrt{n+2}\,\sin\dfrac{\pi}{2(n+2)}}$

此时的$v(z)$称为n阶第二类Airy函数记为

$$\mathrm{Bi}_n(z)=\begin{cases}\dfrac{1}{2\sqrt{n+2}\,\sin\dfrac{\pi}{2(n+2)}}\sqrt{z}\left[I_{-\frac{1}{n+2}}\left(\dfrac{2}{n+2}z^{\frac{n+2}{2}}\right)+I_{\frac{1}{n+2}}\left(\dfrac{2}{n+2}z^{\frac{n+2}{2}}\right)\right] & (z\geqslant 0)\\ \dfrac{1}{2\sqrt{n+2}\,\sin\dfrac{\pi}{2(n+2)}}\sqrt{-z}\left[J_{-\frac{1}{n+2}}\left(\dfrac{2}{n+2}(-z)^{\frac{n+2}{2}}\right)-J_{\frac{1}{n+2}}\left(\dfrac{2}{n+2}(-z)^{\frac{n+2}{2}}\right)\right] & (z<0)\end{cases}\tag{2.10}$$

$z\to-\infty$时

$$\mathrm{Bi}_n(z)=\frac{(-z)^{-\frac{n}{4}}}{\sqrt{\pi}}\cos\left(\frac{2}{n+2}(-z)^{\frac{n+2}{2}}+\frac{\pi}{4}\right)+O\left((-z)^{-\frac{3n+4}{4}}\right)\tag{2.11}$$

$\mathrm{Bi}_n(z)$当$n=1$时，得$\mathrm{Bi}_1(z)$即通常的$\mathrm{Bi}(z)$[3]

$$\mathrm{Bi}(z)=\frac{(-z)^{-\frac{1}{4}}}{\sqrt{\pi}}\cos\left(\frac{2}{3}(-z)^{3/2}+\frac{\pi}{4}\right)+O\left((-z)^{-\frac{7}{4}}\right)\tag{2.12}$$

由渐近式（$x\to+\infty$时）

$$I_\nu(x)=\frac{\exp[x]}{\sqrt{2\pi x}}(1+O(x^{-1}))$$ [2]

知，$z\to+\infty$时

$$\mathrm{Bi}_n(z)=\frac{z^{-\frac{n}{4}}}{2\sqrt{\pi}\,\sin\dfrac{\pi}{2(n+2)}}\exp\left[\frac{2}{n+2}z^{\frac{n+2}{2}}\right]\left(1+O\left(z^{-\frac{n+2}{2}}\right)\right)\tag{2.13}$$

$$\mathrm{Bi}(z)=\frac{z^{-\frac{1}{4}}}{\sqrt{\pi}}\exp\left[\frac{2}{3}z^{3/2}\right]\left(1+O\left(z^{-3/2}\right)\right)\tag{2.14}$$

$\mathrm{Ai}(z)$，$\mathrm{Bi}(z)$的图形见图1。

ii 若n为偶数

$z>0$时，由(2.3)

$$v=A\sum_{m=0}^{\infty}\frac{(-1)^m\left(\dfrac{1}{n+2}\right)^{2m-\frac{1}{n+2}}z^{m(n+2)}}{m!\,\Gamma\left(m+\dfrac{n+1}{n+2}\right)}$$

$$+B\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{1}{n+2}\right)^{2m+\frac{1}{n+2}}z^{m(n+2)+1}}{m!\Gamma\left(m+\frac{n+3}{n+2}\right)}$$

$$v^{(k)}\Big|_{z\to 0^+}=\begin{cases} A\dfrac{(-1)^m\left(\frac{1}{n+2}\right)^{2m-\frac{1}{n+2}}[m(n+2)]!}{m!\Gamma\left(m+\frac{n+1}{n+2}\right)} & (k=m(n+2))\\ B\dfrac{(-1)^m\left(\frac{1}{n+2}\right)^{2m+\frac{1}{n+2}}[m(n+2)+1]!}{m!\Gamma\left(m+\frac{n+3}{n+2}\right)} & (k=m(n+2)+1)\\ 0 & (k=\text{其他})\end{cases}$$

$z<0$时，由(2.4)

$$v=\alpha\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{1}{n+2}\right)^{2m-\frac{1}{n+2}}z^{m(n+2)}}{m!\Gamma\left(m+\frac{n+1}{n+2}\right)}$$

$$-\beta\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{1}{n+2}\right)^{2m+\frac{1}{n+2}}z^{m(n+2)+1}}{m!\Gamma\left(m+\frac{n+3}{n+2}\right)}$$

$$v^{(k)}\Big|_{z\to 0^-}=\begin{cases} \alpha\dfrac{(-1)^m\left(\frac{1}{n+2}\right)^{2m-\frac{1}{n+2}}[m(n+2)]!}{m!\Gamma\left(m+\frac{n+1}{n+2}\right)} & (k=m(n+2))\\ -\beta\dfrac{(-1)^m\left(\frac{1}{n+2}\right)^{2m+\frac{1}{n+2}}[m(n+2)+1]!}{m!\Gamma\left(m+\frac{n+3}{n+2}\right)} & (k=m(n+2)+1)\\ 0 & (k=\text{其他})\end{cases}$$

保持$z=0$处函数及其导数的连续性，得

$$A=\alpha,\quad B=-\beta.$$

取$\alpha=\beta$时，具体取 $\alpha=\dfrac{1}{2\sqrt{n+2}\cos\frac{\pi}{2(n+2)}}$

又取$\alpha=-\beta$时，具体取 $\alpha=\dfrac{1}{2\sqrt{n+2}\sin\frac{\pi}{2(n+2)}}$

仿照n为奇数的情况得二整函数

$$\mathrm{Ai}_n(z)=\begin{cases}\dfrac{1}{2\sqrt{n+2}\cos\dfrac{\pi}{2(n+2)}}\sqrt{z}\left[J_{-\frac{1}{n+2}}\left(\dfrac{2}{n+2}z^{\frac{n+2}{2}}\right)-J_{\frac{1}{n+2}}\left(\dfrac{2}{n+2}z^{\frac{n+2}{2}}\right)\right] & (z\geqslant 0)\\ \dfrac{1}{2\sqrt{n+2}\cos\dfrac{\pi}{2(n+2)}}\sqrt{-z}\left[J_{-\frac{1}{n+2}}\left(\dfrac{2}{n+2}(-z)^{\frac{n+2}{2}}\right)+J_{\frac{1}{n+2}}\left(\dfrac{2}{n+2}(-z)^{\frac{n+2}{2}}\right)\right] & (z<0)\end{cases} \tag{2.15}$$

$$\mathrm{Bi}_n(z)=\begin{cases}\dfrac{1}{2\sqrt{n+2}\sin\dfrac{\pi}{2(n+2)}}\sqrt{z}\left[J_{-\frac{1}{n+2}}\left(\dfrac{2}{n+2}z^{\frac{n+2}{2}}\right)+J_{\frac{1}{n+2}}\left(\dfrac{2}{n+2}z^{\frac{n+2}{2}}\right)\right] & (z\geqslant 0)\\ \dfrac{1}{2\sqrt{n+2}\sin\dfrac{\pi}{2(n+2)}}\sqrt{-z}\left[J_{-\frac{1}{n+2}}\left(\dfrac{2}{n+2}(-z)^{\frac{n+2}{2}}\right)-J_{\frac{1}{n+2}}\left(\dfrac{2}{n+2}(-z)^{\frac{n+2}{2}}\right)\right] & (z<0)\end{cases} \tag{2.16}$$

$z\to-\infty$时

$$\mathrm{Ai}_n(z)=\frac{(-z)^{-\frac{n}{4}}}{\sqrt{\pi}}\sin\left(\frac{2}{n+2}(-z)^{\frac{n+2}{2}}+\frac{\pi}{4}\right)+O\left((-z)^{-\frac{3n+4}{4}}\right) \tag{2.17}$$

$$\mathrm{Bi}_n(z)=\frac{(-z)^{-\frac{n}{4}}}{\sqrt{\pi}}\cos\left(\frac{2}{n+2}(-z)^{\frac{n+2}{2}}+\frac{\pi}{4}\right)+O\left((-z)^{-\frac{3n+4}{4}}\right) \tag{2.18}$$

$z\to+\infty$时

$$\mathrm{Ai}_n(z)=\frac{\mathrm{tg}\dfrac{\pi}{2(n+2)}}{\sqrt{\pi}}z^{-\frac{n}{4}}\cos\left(\frac{2}{n+2}z^{\frac{n+2}{2}}+\frac{\pi}{4}\right)+O\left(z^{-\frac{3n+4}{4}}\right) \tag{2.19}$$

$$\mathrm{Bi}_n(z)=\frac{\mathrm{ctg}\dfrac{\pi}{2(n+2)}}{\sqrt{\pi}}z^{-\frac{n}{4}}\sin\left(\frac{2}{n+2}z^{\frac{n+2}{2}}+\frac{\pi}{4}\right)+O\left(z^{-\frac{3n+4}{4}}\right) \tag{2.20}$$

故

$$\mathrm{Ai}_2(z)=\frac{(-z)^{-1/2}}{\sqrt{\pi}}\sin\left(\frac{1}{2}z^2+\frac{\pi}{4}\right)+O\left((-z)^{-5/2}\right)\qquad(z\to-\infty) \tag{2.21}$$

$$\mathrm{Ai}_2(z)=\frac{\mathrm{tg}\dfrac{\pi}{8}}{\sqrt{\pi}}z^{-1/2}\cos\left(\frac{1}{2}z^2+\frac{\pi}{4}\right)+O(z^{-5/2})\qquad(z\to+\infty) \tag{2.22}$$

$$\mathrm{Bi}_2(z)=\frac{(-z)^{-1/2}}{\sqrt{\pi}}\cos\left(\frac{1}{2}z^2+\frac{\pi}{4}\right)+O\left((-z^{-5/2})\right)\qquad(z\to-\infty) \tag{2.23}$$

$$\mathrm{Bi}_2(z)=\frac{\mathrm{ctg}\dfrac{\pi}{8}}{\sqrt{\pi}}z^{-1/2}\sin\left(\frac{1}{2}z^2+\frac{\pi}{4}\right)+O(z^{-5/2})\qquad(z\to+\infty) \tag{2.24}$$

$Ai_2(z)$，$Bi_2(z)$的图形见图2

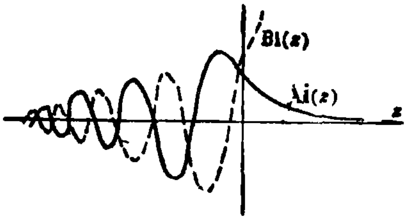

图 1

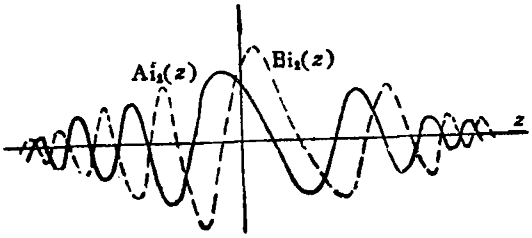

图 2

（Ⅱ） $\frac{d^2v}{dz^2}-z^nv=0$ （n为正的偶数） (2.1)′

叫 n 阶修正Airy方程，仿照（2.1）的求解

当$z>0$时，(2.1)′的通解为

$$v=\sqrt{z}\left[AI_{-\frac{1}{n+2}}\left(\frac{2}{n+2}z^{\frac{n+2}{2}}\right)+BI_{\frac{1}{n+2}}\left(\frac{2}{n+2}z^{\frac{n+2}{2}}\right)\right]$$

当$z<0$时，(2.1)′的通解为

$$v=\sqrt{-z}\left[\alpha I_{-\frac{1}{n+2}}\left(\frac{2}{n+2}(-z)^{\frac{n+2}{2}}\right)+\beta I_{\frac{1}{n+2}}\left(\frac{2}{n+2}(-z)^{\frac{n+2}{2}}\right)\right]$$

即$z>0$时

$$v=A\sum_{m=0}^{\infty}\frac{\left(\frac{1}{n+2}\right)^{2m-\frac{1}{n+2}}z^{m(n+2)}}{m!\Gamma\left(m+\frac{n+1}{n+2}\right)}+B\sum_{m=0}^{\infty}\frac{\left(\frac{1}{n+2}\right)^{2m+\frac{1}{n+2}}z^{m(n+2)+1}}{m!\Gamma\left(m+\frac{n+3}{n+2}\right)}$$

$z<0$时

$$v=\alpha\sum_{m=0}^{\infty}\frac{\left(\frac{1}{n+2}\right)^{2m-\frac{1}{n+2}}z^{m(n+2)}}{m!\Gamma\left(m+\frac{n+1}{n+2}\right)}-\beta\sum_{m=0}^{\infty}\frac{\left(\frac{1}{n+2}\right)^{2m+\frac{1}{n+2}}z^{m(n+2)+1}}{m!\Gamma\left(m+\frac{n+3}{n+2}\right)}$$

保持$z=0$处函数及其导数的连续性，得 $A=\alpha$，$B=-\beta$。而后得到两个整函数，叫第一类 n 阶修正Airy函数和第二类 n 阶修正Airy函数，分别记为$IAi_n(z)$，$IBi_n(z)$。

$$IAi_n(z)=\begin{cases}\frac{1}{2\sqrt{n+2}\cos\frac{\pi}{2(n+2)}}\sqrt{z}\left[I_{-\frac{1}{n+2}}\left(\frac{2}{n+2}z^{\frac{n+2}{2}}\right)\right.\\ \left.\quad -I_{\frac{1}{n+2}}\left(\frac{2}{n+2}z^{\frac{n+2}{2}}\right)\right] \quad (z\geqslant 0)\\ \frac{1}{2\sqrt{n+2}\cos\frac{\pi}{2(n+2)}}\sqrt{-z}\left[I_{-\frac{1}{n+2}}\left(\frac{2}{n+2}(-z)^{\frac{n+2}{2}}\right)\right.\\ \left.\quad +I_{\frac{1}{n+2}}\left(\frac{2}{n+2}(-z)^{\frac{n+2}{2}}\right)\right] \quad (z<0)\end{cases} \tag{2.25}$$

$$
IBi_n(z)=\begin{cases}\dfrac{1}{2\sqrt{n+2}\ \sin\dfrac{\pi}{2(n+2)}}\sqrt{z}\left[I_{-\frac{1}{n+2}}\left(\dfrac{2}{n+2}z^{\frac{n+2}{2}}\right)\right.\\ \qquad\left.+I_{\frac{1}{n+2}}\left(\dfrac{2}{n+2}z^{\frac{n+2}{2}}\right)\right] \quad (z\geqslant 0)\\ \dfrac{1}{2\sqrt{n+2}\ \sin\dfrac{\pi}{2(n+2)}}\sqrt{-z}\left[I_{-\frac{1}{n+2}}\left(\dfrac{2}{n+2}(-z)^{\frac{n+2}{2}}\right)\right.\\ \qquad\left.-I_{\frac{1}{n+2}}\left(\dfrac{2}{n+2}(-z)^{\frac{n+2}{2}}\right)\right] \quad (z<0)\end{cases} \tag{2.26}
$$

$z\to-\infty$时

$$
IAi_n(z)=\frac{(-z)^{-\frac{n}{4}}}{2\sqrt{\pi}\cos\dfrac{\pi}{2(n+2)}}\exp\left[\frac{2}{n+2}(-z)^{\frac{n+2}{2}}\right]\left(1+O([-z]^{-\frac{n+2}{2}})\right) \tag{2.27}
$$

$$
IBi_n(z)=\frac{\cos\dfrac{\pi}{2(n+2)}}{\sqrt{\pi}}(-z)^{-\frac{n}{4}}\exp\left[-\frac{2}{n+2}(-z)^{\frac{n+2}{2}}\right]\left(1+O([-z]^{-\frac{n+2}{2}})\right) \tag{2.28}
$$

$z\to+\infty$时

$$
IAi_n(z)=\frac{\sin\dfrac{\pi}{2(n+2)}}{\sqrt{\pi}}z^{-\frac{n}{4}}\exp\left[-\frac{2}{n+2}(z)^{\frac{n+2}{2}}\right]\left(1+O(z^{-\frac{n+2}{2}})\right) \tag{2.29}
$$

$$
IBi_n(z)=\frac{1}{2\sqrt{\pi}\sin\dfrac{\pi}{2(n+2)}}z^{-\frac{n}{4}}\exp\left[\frac{2}{n+2}z^{\frac{n+2}{2}}\right]\left(1+O(z^{-\frac{n+2}{2}})\right) \tag{2.30}
$$

因而

$$
IAi_2(z)=\frac{1}{2\sqrt{\pi}\cos\dfrac{\pi}{8}}(-z)^{-1/2}\exp\left[\frac{1}{2}z^2\right](1+O(z^{-2})) \quad (z\to-\infty) \tag{2.31}
$$

$$
IAi_2(z)=\frac{\sin\dfrac{\pi}{8}}{\sqrt{\pi}}z^{-1/2}\exp\left[-\frac{1}{2}z^2\right](1+O(z^{-2})) \quad (z\to+\infty) \tag{2.32}
$$

$$
IBi_2(z)=\frac{\cos\dfrac{\pi}{8}}{\sqrt{\pi}}(-z)^{-1/2}\exp\left[-\frac{1}{2}z^2\right]\cdot(1+O(z^{-2})) \quad (z\to-\infty) \tag{2.33}
$$

$$
IBi_2(z)=\frac{1}{2\sqrt{\pi}\sin\dfrac{\pi}{8}}z^{-1/2}\exp\left[\frac{1}{2}z^2\right]\cdot(1+O(z^{-2})) \quad (z\to+\infty) \tag{2.34}
$$

$IAi_2(z)$，$IBi_2(z)$的图形见图3。

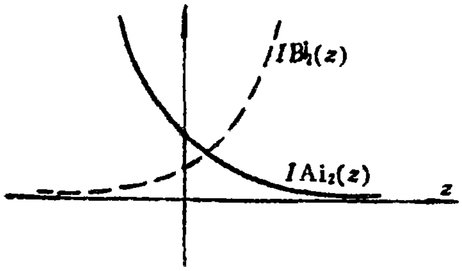

图 3

三、Olver变换

对(1.1)，当$q_1(x)\geqslant 0$时，选择$\zeta=\zeta(z)$

使 $x=\dfrac{q_1(x)}{\zeta'^2}>0$ （只要$\zeta'^2(z)$与$q_1(x)$在相同位置有同阶零点）

若由 $\zeta=\zeta(z)=\int^z\sqrt{q_1(\tau)}\,d\tau$，确定一单调增加或单调减少的隐函数$z=\Phi(x)$。

令 $y=x^{-\frac{1}{4}}v$ $\left(\text{其实}\left|\dfrac{dz}{dx}\right|=x^{1/2}>0,\quad y=\dfrac{v}{\sqrt{|\Phi'(x)|}}\right)$

由(1.1)得 $\dfrac{d^2v}{dz^2}+\lambda^2\zeta'^2v=\delta v$

$$\delta=-\frac{q_2}{x}-x^{-\frac{3}{4}}\frac{d^2x^{-\frac{1}{4}}}{dx^2}=O(1)$$

其首阶近似由相关方程 $\dfrac{d^2v}{dz^2}+\lambda^2\zeta'^2v=0$ (3.1)

给出：

当$q_1(x)\leqslant 0$时，使 $x=\dfrac{-q_1(x)}{\zeta'^2}>0$

若 $\zeta=\zeta(z)=\int^z\sqrt{-q_1(\tau)}\,d\tau$

相关方程为 $\dfrac{d^2v}{dz^2}-\lambda^2\zeta'^2v=0$ (3.2)

例如 $q_1(x)=(x-\mu)^nf(x)$，$f(x)>0$，n为奇数

若选择单调增函数$z=\Phi(x)$

此时$x\leqslant\mu$时$q_1(x)\leqslant 0$ 使

$$\frac{2}{n+2}[-\Phi(x)]^{\frac{n+2}{2}}=\int_x^\mu\sqrt{-q_1(\tau)}\,d\tau \tag{3.3}$$

此时$\zeta'^2=(-z)^n$，相关方程为$\dfrac{d^2v}{dz^2}+\lambda^2z^nv=0$ (3.4)

$x\geqslant\mu$ 时 $q_1(x)\geqslant 0$

使
$$\frac{2}{n+2}[\Phi(x)]^{\frac{n+2}{2}}=\int_\mu^x\sqrt{q_1(\tau)}\,d\tau \tag{3.5}$$

此时$\zeta'^2=z^n$，相关方程仍为 $\dfrac{d^2v}{dz^2}+\lambda^2z^nv=0$ (3.6)

若 n 为偶数相应于(3.3)～(3.6)有类似的结果.

四、具有三个转向点的方程

对含大参数 λ 的方程

$$\frac{d^2y}{dx^2}+[\lambda^2 q_1(x)+q_2(x)]y=0 \tag{4.1}$$

其中$q_1(x)=(x-\mu_1)(x-\mu_2)(x-\mu_3)f(x)$，$f(x)$为正则的（不失一般性假设$f(x)>0$），$q_2(x)$为连续的。

i. 设$\mu_1<\mu_2<\mu_3$

（1） 作函数$z=\Phi_1(x)$

使 $x\leqslant\mu_1$ 时 $\frac{2}{3}[-\Phi_1(x)]^{\frac{3}{2}}=\int_x^{\mu_1}\sqrt{-q_1(\tau)}\,d\tau$，$\Phi_1(x)\leqslant 0$

$\mu_1\leqslant x\leqslant\mu_2$ 时 $\frac{2}{3}[\Phi(x)]^{\frac{3}{2}}=\int_{\mu_1}^{x}\sqrt{q_1(\tau)}\,d\tau$，$\Phi_1(x)\geqslant 0$

其相关方程为 $\frac{d^2v}{dz^2}+\lambda^2 zv=0$

相应的解为
$$y_1=\frac{a_1\mathrm{Ai}(-\lambda^{\frac{2}{3}}\Phi_1)+b_1\mathrm{Bi}(-\lambda^{\frac{2}{3}}\Phi_1)}{\sqrt{\Phi_1'(x)}} \tag{4.2}$$

其中a_1，b_1为常数

（2） 作函数$z=\Phi_2(x)$

使 $\mu_1\leqslant x\leqslant\mu_2$ 时 $\frac{2}{3}[\Phi_2(x)]^{3/2}=\int_x^{\mu_2}\sqrt{q_1(\tau)}\,d\tau$，$\Phi_2(x)\geqslant 0$

$\mu_2<x\leqslant\mu_3$ 时 $\frac{2}{3}[-\Phi_2(x)]^{3/2}=\int_{\mu_2}^{x}\sqrt{-q_1(\tau)}\,d\tau$，$\Phi_2(x)\leqslant 0$

相应地得
$$y_2=\frac{a_2\mathrm{Ai}(-\lambda^{\frac{2}{3}}\Phi_2)+b_2\mathrm{Bi}(-\lambda^{\frac{2}{3}}\Phi_2)}{\sqrt{-\Phi_2'(x)}} \tag{4.3}$$

其中a_2，b_2为常数

（3） 作函数 $z=\Phi_3(x)$

使 $\mu_2\leqslant x\leqslant\mu_3$ $\frac{2}{3}[-\Phi_3(x)]^{3/2}=\int_x^{\mu_3}\sqrt{-q_1(\tau)}\,d\tau$，$\Phi_3(x)\leqslant 0$

$\mu_3<x$ $\frac{2}{3}[\Phi_3(x)]^{3/2}=\int_{\mu_3}^{x}\sqrt{q_1(\tau)}\,d\tau$，$\Phi_3(x)\geqslant 0$

相应地得
$$y_3=\frac{a_3\mathrm{Ai}(-\lambda^{\frac{2}{3}}\Phi_3)+b_3\mathrm{Bi}(-\lambda^{\frac{2}{3}}\Phi_3)}{\sqrt{\Phi_3'(x)}} \tag{4.4}$$

我们用匹配法[4]可确定a_1，b_1，a_2，b_2，a_3，b_3之间的关系。因为(4.2)，(4.3)，在区间$\mu_1+\delta_1<x<\mu_2-\delta_2$是有效的（假设$\delta_1$，$\delta_2$，$\delta_3$为小正数）。

因为$\lambda\to+\infty$时，由(2.7)

$$\mathrm{Ai}(-\lambda^{\frac{2}{3}}\Phi_1)=\frac{\lambda^{-\frac{1}{6}}\Phi_1^{-\frac{1}{4}}}{\sqrt{\pi}}\sin\left(\frac{2}{3}\lambda\Phi_1^{3/2}+\frac{\pi}{4}\right)+O(\lambda^{-\frac{7}{6}})\quad(\Phi_1>0)$$

由(2.12)

$$\mathrm{Bi}(-\lambda^{\frac{2}{3}}\Phi_1)=\frac{\lambda^{-\frac{1}{6}}\Phi_1^{-\frac{1}{4}}}{\sqrt{\pi}}\cos\left(\frac{2}{3}\lambda\Phi_1^{3/2}+\frac{\pi}{4}\right)+O(\lambda^{-7})\quad(\Phi_1>0)$$

$$y_1=\frac{\lambda^{-\frac{1}{6}}}{\sqrt{\pi}\sqrt[4]{q_1(x)}}\left[a_1\sin\left(\frac{2}{3}\lambda\Phi_1^{3/2}+\frac{\pi}{4}\right)+b_1\cos\left(\frac{2}{3}\lambda\Phi_1^{3/2}+\frac{\pi}{4}\right)\right]+O(\lambda^{-\frac{7}{6}})$$

同理

$$y_2=\frac{\lambda^{-\frac{1}{6}}}{\sqrt{\pi}\sqrt[4]{q_1(x)}}\left[a_2\sin\left(\frac{2}{3}\lambda\Phi_2^{3/2}+\frac{\pi}{4}\right)+b_2\cos\left(\frac{2}{3}\lambda\Phi_2^{3/2}+\frac{\pi}{4}\right)\right]+O(\lambda^{-\frac{7}{6}})\quad(\Phi_2>0)$$

令 $$\Lambda_1=\lambda\left(\frac{2}{3}\Phi_1^{3/2}+\frac{2}{3}\Phi_2^{3/2}\right)+\frac{\pi}{2}=\lambda\int_{\mu_1}^{\mu_2}\sqrt{q_1(\tau)}\,d\tau+\frac{\pi}{2}$$

对y_1，y_2进行匹配，比较得

$$\begin{cases}a_1=b_2\sin\Lambda_1-a_2\cos\Lambda_1\\ b_1=a_2\sin\Lambda_1+b_2\cos\Lambda_1\end{cases}\tag{4.5}$$

在区间$\mu_2+\delta_2<x<\mu_3-\delta_3$，将$y_2$，$y_3$匹配如下：

由(2.9) $$\mathrm{Ai}(-\lambda^{\frac{2}{3}}\Phi_2)=\frac{\lambda^{-\frac{1}{6}}(-\Phi_2)^{-\frac{1}{4}}}{2\sqrt{\pi}}\exp\left[-\frac{2}{3}\lambda(-\Phi_2)^{3/2}\right](1+O(\lambda^{-1}))\quad(\Phi_2<0)$$

由(2.14) $$\mathrm{Bi}(-\lambda^{\frac{2}{3}}\Phi_2)=\frac{\lambda^{-\frac{1}{6}}(-\Phi_2)^{-\frac{1}{4}}}{\sqrt{\pi}}\exp\left[\frac{2}{3}\lambda(-\Phi_2)^{3/2}\right](1+O(\lambda^{-1}))\quad(\Phi_2<0)$$

$$y_2=\frac{\lambda^{-\frac{1}{6}}}{\sqrt{\pi}\sqrt[4]{-q_1(x)}}\left[\frac{a_2}{2}\exp\left[-\frac{2}{3}\lambda(-\Phi_2)^{3/2}\right](1+O(\lambda^{-1}))\right.$$
$$\left.+b_2\exp\left[\frac{2}{3}\lambda(-\Phi_2)^{3/2}\right](1+O(\lambda^{-1}))\right]$$

同理

$$y_3=\frac{\lambda^{-\frac{1}{6}}}{\sqrt{\pi}\sqrt[4]{-q_1(x)}}\left[\frac{a_3}{2}\exp\left[-\frac{2}{3}\lambda(-\Phi_3)^{3/2}\right](1+O(\lambda^{-1}))\right.$$
$$\left.+b_3\exp\left[\frac{2}{3}\lambda(-\Phi_2)^{3/2}\right](1+O(\lambda^{-1}))\right]\quad(\Phi_3<0)$$

令 $$\Lambda_2=\lambda\left[\frac{2}{3}(-\Phi_2)^{3/2}+\frac{2}{3}(-\Phi_3)^{3/2}\right]=\lambda\int_{\mu_2}^{\mu_3}\sqrt{-q_1(\tau)}\,d\tau$$

比较而得 $$\begin{cases}b_2=\frac{a_3}{2}\exp[-\Lambda_2]\\ b_3=\frac{a_2}{2}\exp[-\Lambda_2]\end{cases}\tag{4.6}$$

将a_2，a_3视为任意常数，则由(4.5)，(4.6)，a_1，b_1，b_2，b_3随a_2，a_3确定而确定。

此时(4.1)的渐进解当$x<\mu_2$时，用(4.2)表达，当$\mu_1<x<\mu_3$时用(4.3)表达，当$x>\mu_2$时用(4.4)表达。此时渐近解的图形见图4。

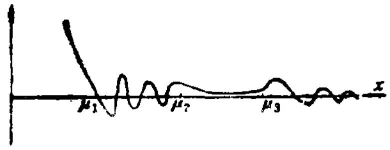

图 4

ii. 设$\mu_1<\mu_2=\mu_3$

（1） 作函数$z=\Phi_1(x)$

当 $x\leqslant\mu_1$ 时，$\frac{2}{3}[-\Phi_1(x)]^{3/2}=\int_x^{\mu_1}\sqrt{-q_1(\tau)}\,d\tau$，　$\Phi_1(x)\leqslant 0$

$\mu_1<x\leqslant\mu_2$ 时，$\frac{2}{3}[\Phi_1(x)]^{3/2}=\int_{\mu_1}^{x}\sqrt{q_1(\tau)}\,d\tau$，　$\Phi_1(x)\geqslant 0$

$$y_1=\frac{a_1\mathrm{Ai}(-\lambda^{\frac{2}{3}}\Phi_1)+b_1\mathrm{Bi}(-\lambda^{\frac{2}{3}}\Phi_1)}{\sqrt{\Phi_1'(x)}}\tag{4.7}$$

（2） 作函数 $z=\Phi_2(x)$

当 $\mu_1\leqslant x\leqslant\mu_2$ 时　$\frac{1}{2}[\Phi_2(x)]^2=\int_x^{\mu_2}\sqrt{q_1(\tau)}\,d\tau$，　$\Phi_2(x)\geqslant 0$

$\mu_2<x$ 时　　$\frac{1}{2}[-\Phi_2(x)]^2=\int_{\mu_2}^{x}\sqrt{q_1(\tau)}\,d\tau$，　$\Phi_2(x)<0$

$$y_2=\frac{a_2\mathrm{Ai}_2(\lambda^{\frac{1}{2}}\Phi_2)+b_2\mathrm{Bi}_2(\lambda^{\frac{1}{2}}\Phi_2)}{\sqrt{-\Phi_2'(x)}}\tag{4.8}$$

$\mu_1<x<\mu_2$ 时

$$y_1=\frac{\lambda^{-\frac{1}{6}}}{\sqrt{\pi}\sqrt[4]{q_1(x)}}\left[a_1\sin\left(\frac{2}{3}\lambda\Phi_1^{\frac{3}{2}}+\frac{\pi}{4}\right)+b_1\cos\left(\frac{2}{3}\lambda\Phi_1^{\frac{3}{2}}+\frac{\pi}{4}\right)\right]+O(\lambda^{-\frac{7}{6}})$$

由(2.22)：

$$\mathrm{Ai}_2(\lambda^{\frac{1}{2}}\Phi_2)=\frac{\mathrm{tg}\frac{\pi}{8}}{\sqrt{\pi}}\lambda^{-\frac{1}{4}}\Phi_2^{-\frac{1}{2}}\cos\left(\frac{1}{2}\lambda\Phi_2^2+\frac{\pi}{4}\right)+O(\lambda^{-\frac{5}{4}})$$

由(2.24)：

$$\mathrm{Bi}_2(\lambda^{\frac{1}{2}}\Phi_2)=\frac{\mathrm{ctg}\frac{\pi}{8}}{\sqrt{\pi}}\lambda^{-\frac{1}{4}}\Phi_2^{-\frac{1}{2}}\sin\left(\frac{1}{2}\lambda\Phi_2^2+\frac{\pi}{4}\right)+O(\lambda^{-\frac{5}{4}})$$

又 $\mu_1<x<\mu_2$ 时

$$y_2=\frac{\lambda^{-\frac{1}{4}}}{\sqrt{\pi}\sqrt[4]{q_1(x)}}\left[a_2\mathrm{tg}\frac{\pi}{8}\cos\left(\frac{1}{2}\lambda\Phi_2^2+\frac{\pi}{4}\right)+b_2\mathrm{ctg}\frac{\pi}{8}\sin\left(\frac{1}{2}\lambda\Phi_2^2+\frac{\pi}{4}\right)\right]+O(\lambda^{-\frac{5}{4}})$$

令　$$\Lambda_1=\lambda\left(\frac{2}{3}\Phi_1^{\frac{3}{2}}+\frac{1}{2}\Phi_2\right)+\frac{\pi}{2}=\lambda\int_{\mu_1}^{\mu_2}\sqrt{q_1(\tau)}\,d\tau+\frac{\pi}{2}$$

比较 y_1，y_2 得：

$$\begin{cases}a_1=\Lambda_1^{-\frac{1}{12}}\left(a_2\mathrm{tg}\frac{\pi}{8}\sin\Lambda_1-b_2\mathrm{ctg}\frac{\pi}{8}\cos\Lambda_1\right)\\ b_1=\Lambda_1^{-\frac{1}{12}}\left(a_2\mathrm{tg}\frac{\pi}{8}\cos\Lambda_1+b_2\mathrm{ctg}\frac{\pi}{8}\sin\Lambda_1\right)\end{cases}\tag{4.9}$$

将 a_2，b_2 视为任意常数，则由（4.9）a_1，b_1 随 a_2，b_2 确定而确定，此时(4.1)的渐近解由（4.7）和（4.8）联合表达。

此时渐近解的图形见图5

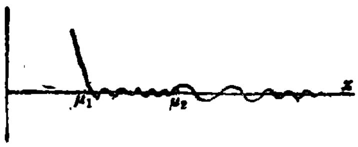

图　5

iii．设 $\mu_1=\mu_2<\mu_3$

（1） 作函数 $z=\Phi_2(x)$

当 $x\leqslant\mu_2$ 时， $\frac{1}{2}[-\Phi_2(x)]^2=\int_x^{\mu_2}\sqrt{-q_1(\tau)}\,d\tau$， $\Phi_2(x)\leqslant 0$

$\mu_2<x\leqslant\mu_3$ 时， $\frac{1}{2}[\Phi_2(x)]^2=\int_{\mu_2}^{x}\sqrt{-q_1(\tau)}\,d\tau$， $\Phi_2(x)\geqslant 0$

$$y_2=\frac{a_2 I\,\mathrm{Ai}_2(\lambda^{\frac{1}{2}}\Phi_2)+b_2 I\,\mathrm{Bi}_2(\lambda^{\frac{1}{2}}\Phi_2)}{\sqrt{\Phi_2'(x)}} \tag{4.10}$$

（2） 作函数 $z=\Phi_3(x)$

当 $\mu_2\leqslant x\leqslant\mu_3$ 时 $\frac{2}{3}[\Phi_3(x)]^{3/2}=\int_x^{\mu_3}\sqrt{-q_1(\tau)}\,d\tau$， $\Phi_3(x)\geqslant 0$

$\mu_3<x$ 时 $\frac{2}{3}[-\Phi_3(x)]^{3/2}=\int_{\mu_3}^{x}\sqrt{q_1(\tau)}\,d\tau$， $\Phi_3(x)\leqslant 0$

$$y_3=\frac{a_3\mathrm{Ai}(\lambda^{\frac{2}{3}}\Phi_3)+b_3\mathrm{Bi}(\lambda^{\frac{2}{3}}\Phi_3)}{\sqrt{-\Phi_3'(x)}} \tag{4.11}$$

当 $\mu_2<x<\mu_3$

由(2.32)

$$I\,\mathrm{Ai}_2(\lambda^{\frac{1}{2}}\Phi_2)=\frac{\lambda^{-\frac{1}{4}}\Phi_2^{-\frac{1}{2}}\sin\frac{\pi}{8}}{\sqrt{\pi}}\exp\left[-\frac{1}{2}\lambda\Phi_2^2\right](1+O(\lambda^{-1}))$$

由(2.34)

$$I\,\mathrm{Bi}_2(\lambda^{\frac{1}{2}}\Phi_2)=\frac{\lambda^{-\frac{1}{4}}\Phi_2^{-\frac{1}{2}}}{2\sqrt{\pi}\sin\frac{\pi}{8}}\exp\left[\frac{1}{2}\lambda\Phi_2^2\right](1+O(\lambda^{-1}))$$

$$y_2=\frac{\lambda^{-\frac{1}{4}}}{\sqrt{\pi}\sqrt[4]{-q_1(x)}}\left[a_2\left(\sin\frac{\pi}{8}\right)\exp\left[-\frac{1}{2}\lambda\Phi_2^2\right](1+O(\lambda^{-1}))\right.$$

$$\left.+b_2\frac{1}{2\sin\frac{\pi}{8}}\exp\left[\frac{1}{2}\lambda\Phi_2^2\right](1+O(\lambda^{-1}))\right]$$

又，当 $\mu_2<x<\mu_3$ 时

$$y_3=\frac{\lambda^{-\frac{1}{6}}}{\sqrt{\pi}\sqrt[4]{-q_1(x)}}\left[\frac{a_3}{2}\exp\left[-\frac{2}{3}\lambda\Phi_3^{3/2}\right](1+O(\lambda^{-1}))\right.$$

$$\left.+b_3\exp\left[\frac{2}{3}\lambda\Phi_3^{3/2}\right](1+O(\lambda^{-1}))\right]$$

令 $\Lambda_2=\lambda\left[\frac{1}{2}\Phi_2^2+\frac{2}{3}\Phi_3^{3/2}\right]=\lambda\int_{\mu_2}^{\mu_3}\sqrt{-q_1(\tau)}\,d\tau$

比较 y_2，y_3 得：

$$\begin{cases} b_2=a_3\exp[-\Lambda_2]\lambda^{\frac{1}{12}}\sin\frac{\pi}{8} \\ b_3=a_2\exp[-\Lambda_2]\lambda^{-\frac{1}{12}}\sin\frac{\pi}{8} \end{cases} \tag{4.12}$$

将 a_2，a_3 视为任意常数，由(4.12)，b_2、b_3 随 a_2、a_3 确定而确定，用(4.10)，(4.11)给

出(4.1)的渐进解．此时渐近解的图形见图6．

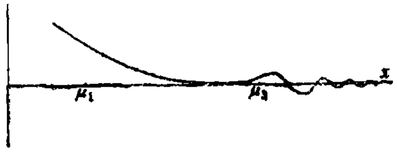

图 6

iv 当 $\mu_1=\mu_2=\mu_3$ 时

作函数 $z=\Phi_1(x)$，

当 $x\leqslant\mu_1$ 时

$$\frac{2}{5}[-\Phi_1(x)]^{5/2}=\int_x^{\mu_1}\sqrt{-q_1(\tau)}\,d\tau,\qquad \Phi_1(x)\leqslant 0$$

$$x>\mu_1\text{ 时}\quad \frac{2}{5}[\Phi_1(x)]^{5/2}=\int_{\mu_1}^{x}\sqrt{q_1(\tau)}\,d\tau,\qquad \Phi_1(x)\geqslant 0$$

得 (4.1) 的渐近解

$$y=\frac{a_1\mathrm{Ai}_3(-\lambda^{\frac{2}{5}}\Phi_1)+b_1\mathrm{Bi}_3(-\lambda^{\frac{2}{5}}\Phi_1)}{\sqrt{\Phi_1'(x)}}$$

此时渐近解的图形见图7．

五、具有 n 个转向点的方程

前述用广义 Airy 函数解决三个转向点问题的方法，可以用来解决一般的 n 个转向点的问题．

具有大参数 λ 的二阶线性微分方程：

$$\frac{d^2y}{dx^2}+[\lambda^2q_1(x)+q_2(x)]y=0,$$

其中 $q_1(x)=(x-\mu_1)^{\nu_1}(x-\mu_2)^{\nu_2}\cdots(x-\mu_n)^{\nu_n}f(x)$

（不失一般性假设 $f(x)>0$），$\nu_1,\nu_2,\cdots,\nu_n$ 为 n 个正整数，为一般的具有 n 个转向点问题，为简便起见，具体考虑 $q_1(x)=(x-\mu_1)(x-\mu_2)^2(x-\mu_3)^3(x-\mu_4)^4$，$(\mu_1<\mu_2<\mu_3<\mu_4)$

i．作函数 $z=\Phi_1(x)$

$$\text{当 } x\leqslant\mu_1\text{ 时，使}\frac{2}{3}[-\Phi_1(x)]^{3/2}=\int_x^{\mu_1}\sqrt{q_1(\tau)}\,d\tau,\qquad \Phi_1(x)\leqslant 0$$

$$\mu_1<x\leqslant\mu_2,\qquad \frac{2}{3}[\Phi_1(x)]^{3/2}=\int_{\mu_1}^{x}\sqrt{-q_1(\tau)}\,d\tau,\qquad \Phi_1(x)\geqslant 0$$

$$y_1=\frac{a_1\mathrm{Ai}(\lambda^{\frac{2}{3}}\Phi_1)+b_1\mathrm{Bi}(\lambda^{\frac{2}{3}}\Phi_1)}{\sqrt{\Phi_1'(x)}}\tag{5.1}$$

ii．作函数 $z=\Phi_2(x)$

$$\text{当}\mu_1\leqslant x\leqslant\mu_2\text{时，使 }\frac{1}{2}[\Phi_2(x)]^2=\int_x^{\mu_2}\sqrt{-q_1(\tau)}\,d\tau,\qquad \Phi_2(x)\geqslant 0$$

$$\mu_2\leqslant x\leqslant\mu_3\qquad \frac{1}{2}[-\Phi_2(x)]^2=\int_{\mu_2}^{x}\sqrt{-q_1(\tau)}\,d\tau,\qquad \Phi_2(x)\leqslant 0$$

$$y_2=\frac{a_2I\mathrm{Ai}_2(\lambda^{\frac{1}{2}}\Phi_2)+b_2I\mathrm{Bi}_2(\lambda^{\frac{1}{2}}\Phi_2)}{\sqrt{-\Phi_2'(x)}}\tag{5.2}$$

iii．作函数 $z=\Phi_3(x)$

当 $\mu_2 \leqslant x \leqslant \mu_3$ 时，　$\frac{2}{5}[-\Phi_3(x)]^{5/2}=\int_x^{\mu_3}\sqrt{-q_1(\tau)}\,d\tau$，　$\Phi_3(x)\leqslant 0$

$\mu_3 < x \leqslant \mu_4$　　$\frac{2}{5}[\Phi_3(x)]^{5/2}=\int_{\mu_3}^{x}\sqrt{q_1(\tau)}\,d\tau$，　$\Phi_3(x)\geqslant 0$

$$y_3=\frac{a_3\mathrm{Ai}_3(-\lambda^{\frac{2}{5}}\Phi_3)+b_3\mathrm{Bi}_3(-\lambda^{\frac{2}{5}}\Phi_3)}{\sqrt{\Phi_3'(x)}} \tag{5.3}$$

iv. 作函数 $z=\Phi_4(x)$

当 $\mu_3 \leqslant x \leqslant \mu_4$ 时，　$\frac{1}{3}[\Phi_4(x)]^3=\int_x^{\mu_4}\sqrt{q_1(\tau)}\,d\tau$，　$\Phi_4(x)\geqslant 0$

$x>\mu_4$　　$\frac{1}{3}[-\Phi_4(x)]^3=\int_{\mu_4}^{x}\sqrt{q_1(\tau)}\,d\tau$，　$\Phi_4(x)\leqslant 0$

$$y_4=\frac{a_4\mathrm{Ai}_4(\lambda^{\frac{1}{3}}\Phi_4)+b_4\mathrm{Bi}_4(\lambda^{\frac{1}{3}}\Phi_4)}{\sqrt{-\Phi_4'(x)}} \tag{5.4}$$

将y_1、y_2作匹配。令 $\Lambda_1=\lambda\left(\frac{2}{3}\Phi_1^{3/2}+\frac{1}{2}\Phi_2^2\right)=\lambda\int_{\mu_1}^{\mu_2}\sqrt{-q_1(\tau)}\,d\tau$ 利用(2.9)、(2.14)、(2.32)、(2.34)得：

$$\begin{cases} b_2=a_1\exp[-\Lambda_1]\lambda^{\frac{1}{12}}\sin\frac{\pi}{8} \\ b_1=a_2\exp[-\Lambda_1]\lambda^{-\frac{1}{12}}\sin\frac{\pi}{8} \end{cases} \tag{5.5}$$

将y_2、y_3作匹配，令$\Lambda_2=\lambda\left[\frac{1}{2}(-\Phi_2)^2+\frac{2}{5}(-\Phi_3)^{5/2}\right]=\lambda\int_{\mu_2}^{\mu_3}\sqrt{-q_1(\tau)}\,d\tau$利用(2.31)、(2.33)、(2.8)、(2.13)得：

$$\begin{cases} a_2=a_3\exp[-\Lambda_2]\lambda^{-\frac{1}{20}}2\cos\frac{\pi}{8}\sin\frac{\pi}{10} \\ b_3=b_2\exp[-\Lambda_2]\lambda^{\frac{1}{20}}2\cos\frac{\pi}{8}\sin\frac{\pi}{10} \end{cases} \tag{5.6}$$

将y_3，y_4作匹配，令 $\Lambda_3=\lambda\left(\frac{2}{5}\Phi_3^{5/2}+\frac{1}{3}\Phi_4^3\right)+\frac{\pi}{2}=\int_{\mu_3}^{\mu_4}\sqrt{q_1(\tau)}\,d\tau+\frac{\pi}{2}$利用(2.6)、(2.11)、(2.19)、(2.20)得：

$$\begin{cases} a_4=\lambda^{-\frac{1}{30}}(a_3\sin\Lambda_3+b_3\cos\Lambda_3)\mathrm{ctg}\frac{\pi}{12} \\ b_4=\lambda^{-\frac{1}{30}}(b_3\sin\Lambda_3-a_3\cos\Lambda_3)\mathrm{tg}\frac{\pi}{12} \end{cases} \tag{5.7}$$

其中a_1，a_3（限制$a_3=O(\lambda^{-\frac{1}{30}})$）为任意常数，其余$b_1$，$a_2$，$b_2$，$b_3$，$a_4$，$b_4$由(5.5)、(5.6)、(5.7)。随$a_1$，$a_3$确定而确定，此时所求渐近解由(5.1)，(5.2)，(5.3)，(5.4)联合表示。此时渐近解的图形见图8。

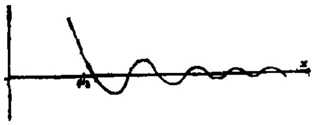

图 7

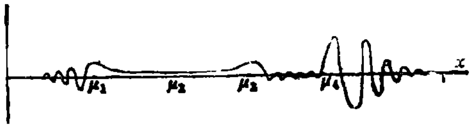

图 8

参 考 文 献

[1] 蒲富全，具有二阶转向点方程的一致有效渐近解问题，清华大学学报，20 (3) (1980)，103—109.

[2] 郭敦仁，《数学物理方程》，人民出版社 (1979).

[3] Olver, F. W. J., *Asymptotics and Special Functions* (1974).

[4] Nayfeh, A. H., *Perturbation Methods* (1973).

[5] Sibuya, Y., *Global Theory of a Second Order Linear Ordinary Differential Equation with a Polynomial Coefficient*, North-Holland Publishing Company (1975).

Extended Airy Function and Differential Equations with n-Turning Points

Zhang Ju-ling

(Sichuan Institute of Light Industry and Chemical Technology, Zigong)

Abstract

This paper studies a second order linear ordinary differential equation with n-turning points

$$\frac{d^2y}{dx^2}+(\lambda^2q_1(x)+q_2(x))y=0$$

where $q_1(x)=(x-\mu_1)(x-\mu_2)\cdots(x-x_n)f(x)$, $f(x)\neq 0$, and λ is a large parameter. The formal uniformly valid asymptotic solution of the equation is obtained based on the analysis of the three points by means of the matched method. By the work a method is developed and the applicability of this method to the n turning points is demonstrated.

Key words uniformly valid, asymptotic solution, turning point