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弹性矩形薄板受迫振动的功的互等定理法（Ⅲ）—悬臂矩形板*

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摘　要

本文应用功的互等定理法给出了在均布谐载和在任意点受集中谐载作用下悬臂矩形板受迫振动的稳态解，并给出了有关弯矩和挠度幅值的图表。

关键词　功的互等定理法　受迫振动　悬臂矩形板　弯矩幅值　挠度幅值

悬臂矩形板静力弯曲问题的求解是非常困难的，但由于世界各国科学工作者的努力，它已获解决。然而，对于悬臂矩形板的受迫振动问题至今尚无人问津。本文将给出悬臂矩形板受迫振动的稳态解。

一、受简谐均布载荷作用的悬臂矩形板

让我们考虑如图1所示的悬臂矩形板。干扰力为一简谐均布载荷 $F(x,y,t)=q\sin\omega t$。解除沿固定边的弯曲约束并以M_{y0}代之，则得到作为实际系统的如图2所示的矩形板。

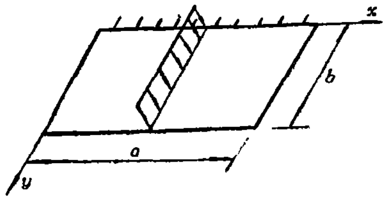

图　1

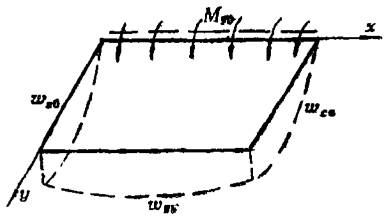

图　2

这是一对称问题，所以我们可以假设固定边的弯矩幅值为

$$M_{y0}=\sum_{m=1,3}^{\infty}G_m\sin k_m x \tag{1.1}$$

* 钱伟长推荐，1987年9月8日收到。

和三个自由边的位移幅值为

$$w_{yb}=k+\sum_{m=1,3}^{\infty}B_m\sin k_m x$$

$$w_{x0}=w_{xa}=\frac{y}{b}k+\sum_{n=1,2}^{\infty}A_n\sin k_n y \tag{1.2}$$

在文[14]的图1所示基本系统与本文图2所示实际系统之间应用功的互等定理，则得

$$w(\xi,\ \eta)=\int_0^a\int_0^b qw_1\,dxdy+\int_0^a M_{y0}\left(\frac{\partial w_1}{\partial y}\right)_{y=0}dx-\int_0^a(v_{1y})_{y=b}w_{yb}dy$$

$$+\int_0^b[(v_{1x})_{x=0}-(v_{1x})_{x=a}]w_{x_0}dy+[(R_1)_{\substack{x=a\\y=b}}-(R_1)_{\substack{x=0\\y=0}}]k \tag{1.3}$$

当$\lambda<k_m^2$和$\lambda<k_n^2$时，我们得到

$$w(\xi,\ \eta)=\sum_{m=1,3}^{\infty}\frac{4q}{m\pi D}\left\{\frac{1}{\alpha_m^2-\beta_m^2}\left[\frac{\mathrm{ch}\alpha_m\left(\frac{b}{2}-\eta\right)}{\alpha_m^2\mathrm{ch}\alpha_m\frac{b}{2}}-\frac{\mathrm{ch}\beta_m\left(\frac{b}{2}-\eta\right)}{\beta_m^2\mathrm{ch}\beta_m\frac{b}{2}}\right]+\frac{1}{\alpha_m^2\beta_m^2}\right\}\sin k_m\xi$$

$$\left(或\sum_{n=1,3}^{\infty}\frac{4q}{n\pi D}\left\{\frac{1}{\alpha_n^2-\beta_n^2}\left[\frac{\mathrm{ch}\alpha_n\left(\frac{a}{2}-\xi\right)}{\alpha_n^2\mathrm{ch}\alpha_n\frac{a}{2}}-\frac{\mathrm{ch}\beta_n\left(\frac{a}{2}-\xi\right)}{\beta_n^2\mathrm{ch}\beta_n\frac{a}{2}}\right]+\frac{1}{\alpha_n^2\beta_n^2}\right\}\sin k_n\eta\right)$$

$$+\sum_{m=1,3}^{\infty}\frac{G_m}{\alpha_m^2-\beta_m^2}\left[-\frac{\mathrm{sh}\alpha_m(b-\eta)}{\mathrm{sh}\alpha_m b}+\frac{\mathrm{sh}\beta_m(b-\eta)}{\mathrm{sh}\beta_m b}\right]\sin k_m\xi$$

$$+\sum_{m=1,3}^{\infty}\frac{\beta_m}{\alpha_m^2-\beta_m^2}\left\{\frac{[\alpha_m^2-k_m^2(2-\mu)]}{\mathrm{sh}\alpha_m b}\mathrm{sh}\,\alpha_m\eta-\frac{[\beta_m^2-k_m^2(2-\mu)]}{\mathrm{sh}\beta_m b}\mathrm{sh}\beta_m\eta\right\}\sin k_m\xi$$

$$+\sum_{n=1,2}^{\infty}\frac{A_n}{\alpha_n^2-\beta_n^2}\left\{\frac{[\alpha_n^2-k_n^2(2-\mu)]}{\mathrm{ch}\alpha_n\frac{a}{2}}\mathrm{ch}\,\alpha_n\left(\frac{a}{2}-\xi\right)-\frac{[\beta_n^2-k_n^2(2-\mu)]}{\mathrm{ch}\beta_n\frac{a}{2}}\right.$$

$$\left.\cdot\,\mathrm{ch}\alpha_n\left(\frac{a}{2}-\xi\right)\right\}\sin k_n\eta$$

$$+\sum_{m=1,3}^{\infty}\frac{4k\lambda^2}{m\pi}\left\{\frac{1}{\alpha_m^2-\beta_m^2}\left[\frac{\mathrm{sh}\alpha_m\eta}{\alpha_m^2\mathrm{sh}\alpha_m b}-\frac{\mathrm{sh}\beta_m\eta}{\beta_m^2\mathrm{sh}\beta_m b}\right]+\frac{\eta}{\alpha_m^2\beta_m^2 b}\right\}\sin k_m\xi+\frac{\eta}{b}k$$

$$\left(或-\sum_{n=1,2}^{\infty}\frac{2k\lambda^2}{n\pi}\left\{\frac{1}{\alpha_n^2-\beta_n^2}\left[\frac{\mathrm{ch}\alpha_n\left(\frac{a}{2}-\xi\right)}{\alpha_n^2\mathrm{ch}\alpha_n\frac{a}{2}}-\frac{\mathrm{ch}\beta_n\left(\frac{a}{2}-\xi\right)}{\beta_n^2\mathrm{ch}\beta_n\frac{a}{2}}\right]+\frac{1}{\alpha_n^2\beta_n^2}\right\}\sin k_n\eta\cos n\pi\right.$$

$$\left.+\frac{\eta}{b}k\right) \tag{1.4}$$

而当$\lambda>k_m^2$和$\lambda>k_n^2$时，我们只需将$\beta_m\to i\beta_m'$和$\beta_n\to i\beta_n'$代入(1.4)，即可得到相应的表达式，故可略去。

满足板的边界条件，对于$\lambda<k_m^2$和$\lambda<k_n^2$的情况，我们得到

$$\frac{4q}{m\pi D}\left(-\frac{\text{th}\alpha_m\frac{b}{2}}{\alpha_m}+\frac{\text{th}\beta_m\frac{b}{2}}{\beta_m}\right)+\frac{G_m}{D}(\alpha_m(\text{th}\alpha_m b-\beta_m\text{cth}\beta_m b)+B_m\left\{\frac{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]}{\text{sh}\alpha_m b}\right.$$

$$\left.-\frac{\beta_m[\beta_m^2-k_m^2(2-\mu)]}{\text{sh}\beta_m b}\right\}+\sum_{n=1,2}^{\infty}\frac{8\lambda}{a}A_n\frac{k_mk_n[k_m^2+k_n^2(2-\mu)]}{k_{mn}}+\frac{4\lambda}{m\pi}k\left[\frac{2}{b}\left(1+\frac{\lambda^2}{\alpha_m^2\beta_m^2}\right)\right.$$

$$\left.+\lambda\left(\frac{1}{\alpha_m\text{sh}\alpha_m b}-\frac{1}{\beta_m\text{sh}\beta_m b}\right)\right]=0 \tag{1.5}$$

$$\frac{4q}{m\pi D}\left\{\frac{[\alpha_m^2-k_m^2(2-\mu)]}{\alpha_m^2}\text{th}\frac{\alpha_m b}{2}-\frac{[\beta_m^2-k_m^2(2-\mu)]}{\beta_m^2}\text{th}\frac{\beta_m b}{2}\right\}+\frac{G_m}{D}\left\{\frac{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]}{\text{sh}\alpha_m b}\right.$$

$$\left.-\frac{\beta_m[\beta_m^2-k_m^2(2-\mu)]}{\text{sh}\beta_m b}\right\}+B_m\{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]^2\text{cth}\alpha_m b-\beta_m[\beta_m^2-k_m^2(2-\mu)]^2$$

$$\cdot\text{cth}\beta_m b\}-\sum_{n=1,2}^{\infty}\frac{8\lambda}{a}A_n\frac{k_mk_n[\lambda^2(2-\mu)+(1-\mu)^2k_m^2k_n^2]}{k_{mn}}\cos n\pi+\frac{4\lambda^2}{m\pi}k$$

$$\cdot\left\{\frac{[\alpha_m^2-k_m^2(2-\mu)]}{\alpha_m}\text{cth}\alpha_m b-\frac{[\beta_m^2-k_m^2(2-\mu)]}{\beta_m}\text{cth}\beta_m b-\frac{2\lambda(2-\mu)k_m^2}{\alpha_m^2\beta_m^2 b}\right\}=0 \tag{1.6}$$

$$-\frac{2q(1-\cos n\pi)}{n\pi D}\left\{\frac{[\alpha_n^2-k_n^2(2-\mu)]}{\alpha_n^2}\text{th}\alpha_n\frac{a}{2}-\frac{[\beta_n^2-k_n^2(2-\mu)]}{\beta_n^2}\text{th}\beta_n\frac{a}{2}\right\}-\frac{4\lambda}{bD}\sum_{m=1,3}^{\infty}G_m$$

$$\cdot\frac{k_mk_n[k_m^2+k_n^2(2-\mu)]}{k_{mn}}+\frac{4\lambda}{b}\sum_{m=1,3}^{\infty}B_m\frac{k_mk_n[\lambda^2(2-\mu)+(1-\mu)^2k_m^2k_n^2]}{k_{mn}}\cos n\pi-A_n\left\{\alpha_n\right.$$

$$\left.\cdot[\alpha_n^2-k_n^2(2-\mu)]^2\text{th}\alpha_n\frac{a}{2}-\beta_n[\beta_n^2-k_n^2(2-\mu)]^2\text{th}\beta_n\frac{a}{2}\right\}+\frac{2\lambda^2}{n\pi}k\left\{\frac{[\alpha_n^2-k_n^2(2-\mu)]}{\alpha_n}\right.$$

$$\left.\cdot\text{th}\alpha_n\frac{a}{2}-\frac{[\beta_n^2-k_n^2(2-\mu)]}{\beta_n}\text{th}\beta_n\frac{a}{2}\right\}\cos n\pi=0 \tag{1.7}$$

$$\frac{4q}{aD}\sum_{m=1,3}^{\infty}\left(\frac{\text{th}\alpha_m\frac{b}{2}}{\alpha_m}-\frac{\text{th}\beta_m\frac{b}{2}}{\beta_m}\right)+\frac{1}{D}\sum_{m=1,3}^{\infty}G_mk_m\left(\frac{\alpha_m}{\text{sh}\alpha_m b}-\frac{\beta_m}{\text{sh}\beta_m b}\right)+\sum_{m=1,3}^{\infty}B_mk_m$$

$$\{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]\text{cth}\alpha_m b-\beta_m[\beta_m^2-k_m^2(2-\mu)]\text{cth}\beta_m b\}-\sum_{n=1,2}^{\infty}A_nk_n\left\{\alpha_n[\alpha_n^2\right.$$

$$\left.-k_n^2(2-\mu)]\text{th}\alpha_n\frac{a}{2}-\beta_n[\beta_n^2-k_n^2(2-\mu)]\text{th}\beta_n\frac{a}{2}\right\}\cos n\pi+\sum_{m=1,3}^{\infty}\frac{4\lambda^2}{a}k\left[\left(\frac{\text{cth}\alpha_m b}{\alpha_m}\right.\right.$$

$$\left.\left.-\frac{\text{cth}\beta_m b}{\beta_m}\right)+\frac{2\lambda}{\alpha_m^2\beta_m^2 b}\right]=0 \tag{1.8}$$

而对于$\lambda>k_m^2$和$\lambda>k_n^2$，(1.5)～(1.8)变为

$$\frac{4q}{m\pi D}\left(-\frac{\text{th}\alpha_m\frac{b}{2}}{\alpha_m}+\frac{\text{tg}\beta_m'\frac{b}{2}}{\beta_m'}\right)+\frac{G_m}{D}(\alpha_m\text{cth}\alpha_m b-\beta_m'\text{ctg}\beta_m' b)+B_m\left\{\frac{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]}{\text{sh}\alpha_m b}\right.$$

$$+\frac{\beta'_m[\beta'^2_m+k^2_m(2-\mu)]}{\sin\beta'_m b}\Bigg\}+\sum_{n=1,2}^{\infty}\frac{8\lambda}{a}A_n\frac{k_mk_n[k^2_m+k^2_n(2-\mu)]}{k_{mn}}-\frac{4\lambda}{m\pi}k\Bigg[\frac{2}{b}\Big(1-\frac{\lambda^2}{\alpha^2_m\beta'^2_m}\Big)$$

$$+\lambda\Big(\frac{1}{\alpha_m\mathrm{sh}\alpha_m b}-\frac{1}{\beta'_m\sin\beta'_m b}\Big)\Bigg]=0 \tag{1.9}$$

$$\frac{4q}{m\pi D}\Bigg\{\frac{[\alpha^2_m-k^2_m(2-\mu)]}{\alpha_m}\mathrm{th}\alpha_m\frac{b}{2}+\frac{[\beta'^2_m+k^2_m(2-\mu)]}{\beta'_m}\mathrm{tg}\beta'_m\frac{b}{2}\Bigg\}+\frac{G_m}{D}\Bigg\{\frac{\alpha_m[\alpha^2_m-k^2_m(2-\mu)]}{\mathrm{sh}\alpha_m b}$$

$$+\frac{\beta'_m[\beta'^2_m+k^2_m(2-\mu)]}{\sin\beta'_m b}\Bigg\}+B_m\{\alpha_m[\alpha^2_m-k^2_m(2-\mu)]^2\mathrm{ctb}\alpha_m b-\beta'_m[\beta'^2_m+k^2_m(2-\mu)]^2$$

$$\mathrm{ctg}\beta'_m b\}-\sum_{n=1,2}^{\infty}\frac{8\lambda}{a}A_n\frac{k_mk_n[\lambda^2(2-\mu)+(1-\mu)^2k^2_mk]}{k_{mn}}\cos n\pi+\frac{4\lambda^2}{m\pi}k\Bigg\{\frac{[\alpha^2_m-k^2_m(2-\mu)]}{\alpha_m}$$

$$\mathrm{cth}\,\alpha_m b-\frac{[\beta'^2_m+k^2_m(2-\mu)]}{\beta'_m}\mathrm{ctg}\beta'_m b+\frac{2\lambda(2-\mu)k^2_m}{\alpha^2_m\beta'^2_m b}\Bigg\}=0 \tag{1.10}$$

$$-\frac{2q(1-\cos n\pi)}{n\pi D}\Bigg\{\frac{[\alpha^2_n-k^2_n(2-\mu)]}{\alpha_n}\mathrm{th}\alpha_n\frac{a}{2}-\frac{[\beta'^2_n+k^2_n(2-\mu)]}{\beta'_n}\mathrm{tg}\beta'_n\frac{a}{2}\Bigg\}-\frac{4\lambda}{bD}\sum_{m=1,3}^{\infty}G_m$$

$$\frac{k_mk_n[k^2_m+k^2_n(2-\mu)]}{k_{mn}}+\frac{4\lambda}{b}\sum_{m=1,3}^{\infty}B_m\frac{k_mk_n[\lambda^2(2-\mu)+(1-\mu)^2k^2_mk^2_n]}{k_{mn}}\cos n\pi-A_n$$

$$\{\alpha_n[\alpha^2_n-k^2_n(2-\mu)]^2\mathrm{th}\frac{a}{2}-\beta'_n[\beta'^2_n+k^2_n(2-\mu)]^2\mathrm{tg}\beta'_n\frac{a}{2}\Bigg\}+\frac{2\lambda^2}{n\pi}k\Bigg\{\frac{[\alpha^2_n-k^2_n(2-\mu)]}{\alpha_n}$$

$$\mathrm{th}\alpha_n\frac{a}{2}+\frac{[\beta'^2_n+k^2_n(2-\mu)]}{\beta'_n}\mathrm{tg}\beta'_n\frac{a}{2}\Bigg\}\cos n\pi=0 \tag{1.11}$$

$$\frac{4q}{aD}\sum_{m=1,3}^{\infty}\left(\frac{\mathrm{th}\alpha_m\frac{b}{2}}{\alpha_m}-\frac{\mathrm{tg}\beta'_m\frac{b}{2}}{\beta'_m}\right)+\frac{1}{D}\sum_{m=1,3}^{\infty}G_mk_m\Big(\frac{\alpha_m}{\mathrm{sh}\alpha_m b}-\frac{\beta'_m}{\sin\beta'_m b}\Big)$$

$$+\sum_{m=1,3}^{\infty}B_mk_m\{\alpha_m[\alpha^2_m-k^2_m(2-\mu)]\mathrm{cth}\alpha_m b+\beta'_m[\beta'^2_m+k^2_m(2-\mu)]\mathrm{ctg}\beta'_m b\}$$

$$-\sum_{n=1,2}^{\infty}A_nk_n\Big\{\alpha_n[\alpha^2_n-k^2_n(2-\mu)]\mathrm{th}\alpha_n\frac{a}{2}-\beta'_n[\beta'^2_n+k^2_n(2-\mu)]\mathrm{tg}\beta'_n\frac{a}{2}\Big\}\cos n\pi$$

$$+\sum_{m=1,3}^{\infty}\frac{4\lambda^2}{a}k\Bigg[\Big(\frac{\mathrm{cth}\alpha_m b}{\alpha_m}-\frac{\mathrm{cth}\beta'_m b}{\beta'_m}\Big)-\frac{2\lambda}{\alpha^2_m\beta'^2_m b}\Bigg]=0 \tag{1.12}$$

于是我们得到四组无穷联立方程(1.5)～(1.8)或(1.9)～(1.12)。从A_m、B_n和G_m各截取24个系数，形成一个73×73阶线性方程组。取$\mu=1/6$，经计算，我们得到图3～5和表1。

二、在任意点受简谐集中力作用的悬臂矩形板

最后，让我们考虑一个在简谐集中载荷$F(x,y,t)=P\delta(x-x_0,y-y_0)\sin\omega t$作用下的

表 1 固定边弯矩(单位qa^2)**和自由边振幅**(单位qa^4/D) $a/b=1$

ω/ω_{11}	x/a (y/b) M (w)	0.05	0.15	0.35	0.5	0.7	0.9	1.0
	M_{y0}	−0.45689	−0.53547	−0.53285	−0.51623	−0.52173	−0.53269	0.0
0.0	w_{yb}	0.12903	0.12925	0.12956	0.12963	0.12950	0.12914	0.128909
	w_{x0}	0.000453	0.004668	0.02385	0.04467	0.07702	0.11149	0.128909
	M_{y0}	−0.49818	−0.58436	−0.58183	−0.56384	−0.56974	−0.58105	0.0
0.3	w_{yb}	0.14238	0.14263	0.14298	0.14306	0.142914	0.14251	0.142241
	w_{x0}	0.000494	0.005101	0.02615	0.04909	0.084824	0.12297	0.142241
	M_{y0}	−0.59699	−0.70139	−0.69906	−0.67770	−0.68467	−0.69681	0.0
0.5	w_{yb}	0.17435	0.17467	0.17512	0.17523	0.17504	0.17452	0.174171
	w_{x0}	0.000591	0.006137	0.031677	0.05969	0.10253	0.15045	0.174171
	M_{y0}	−1.23827	−1.46093	−1.46005	−1.41751	−1.4307	−1.44805	0.0
0.8	w_{yb}	0.38207	0.38289	0.38398	0.38424	0.38378	0.38249	0.381634
	w_{x0}	0.001219	0.012858	0.067545	0.12849	0.22500	0.32900	0.381634

悬臂矩形板，这里(x_0, y_0)是该板上的任意一点。这是一非对称问题。

解除该板固定边的弯曲约束代以弯矩M_{y0}，我们得到如图 7 所示的实际系统。假设

$$M_{y0}=\sum_{m=1,2}^{\infty}G_m\sin k_m x \qquad (2.1)$$

和自由边的位移振幅为

$$w_{x0}=\frac{y}{b}k_1+\sum_{n=1,2}^{\infty}A_{1n}\sin k_n y$$

$$w_{xa}=\frac{y}{b}k_2+\sum_{n=1,2}^{\infty}A_{2n}\sin k_n y \qquad (2.2)$$

$$w_{yb}=k_1+\frac{k_2-k_1}{a}x+\sum_{m=1,2}^{\infty}B_m\sin k_m x$$

在文[14]图 1 所示基本系统与本文图 7 所示实际系统之间应用功的互等定理，则得

$$w(\xi,\ \eta)=Pw_1(x_0,\ y_0;\ \xi,\ \eta)$$

$$+\int_0^a\left(\frac{\partial w_1}{\partial y}\right)_{y=0}M_{y0}dx-\int_0^a(v_{1y})_{y=b}w_{yb}dx$$

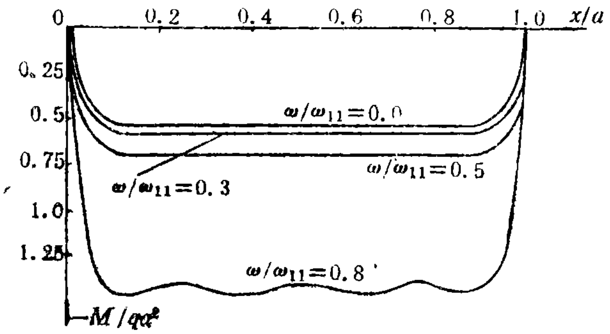

图3 固定边$y=0$弯矩分布曲线

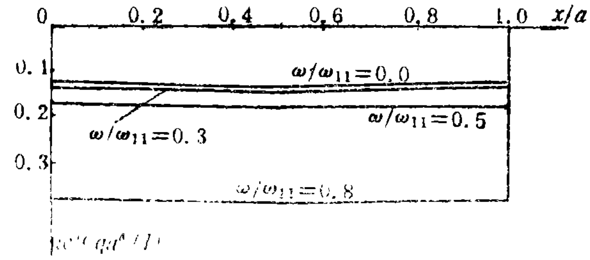

图4 自由边$y=b$振幅曲线

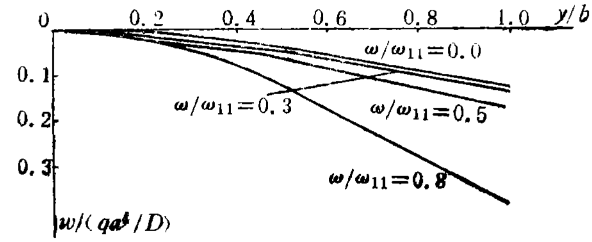

图5 自由边$x=0$振幅曲线

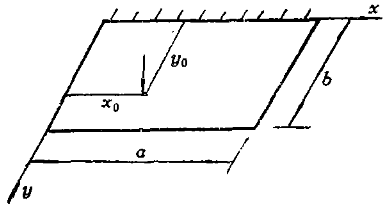

图 6

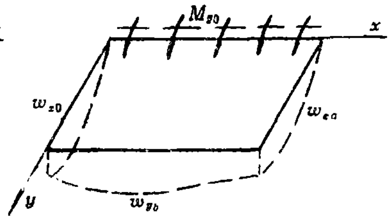

图 7

$$-\int_0^b (v_{1x})_{x=a}\ w_{xa}dy+\int_0^b (v_{1x})_{x=0}w_{x0}dy+[(R_1)_{\substack{x=a\\y=b}}k_2-(R_1)_{\substack{x=0\\y=b}}k_1] \tag{2.3}$$

当$\lambda<k_m^2$和$\lambda<k_n^2$时，式(2.3)成为

$$w(\xi,\eta)=w_a[14](4.11)]+\sum_{m=1,2}^{\infty}\frac{G_m}{(\alpha_m^2-\beta_m^2)D}\left[-\frac{\operatorname{sh}\alpha_m(b-\eta)}{\operatorname{sh}\alpha_m b}+\frac{\operatorname{sh}\beta_m(b-\eta)}{\operatorname{sh}\beta_m b}\right]\sin k_m\xi$$

$$+\sum_{m=1,2}^{\infty}\frac{B_m}{\alpha_m^2-\beta_m^2}\left\{\frac{[\alpha_m^2-k_m^2(2-\mu)]}{\operatorname{sh}\alpha_m b}\operatorname{sh}\alpha_m\eta-\frac{[\beta_m^2-k_m^2(2-\mu)]}{\operatorname{sh}\beta_m b}\operatorname{sh}\beta_m\eta\right\}\sin k_m\xi$$

$$+\sum_{n=1,2}^{\infty}\frac{A_{2n}}{\alpha_n^2-\beta_n^2}\left\{\frac{[\alpha_n^2-k_n^2(2-\mu)]}{\operatorname{sh}\alpha_n a}\operatorname{sh}\alpha_n\xi-\frac{[\beta_n^2-k_n^2(2-\mu)]}{\operatorname{sh}\beta_n a}\operatorname{sh}\beta_n\xi\right\}\sin k_n\eta$$

$$+\sum_{n=1,2}^{\infty}\frac{A_{1n}}{\alpha_n^2-\beta_n^2}\left\{[\alpha_n^2-k_n^2(2-\mu)]\frac{\operatorname{sh}\alpha_n(a-\xi)}{\operatorname{sh}\alpha_n a}-[\beta_n^2-k_n^2(2-\mu)]\frac{\operatorname{sh}\beta_n(a-\xi)}{\operatorname{sh}\beta_n a}\right\}\sin k_n\eta$$

$$+\frac{2\lambda^2}{\pi}k_1\sum_{m=1,2}^{\infty}\frac{1}{m}\left\{\frac{1}{\alpha_m^2-\beta_m^2}\left[\frac{\operatorname{sh}\alpha_m\eta}{\alpha_m^2\operatorname{sh}\alpha_m b}-\frac{\operatorname{sh}\beta_m\eta}{\beta_m^2\operatorname{sh}\beta_m b}\right]+\frac{\eta}{\alpha_m^2\beta_m^2 b}\right\}\sin k_m\xi+\frac{\eta(a-\xi)}{ab}k_1$$

$$\left(\text{或}-\frac{2\lambda^2}{\pi}k_1\sum_{n=1,2}^{\infty}\frac{1}{n}\left\{\frac{1}{\alpha_n^2-\beta_n^2}\left[\frac{\operatorname{sh}\alpha_n(a-\xi)}{\alpha_n^2\operatorname{sh}\alpha_n a}-\frac{\operatorname{sh}\beta_n(a-\xi)}{\beta_n^2\operatorname{sh}\beta_n a}\right]\right.\right.$$

$$\left.\left.+\frac{a-\xi}{\alpha_n^2\beta_n^2 a}\right\}\sin k_n\eta\cos n\pi+\frac{\eta(a-\xi)}{ab}k_1\right)$$

$$-\frac{2\lambda^2}{\pi}k_2\sum_{m=1,2}^{\infty}\frac{1}{m}\left\{\frac{1}{\alpha_m^2-\beta_m^2}\left[\frac{\operatorname{sh}\beta_m\eta}{\alpha_m^2\operatorname{sh}\alpha_m b}-\frac{\operatorname{sh}\beta_m\eta}{\beta_m^2\operatorname{sh}\beta_m b}+\frac{\eta}{\alpha_m^2\beta_m^2 b}\right\}\right.$$

$$\cdot\sin k_m\xi\cos m\pi+\frac{\xi\eta}{ab}k_2$$

$$\left(\text{或}-\frac{2\lambda^2}{\pi}k_2\sum_{n=1,2}^{\infty}\frac{1}{n}\left\{\frac{1}{\alpha_n^2-\beta_n^2}\left[\frac{\operatorname{sh}\alpha_n\xi}{\alpha_n^2\operatorname{sh}\alpha_n a}-\frac{\operatorname{sh}\beta_n\xi}{\beta_n^2\operatorname{sh}\alpha_n a}\right]\right.\right.$$

$$\left.\left.+\frac{\xi}{\alpha_n^2\beta_n^2 a}\right\}\sin k_n\eta\cos n\pi+\frac{\xi\eta}{ab}k_2\right) \tag{2.4}$$

当$\lambda>k_m^2$和$\lambda>k_n^2$时，将$\beta_m\to i\beta_m'$和$\beta_n\to i\beta_n'$代入式(2.4)，可得相应的表达式，故可将其省略。满足相应的边界条件，对于$\lambda<k_m^2$和$\lambda<k_n^2$，可得

$$\frac{2P}{aD}\left[-\frac{\operatorname{sh}\alpha_m(b-y_0)}{\operatorname{sh}\alpha_m b}+\frac{\operatorname{sh}\beta_m(b-y_0)}{\operatorname{sh}\beta_m b}\right]\sin k_m x_0+\frac{G_m}{D}(\alpha_m\operatorname{cth}\alpha_m b-\beta_m\operatorname{cth}\beta_m b)+B_m$$

$$\cdot\left\{\frac{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]}{\mathrm{sh}\alpha_m b}-\frac{\beta_m[\beta_m^2-k_m^2(2-\mu)]}{\mathrm{sh}\beta_m b}\right\}-\frac{4\lambda}{a}\sum_{n=1,2}^{\infty}A_{2n}\frac{k_mk_n[k_m^2+k_n^2(2-\mu)]}{k_{mn}}\cos m\pi$$

$$+\frac{4\lambda}{a}\sum_{n=1,2}^{\infty}A_{1n}\frac{k_mk_n[k_m^2+k_n^2(2-\mu)]}{k_{mn}}+\frac{4\lambda}{m\pi}k_1\left[\frac{1}{b}+\frac{\lambda^2}{\alpha_m^2-\beta_m^2}\left(\frac{1}{\alpha_m\mathrm{sh}\alpha_m b}-\frac{1}{\beta_m\mathrm{sh}\beta_m b}\right)\right.$$

$$\left.+\frac{\lambda^2}{\alpha_m^2\beta_m^2 b}\right]-\frac{4\lambda}{m\pi}k_2\left[\frac{1}{b}+\frac{\lambda^2}{\alpha_m^2-\beta_m^2}\left(\frac{1}{\alpha_m\mathrm{sh}\alpha_m b}\frac{1}{\beta_m\mathrm{sh}\beta_m b}\right)+\frac{\lambda^2}{\alpha_m^2\beta_m^2 b}\right]\cos m\pi=0 \quad (2.5)$$

$$\frac{2P}{aD}\left\{\frac{[\alpha_m^2-k_m^2(2-\mu)]}{\mathrm{sh}\alpha_m b}\mathrm{sh}\alpha_m y_0-\frac{[\beta_m^2-k_m^2(2-\mu)]}{\mathrm{sh}\beta_m b}\mathrm{sh}\beta_m y_0\right\}\sin k_m x_0+\frac{G_m}{D}$$

$$\cdot\left\{\frac{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]}{\mathrm{sh}\alpha_m b}-\frac{\beta_m[\beta_m^2-k_m^2(2-\mu)]}{\mathrm{sh}\beta_m b}\right\}+B_m\{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]^2\mathrm{cth}\alpha_m b$$

$$-\beta_m[\beta_m^2-k_m^2(2-\mu)]^2\mathrm{cth}\alpha_m b\}+\frac{4\lambda}{a}\sum_{n=1,2}^{\infty}\frac{A_{2n}k_mk_n}{k_{mn}}[\lambda^2(2-\mu)+(1-\mu)^2k_m^2k_n^2]\cos m\pi\cos n\pi$$

$$-\frac{4\lambda}{a}\sum_{n=1,2}^{\infty}\frac{A_{1n}k_mk_n}{k_{mn}}[\lambda^2(2-\mu)+(1-\mu)^2k_m^2k_n^2]\cos n\pi+\frac{2\lambda^2}{m\pi}k_1\left\{\frac{[\alpha_m^2-k_m^2(2-\mu)]}{\alpha_m}\mathrm{cth}\alpha_m b\right.$$

$$\left.-\frac{[\beta_m^2-k_m^2(2-\mu)]}{\beta_m}\mathrm{cth}\beta_m b-\frac{2\lambda(2-\mu)k_m^2}{\alpha_m^2\beta_m^2 b}\right\}-\frac{2\lambda^2}{m\pi}k_2\left\{\frac{[\alpha_m^2-k_m^2(2-\mu)]}{\alpha_m}\mathrm{cth}\,\alpha_m b\right.$$

$$\left.-\frac{[\beta_m^2-k_m^2(2-\mu)]}{\beta_m}\mathrm{cth}\beta_m b-\frac{2\lambda(2-\mu)k_m^2}{\alpha_m^2\beta_m^2 b}\right\}\cos m\pi=0 \quad (2.6)$$

$$\frac{2P}{bD}\left\{-\frac{[\alpha_n^2-k_n^2(2-\mu)]}{\mathrm{sh}\alpha_n a}\mathrm{sh}\alpha_n(a-x_0)+\frac{[\beta_n^2-k_n^2(2-\mu)]}{\mathrm{sh}\beta_n a}\mathrm{sh}\beta_n(a-x_0)\right\}\sin k_n y_0$$

$$-\frac{4\lambda}{bD}\sum_{m=1,2}^{\infty}\frac{G_mk_mk_n}{k_{mn}}[k_m^2+k_n^2(2-\mu)]+\frac{4\lambda}{b}\sum_{m=1,2}^{\infty}\frac{B_mk_mk_n}{k_{mn}}[\lambda^2(2-\mu)+(1-\mu)^2k_m^2k_n^2]\cos n\pi$$

$$+A_{2n}\left\{\frac{\alpha_n[\alpha_n^2-k_n^2(2-\mu)]^2}{\mathrm{sh}\alpha_n a}-\frac{\beta_n[\beta_n^2-k_n^2(2-\mu)]^2}{\mathrm{sh}\beta_n a}\right\}-A_{1n}\{\alpha_n[\alpha_n^2-k_n^2(2-\mu)]^2\mathrm{cth}\alpha_n a$$

$$\left.-\beta_n[\beta_n^2-k_n^2(2-\mu)]^2\mathrm{cth}\beta_n a\right\}+\frac{2\lambda^2}{n\pi}k_1\left\{\frac{[\alpha_n^2-k_n^2(2-\mu)]}{\alpha_n}\mathrm{cth}\alpha_n a-\frac{[\beta_n^2-k_n^2(2-\mu)]}{\beta_n}\right.$$

$$\left.\cdot\,\mathrm{cth}\beta_n a-\frac{2\lambda(2-\mu)k_n^2}{\alpha_n^2\beta_n^2 a}\right\}\cos n\pi-\frac{2\lambda^2}{n\pi}k_2\left\{\frac{[\alpha_n^2-k_n^2(2-\mu)]}{\alpha_n\mathrm{sh}\alpha_n a}-\frac{[\beta_n^2-k_n^2(2-\mu)]}{\beta_n\mathrm{sh}\beta_n a}\right.$$

$$\left.-\frac{2\lambda(2-\mu)k_n^2}{\alpha_n^2\beta_n^2 a}\right\}\cos n\pi=0 \quad (2.7)$$

$$\frac{2P}{bD}\left\{\frac{[\alpha_n^2-k_n^2(2-\mu)]}{\mathrm{sh}\alpha_n a}\mathrm{sh}\alpha_n x_0-\frac{[\beta_n^2-k_n^2(2-\mu)]}{\mathrm{sh}\beta_n a}\mathrm{sh}\beta_n x_0\right\}\sin k_n y_0-\frac{4\lambda}{bD}\sum_{m=1,2}^{\infty}\frac{G_m}{k_{mn}}$$

$$\cdot\,k_mk_n[k_m^2+k_n^2(2-\mu)]\cos m\pi+\frac{4\lambda}{b}\sum_{m=1,2}^{\infty}\frac{B_mk_mk_n}{k_{mn}}[\lambda^2(2-\mu)+(1-\mu)^2k_m^2k_n^2]\cos m\pi\cos n\pi$$

$$+A_{2n}\{\alpha_n[\alpha_n^2-k_n^2(2-\mu)]^2\mathrm{cth}\alpha_n a-\beta_n[\beta_n^2-k_n^2(2-\mu)]^2\mathrm{cth}\beta_n a\}$$

$$-A_{1n}\left\{\frac{\alpha_n[\alpha_n^2-k_n^2(2-\mu)]^2}{\mathrm{sh}\alpha_n a}-\frac{\beta_n[\beta_n^2-k_n^2(2-\mu)]^2}{\mathrm{sh}\beta_n a}\right\}+\frac{2\lambda^2}{n\pi}k_1\left\{\frac{[\alpha_n^2-k_n^2(2-\mu)]}{\alpha_n\mathrm{sh}\alpha_n a}\right.$$

$$\left.-\frac{[\beta_n^2-k_n^2(2-\mu)]}{\beta_n\mathrm{sh}\beta_n a}-\frac{2\lambda(2-\mu)k_n^2}{\alpha_n^2\beta_n^2 a}\right\}\cos n\pi-\frac{2\lambda^2}{n\pi}k_2\left\{\frac{[\alpha_n^2-k_n^2(2-\mu)]}{\alpha_n}\mathrm{cth}\alpha_n a\right.$$

$$\left.-\frac{[\beta_n^2-k_n^2(2-\mu)]}{\beta_n}\mathrm{cth}\beta_n a-\frac{2\lambda(2-\mu)k_n^2}{\alpha_n^2\beta_n^2 a}\right\}\cos n\pi=0 \tag{2.8}$$

$$\frac{2P}{aD}\sum_{m=1,2}^{\infty}k_m\left(\frac{\mathrm{sh}\alpha_m y_0}{\mathrm{sh}\alpha_m b}-\frac{\mathrm{sh}\beta_m y_0}{\mathrm{sh}\beta_m b}\right)\sin k_m x_0+\frac{1}{D}\sum_{m=1,2}^{\infty}G_m k_m\left(\frac{\alpha_m}{\mathrm{sh}\alpha_m b}-\frac{\beta_m}{\mathrm{sh}\beta_m b}\right)$$

$$+\sum_{m=1,2}^{\infty}B_m k_m\{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]\mathrm{cth}\alpha_m b-\beta_m[\beta_m^2-k_m^2(2-\mu)]\mathrm{cth}\beta_m b\}+\sum_{n=1,2}^{\infty}A_{2n}k_n$$

$$\left\{\frac{\alpha_n[\alpha_n^2-k_n^2(2-\mu)]}{\mathrm{sh}\alpha_n a}-\frac{\beta_m[\beta_n^2-k_n^2(2-\mu)]}{\mathrm{sh}\beta_n a}\right\}\cos n\pi-\sum_{n=1,2}^{\infty}A_{1n}k_n\{\alpha_n[\alpha_n^2-k_n^2(2-\mu)]$$

$$\cdot\mathrm{cth}\alpha_n a-\beta_n[\beta_n^2-k_n^2(2-\mu)]\mathrm{cth}\beta_n a\}\cos n\pi+k_1\left\{-\frac{2\lambda}{ab}+\sum_{m=1,2}^{\infty}\frac{2\lambda^2}{a}\left[\left(\frac{\mathrm{cth}\alpha_m b}{\alpha_m}\right.\right.\right.$$

$$\left.\left.\left.-\frac{\mathrm{cth}\beta_m b}{\beta_m}\right)+\frac{2\lambda}{\alpha_m^2\beta_m^2 b}\right]\right\}+k_2\left\{\frac{2\lambda}{ab}-\sum_{m=1,2}^{\infty}\frac{2\lambda^2}{a}\left[\left(\frac{\mathrm{cth}\alpha_m b}{\alpha_m}-\frac{\mathrm{cth}\beta_m b}{\beta_m}\right)+\frac{2\lambda}{\alpha_m^2\beta_m^2 b}\right]\right.$$

$$\left.\cdot\cos m\pi\right\}=0 \tag{2.9}$$

$$\frac{2P}{aD}\sum_{m=1,2}^{\infty}k_m\left(\frac{\mathrm{sh}\alpha_m y_0}{\mathrm{sh}\alpha_m b}-\frac{\mathrm{sh}\beta_m y_0}{\mathrm{sh}\beta_m b}\right)\sin k_m x_0\cos m\pi+\frac{1}{D}\sum_{m=1,2}^{\infty}G_m k_m\left(\frac{\alpha_m}{\mathrm{sh}\alpha_m b}-\frac{\beta_m}{\mathrm{sh}\beta_m b}\right)$$

$$\cdot\cos m\pi+\sum_{m=1,2}^{\infty}B_m k_m\{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]\mathrm{cth}\alpha_m b-\beta_m[\beta_m^2-k_m^2(2-\mu)]\mathrm{cth}\beta_m b\}\cos m\pi$$

$$+\sum_{n=1,2}^{\infty}A_{2n}k_n\{\alpha_n[\alpha_n^2-k_n^2(2-\mu)]\mathrm{cth}\alpha_n a-\beta_n[\beta_n^2-k_n^2(2-\mu)]\mathrm{cth}\beta_n a\}\cos n\pi$$

$$-\sum_{n=1,2}^{\infty}A_{1n}k_n\left\{\frac{\alpha_n[\alpha_n^2-k_n^2(2-\mu)]}{\mathrm{sh}\alpha_n a}-\frac{\beta_n[\beta_n^2-k_n^2(2-\mu)]}{\mathrm{sh}\beta_n a}\right\}\cos n\pi+k_1\left\{-\frac{2\lambda}{ab}\right.$$

$$\left.+\sum_{m=1,2}^{\infty}\frac{2\lambda^2}{a}\left[\left(\frac{\mathrm{cth}\beta_m b}{\alpha_m}-\frac{\mathrm{cth}\alpha_m b}{\beta_m}\right)+\frac{2\lambda}{\alpha_m^2\beta_m^2 b}\right]\cos m\pi\right\}+k_2\left\{\frac{2\lambda}{ab}-\sum_{m=1,2}^{\infty}\frac{2\lambda^2}{a}\right.$$

$$\left.\cdot\left[\left(\frac{\mathrm{cth}\alpha_m b}{\alpha_m}-\frac{\mathrm{cth}\beta_m b}{\beta_m}\right)+\frac{2\lambda}{\alpha_m^2\beta_m^2 b}\right]\right\}=0 \tag{2.10}$$

而当$\lambda>k_m^2$和$\lambda>k_n^2$时，(2.5)～(2.10)分别变为

$$\frac{2P}{aD}\left[-\frac{\text{sh}\alpha_m(b-y_0)}{\text{sh}\alpha_m b}+\frac{\sin\beta'_m(b-y_0)}{\sin\beta'_m b}\right]\sin k_m x_0+\frac{G_m}{D}(\alpha_m\text{cth}\alpha_m b-\beta_m^2\text{ctg}\beta_m^2 b)$$

$$+B_m\left\{\frac{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]}{\text{sh}\alpha_m b}+\frac{\beta'_m[\beta'^2_m+k_m^2(2-\mu)]}{\sin\beta'_m b}\right\}-\frac{4\lambda}{a}\sum_{n=1,2}^{\infty}\frac{A_{2n}k_m k_n[k_m^2+k_n^2(2-\mu)]}{k_{mn}}$$

$$\cdot\cos m\pi+\frac{4\lambda}{a}\sum_{n=1,2}^{\infty}\frac{A_{1n}k_m k_n[k_m^2+k_n^2(2-\mu)]}{k_{mn}}+\frac{4\lambda}{m\pi}k_1\left[\frac{1}{b}+\frac{\lambda^2}{\alpha_m^2+\beta'^2_m}\left(\frac{1}{\alpha_m\text{sh}\,\alpha_m b}\right.\right.$$

$$\left.\left.+\frac{1}{\beta'_m\sin\beta'_m b}\right)-\frac{\lambda^2}{\alpha_m^2\beta'^2_m b}\right]-\frac{4\lambda}{m\pi}k_2\left[\frac{1}{b}+\frac{\lambda^2}{\alpha_m^2+\beta'^2_m}\left(\frac{1}{\alpha_m\text{sh}\alpha_m b}+\frac{1}{\beta'_m\sin\beta_m b}\right)\right.$$

$$\left.-\frac{\lambda^2}{\alpha_m^2\beta_m^2 b}\right]\cos m\pi=0 \tag{2.11}$$

$$\frac{2P}{aD}\left\{\frac{[\alpha_m^2-k_m^2(2-\mu)]}{\text{sh}\alpha_m b}\text{sh}\alpha_m y_0+\frac{[\beta'^2_m+k_m^2(2-\mu)]}{\sin\beta'_m b}\sin\beta'_m y_0\right\}\sin k_m x_0$$

$$+\frac{G_m}{D}\left\{\frac{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]}{\text{sh}\alpha_m b}+\frac{\beta'_m[\beta'^2_m+k_m^2(2-\mu)]}{\sin\beta'_m b}\right\}+B_m\{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]^2\text{cth}\alpha_m b$$

$$-\beta'_m[\beta'^2_m+k_m^2(2-\mu)]^2\text{ctg}\beta'_m b\}+\frac{4\lambda}{a}\sum_{n=1,2}^{\infty}A_{2n}\frac{k_m k_n}{k_{mn}}[\lambda^2(2-\mu)+(1-\mu)^2k_m^2k_n^2]\cos m\pi\cos n\pi$$

$$-\frac{4\lambda}{a}\sum_{n=1,2}^{\infty}A_{1n}\frac{k_m k_n}{k_{mn}}[\lambda^2(2-\mu)+(1-\mu)^2k_m^2k_n^2]\cos n\pi+\frac{2\lambda^2}{m\pi}k_1\left\{\frac{[\alpha_m^2-k_m^2(2-\mu)]}{\alpha_m}\text{cth}\alpha_m b\right.$$

$$\left.-\frac{[\beta'^2_m+k_m^2(2-\mu)]}{\beta'_m}\text{ctg}\beta'_m b+\frac{2\lambda(2-\mu)k_m^2}{\alpha_m^2\beta'^2_m b}\right\}-\frac{2\lambda^2}{m\pi}k_2\left\{\frac{[\alpha_m^2-k_m^2(2-\mu)]}{\alpha_m}\text{cth}\alpha_m b\right.$$

$$\left.-\frac{[\beta'^2_m+k_m^2(2-\mu)]}{\beta'_m}\text{ctg}\beta'_m b+\frac{2\lambda(2-\mu)k_m^2}{\alpha_m^2\beta'^2_m b}\right\}\cos m\pi=0 \tag{2.12}$$

$$-\frac{2P}{bD}\left\{\frac{[\alpha_n^2-k_n^2(2-\mu)]}{\text{sh}\alpha_n a}\text{sh}\alpha_n(a-x_0)+\frac{[\beta'^2_n+k_n^2(2-\mu)]}{\sin\beta'_n a}\sin\beta'_n(a-x_0)\right\}-\frac{4\lambda}{bD}\sum_{m=1,2}^{\infty}$$

$$\cdot\frac{G_m k_m k_n}{k_{mn}}[k_m^2+k_n^2(2-\mu)]+\frac{4\lambda}{b}\sum_{m=1,2}^{\infty}\frac{B_m k_m k_n}{k_{mn}}[\lambda^2(2-\mu)+(1-\mu)^2k_m^2k_n^2]\cos n\pi$$

$$+A_{2n}\left\{\frac{\alpha_n[\alpha_n^2-k_n^2(2-\mu)]}{\text{sh}\alpha_n a}-\frac{\beta'_n[\beta'^2_n+k'_n(2-\mu)]^2}{\sin\beta'_n a}\right\}-A_{1n}\{\alpha_n[\alpha_n^2-k_n^2(2-\mu)]^2\text{cth}\alpha_n a$$

$$-\beta'_n[\beta'^2_n+k_n^2(2-\mu)]^2\text{ctg}\beta'_n a\}+\frac{2\lambda^2}{n\pi}k_1\left\{\frac{[\alpha_n^2-k_n^2(2-\mu)]}{\alpha_n}\text{cth}\alpha_n a\right.$$

$$\left.-\frac{[\beta'^2_n+k_n^2(2-\mu)]}{\beta'_n}\text{ctg}\beta'_n a+\frac{2\lambda(2-\mu)k_n^2}{\alpha_n^2\beta'^2_n a}\right\}\cos n\pi-\frac{2\lambda^2}{n\pi}k_2\left\{\frac{[\alpha_n^2-k_n^2(2-\mu)]}{\alpha_n\text{sh}\alpha_n a}\right.$$

$$\left.-\frac{[\beta'^2_n+k_n^2(2-\mu)]}{\beta'_n\sin\beta'_n a}+\frac{2\lambda(2-\mu)k_n^2}{\alpha_n^2\beta'^2_n a}\right\}\cos n\pi=0 \tag{2.13}$$

$$\frac{2P}{bD}\left\{\frac{[\alpha_n^2-k_n^2(2-\mu)]}{\mathrm{sh}\alpha_n a}\mathrm{sh}\alpha_n x_0+\frac{[\beta_n'^2+k_n^2(2-\mu)]}{\sin\beta_n' a}\sin\beta_n' x_0\right\}\sin k_n y_0-\frac{4\lambda}{bD}\sum_{m=1,2}^{\infty}$$

$$\cdot\frac{Gk_m k_n}{k_{mn}}[k_m^n+k_n^2(2-\mu)]\cos m\pi+\frac{4\lambda}{b}\sum_{m=1,2}^{\infty}\frac{B_m k_m k_n}{k_{mn}}[\lambda^2(2-\mu)+(1-\mu)^2k_m^2k_n^2]\cos m\pi\cos n\pi$$

$$+A_{2n}\{\alpha_n[\alpha_n^2-k_n^2(2-\mu)]^2\mathrm{cth}\alpha_n a-\beta_n'[\beta_n'^2+k_n^2(2-\mu)]^2\mathrm{ctg}\beta_n' a\}$$

$$-A_{1n}\frac{\alpha_n[\alpha_n^2-k_n^2(2-\mu)]^2}{\mathrm{sh}\alpha_n a}-\frac{\beta_n'[\beta_n'^2+k_n^2(2-\mu)]^2}{\sin\beta_n a}\Big\}+\frac{2\lambda^2}{n\pi}k_1\left\{\frac{[\alpha_n^2-k_n^2(2-\mu)]^2}{\alpha_n\mathrm{sh}\alpha_n a}\right.$$

$$-\frac{[\beta_n'^2+k_n^2(2-\mu)]^2}{\beta_n'\sin\beta_n' a}+\left.\frac{2\lambda(2-\mu)k_n^2}{\alpha_n^2\beta_n' a}\right\}\cos n\pi-\frac{2\lambda^2}{n\pi}k_2\left\{\frac{[\alpha_n^2-k_n^2(2-\mu)]^2}{\alpha_n}\mathrm{cth}\alpha_n a\right.$$

$$-\frac{[\beta_n'^2+k_n^2(2-\mu)]^2}{\beta_n'}\mathrm{ctg}\beta_n' a+\left.\frac{2\lambda(2-\mu)k_n^2}{\alpha_n^2\beta_n'^2 a}\right\}\cos n\pi=0 \tag{2.14}$$

$$\frac{2P}{aD}\sum_{m=1,2}^{\infty}k_m\left(\frac{\mathrm{sh}\alpha_m y_0}{\mathrm{sh}\alpha_m b}-\frac{\sin\beta_m' y_0}{\sin\beta_m' b}\right)\sin k_m x_0+\frac{1}{D}\sum_{m=1,2}^{\infty}G_m k_m\left(\frac{\alpha_m}{\mathrm{sh}\alpha_m b}-\frac{\beta_m'}{\sin\beta_m' b}\right)$$

$$+\sum_{m=1,2}^{\infty}B_m k_m\{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]\mathrm{cth}\alpha_m b+\beta_m'[\beta_m'^2+k_m^2(2-\mu)]\mathrm{ctg}\beta_m' b\Big\}+\sum_{n=1,2}^{\infty}A_{2n}$$

$$k_n\left\{\frac{\alpha_n[\alpha_n^2-k_n^2(2-\mu)]}{\mathrm{sh}\alpha_n a}+\frac{\beta_m'[\beta_m'^2+k_n^2(2-\mu)]}{\sin\beta_m' a}\right\}\cos n\pi-\sum_{n=1,2}^{\infty}A_{1n}k_n\{\alpha_n[\alpha_n^2-k_n^2$$

$$\cdot(2-\mu)]\mathrm{cth}\alpha_n a+\beta_n'[\beta_n'^2+k_n^2(2-\mu)]\mathrm{ctg}\beta_n' a\}\cos n\pi+k_1\left\{-\frac{2\lambda}{ab}+\sum_{m=1,2}^{\infty}\frac{2\lambda^2}{a}\right.$$

$$\cdot\left.\left[\left(\frac{\mathrm{cth}\alpha_m b}{\alpha_m}+\frac{\mathrm{cth}\beta_m' b}{\beta_m'}\right)-\frac{2\lambda}{\alpha_m^2\beta_m'^2 b}\right]\cos m\pi\right\}=0 \tag{2.15}$$

$$\frac{2P}{aD}\sum_{m=1,2}^{\infty}k_m\left(\frac{\mathrm{sh}\alpha_m y_0}{\mathrm{sh}\alpha_m b}-\frac{\sin\beta_m' y_0}{\sin\beta_m' b}\right)\sin k_m x_0\cos m\pi+\frac{1}{D}\sum_{m=1,2}^{\infty}G_m k_m\left(\frac{\alpha_m}{\mathrm{sh}\alpha_m b}-\frac{\beta_m'}{\sin\beta_m' b}\right)$$

$$\cdot\cos m\pi+\sum_{m=1,2}^{\infty}B_m\{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]\mathrm{cth}\alpha_m b+\beta_m'[\beta_m'^2+k_m^2(2-\mu)]\mathrm{ctg}\beta_m' b\}k_m\cos m\pi$$

$$+\sum_{n=1,2}^{\infty}A_{2n}k_n\{\alpha_n[\alpha_n^2-k_n^2(2-\mu)]\mathrm{cth}\alpha_n a+\beta_n'[\beta_n'^2+k_n^2(2-\mu)]\mathrm{ctg}\beta_n' a\}\cos n\pi$$

$$-\sum_{n=1,2}^{\infty}A_{1n}k_n\left\{\frac{\alpha_n[\alpha_n^2-k_n^2(2-\mu)]}{\mathrm{sh}\alpha_n a}+\frac{\beta_n'[\beta_n'^2+k_n^2(2-\mu)]}{\sin\beta_n' a}\right\}\cos n\pi+k_1\left\{-\frac{2\lambda}{ab}\right.$$

$$+\sum_{m=1,2}^{\infty}\frac{2\lambda^2}{a}\left[\left(\frac{\mathrm{cth}\alpha_m b}{\alpha_m}+\frac{\mathrm{ctg}\beta_m' b}{\beta_m'}\right)-\frac{2\lambda}{\alpha_m^2\beta_m'^2 b}\right]\cos m\pi\Big\}+k_2\left\{\frac{2\lambda}{ab}-\sum_{m=1,2}^{\infty}\frac{2\lambda^2}{a}\right.$$

$$\cdot\left[\left(\frac{\text{cth}\alpha_m b}{\alpha_m}+\frac{\text{ctg}\beta'_m b}{\beta'_m}\right)-\frac{2\lambda}{\alpha_m^2\beta_m'^2 b}\right\}=0 \tag{2.16}$$

这样，我们便得到了关于 A_{1n}，A_{2n}，B_m，G_m，k_1 和 k_2 的六组无穷联立方程(2.5)～(2.10)或(2.11)～(2.16)。

下面我们将计算三种情况：①、集中载荷作用于 $(a/2,\ b/2)$ 点；②、集中载荷作用于 $(a/2,\ b)$点；③、集中载荷作用于$(3a/4,b)$点；对于①和②，我们从A_{1n}，A_{2n}，B_m和G_m各截取20个系数；对于③，我们各截取25个系数。而对于②～③，均取$\mu=1/6$。通过计算，我们得到下述图表。

表 2　固定边弯矩(单位p)和自由边振幅① (单位pa^2/D)

ω/ω_{11}	$x/a\ (y/b)$ M (w)	0.05	0.15	0.35	0.5	0.7	0.9	1.0
0.0	M_{y0}	−0.48009	−0.51326	−0.57461	−0.57019	−0.52856	−0.41526	0.0
	w_{yb}	0.10553	0.10698	0.10925	0.10985	0.10881	0.10627	0.10477
	w_{x0}	0.000368	0.003910	0.02057	0.03827	0.06452	0.09138	0.10477
0.3	M_{y0}	−0.52662	−0.55773	−0.61821	−0.61075	−0.56872	−0.45112	0.0
	w_{yb}	0.11704	0.11853	0.12083	0.12145	0.12039	0.11779	0.11627
	w_{x0}	0.000406	0.004286	0.022567	0.04210	0.071268	0.10128	0.11627
0.5	M_{y0}	−0.63779	−0.66399	−0.72236	−0.70762	−0.66465	−0.53682	0.0
	w_{yb}	0.14460	0.14616	0.14856	0.14919	0.14809	0.14539	0.14379
	w_{x0}	0.000496	0.00519	0.02735	0.05125	0.087406	0.12498	0.14379
0.8	M_{y0}	−1.35547	−1.34971	−1.39407	−1.33219	−1.28340	−1.08999	0.0
	w_{yb}	0.32311	0.32510	0.32809	0.32887	0.32752	0.32413	0.32207
	w_{x0}	0.001078	0.011006	0.05823	0.11046	0.19186	0.27843	0.32207

表 3　固定边弯矩(单位p)和自由边振幅② (单位pa^2/D)

ω/ω_{11}	$x/a\ (y/b)$ M (w)	0.05	0.15	0.35	0.5	0.7	0.9	1.0
0.0	M_{y0}	−1.05425	−1.08979	−1.12059	−1.06843	−1.04435	−0.89106	0.0
	w_{yb}	0.33359	0.33906	0.35032	0.35549	0.34761	0.33628	0.33098
	w_{x0}	0.000827	0.00897	0.05069	0.10052	0.18466	0.28091	0.33098
0.3	M_{y0}	−1.18963	−1.21937	−1.24762	−1.18661	−1.16141	−0.99563	0.0
	w_{yb}	0.36756	0.37311	0.38449	0.38969	0.38176	0.37029	0.36489
	w_{x0}	0.000937	0.01007	0.056535	0.11175	0.20450	0.31009	0.36489
0.5	M_{y0}	−1.51413	−1.5298	−1.5519	−1.4696	−1.4417	−1.2461	0.0
	w_{yb}	0.44883	0.45458	0.46624	0.47152	0.46347	0.45166	0.44605
	w_{x0}	0.00120	0.012709	0.07054	0.13862	0.25199	0.37992	0.44605
0.8	M_{y0}	−3.6204	−3.5430	−3.4238	−3.3032	−3.2584	−2.8704	0.0
	w_{yb}	0.97475	0.98177	0.99518	1.0008	0.99209	0.97824	0.97128
	w_{x0}	0.002906	0.02980	0.16134	0.31279	0.55947	0.93192	0.97128

表　4　　固定边弯矩(单位p)和自由边振幅③（单位pa^2/D）

ω/ω_{11}	$x/a\,(y/b)$ $M\,(w)$	0.0	0.1	0.25	0.5	0.75	0.9	1.0
0.0	M_{y0}	0.0	−0.75461	−0.91996	−1.07623	−1.16301	−1.24897	0.0
	w_{yb}	0.27858	0.28876	0.30553	0.33843	0.37117	0.38021	0.38451
	w_{x0}	0.0	0.002461	0.019103	0.079858	0.17121	0.23466	0.27858
	w_{xa}	0.0	0.005305	0.033180	0.121746	0.245396	0.32814	0.38451
0.3	M_{y0}	0.0	−0.8634	−1.03256	−1.19424	−1.27986	−1.366606	0.0
	w_{yb}	0.30960	0.32001	0.33711	0.37044	0.40342	0.41249	0.41678
	w_{x0}	0.0	0.002882	0.021859	0.090055	0.19141	0.26133	0.30960
	w_{xa}	0.0	0.005774	0.036167	0.13258	0.26650	0.35597	0.41678
0.5	M_{y0}	0.0	−1.12264	−1.29970	−1.47304	−1.55482	−1.6410	0.0
	w_{yb}	0.38314	0.39404	0.411804	0.44599	0.47936	0.48842	0.49263
	w_{x0}	0.0	0.00389	0.028429	0.11430	0.23935	0.32457	0.38314
	w_{xa}	0.0	0.00687	0.043165	0.15800	0.31628	0.42136	0.49263
0.8	M_{y0}	0.0	−2.7168	−2.92777	−3.1582	−3.20364	−3.2766	0.0
	w_{yb}	0.83077	0.84389	0.86459	0.90185	0.93544	0.9433	0.94636
	w_{x0}	0.0	0.01013	0.06886	0.26273	0.53183	0.70986	0.83077
	w_{xa}	0.0	0.01335	0.08473	0.30955	0.61366	0.81233	0.94636

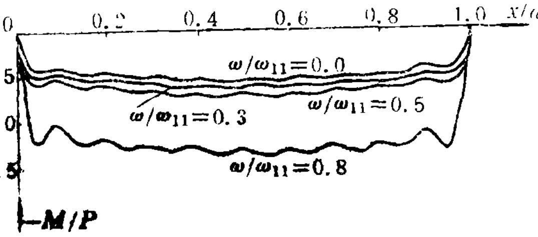

图8　固定边$y=0$弯矩分布曲线①

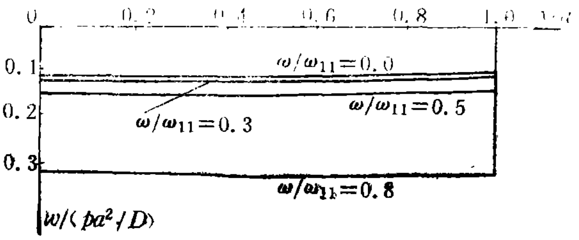

图9　自由边$y=b$振幅曲线①

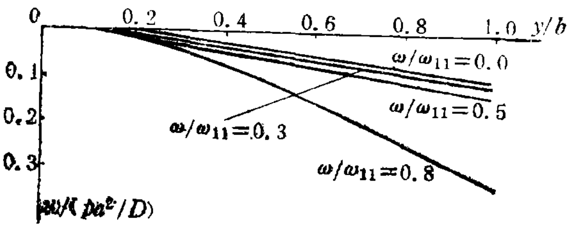

图10　自由边$x=0$振幅曲线①

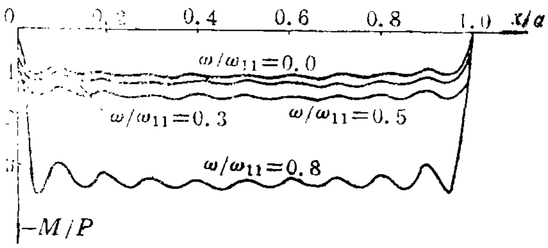

图11　固定边$y=0$弯矩分布曲线②

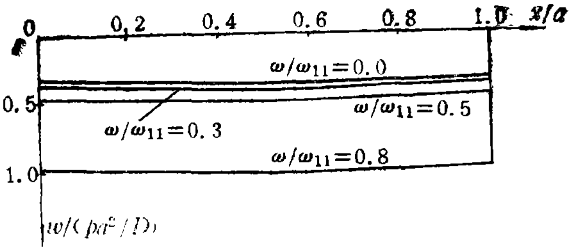

图12　自由边$y=b$振幅曲线②

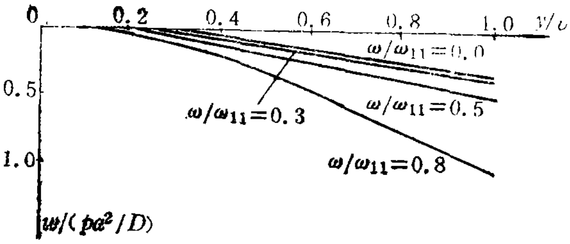

图13　自由边$x=0$振幅曲线②

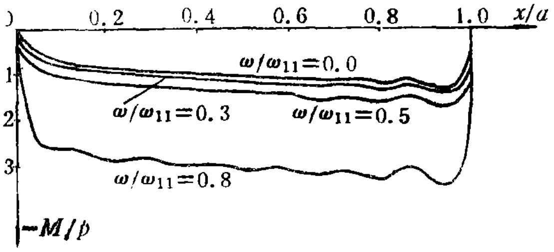

图14　固定边$y=0$弯矩分布曲线③

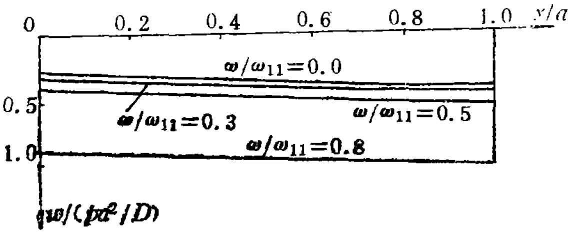

图15　自由边$y=b$振幅曲线③

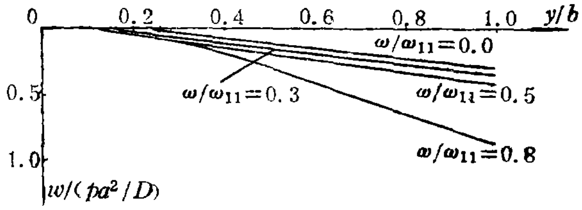

图16　自由边$x=0$振幅曲线③

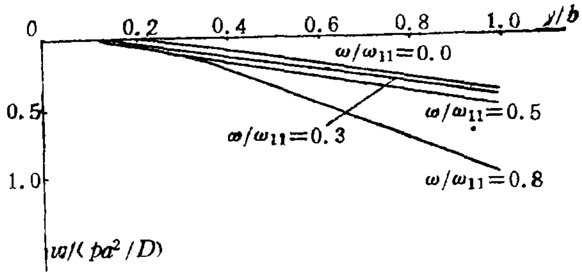

图17　自由边$x=a$振幅曲线③

附　录　Ⅰ

在建立矩形板受迫振动振幅挠曲面方程的过程中，为消除三角级数为满足边界条件在边界上出现的第一类间断点以及加快级数的收敛，必须把三角级数转换成双曲函数。应用下述两种不同的方法求解同一梁的挠曲方程，我们将能实现这些转换：

图18表示一在弹性基上的简支梁，它受到左端的集中弯矩M_0和拉伸力组合作用。

在这种情况下，有微分方程

$$\frac{d^4w}{dx^4}-\frac{N}{EJ}\frac{d^2w}{dx^2}+\frac{k}{EJ}w=0 \qquad (A1.1)$$

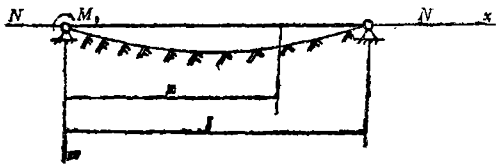

图　18

记　$2\eta=\frac{N}{EJ}$，$\rho^2=\frac{k}{EJ}$

则方程(A1.1)成为

$$\frac{d^4w}{dx^4}-2\eta\frac{d^2w}{dx^2}+\rho^2w=0 \qquad (A1.2)$$

当$\eta>\rho^2$时，方程(A1.2)的解为

$$w(x)=A\text{sh}\alpha x+B\text{ch}\alpha x+C\text{sh}\beta x+D\text{ch}\beta x \qquad (A1.3)$$

这里 $\alpha=\sqrt{\eta+\sqrt{\eta^2-\rho^2}}$，$\beta=\sqrt{\eta-\sqrt{\eta^2-\rho^2}}$

解(A1.3)必须满足边界条件

$$w(0)=w(l)=w''(l)=0 \text{ 和 } w''(0)=-\frac{M_0}{EJ} \qquad (A1.4)$$

执行边界条件(A1.4)，则(A1.3)成为

$$w(x)=\frac{M_0}{(\alpha^2-\beta^2)EJ}\left[-\frac{\text{sh}\alpha(l-x)}{\text{sh}\alpha l}+\frac{\text{sh}\beta(l-x)}{\text{sh}\beta l}\right] \qquad (A1.5)$$

该问题还能用能量法求解。在此情况下，总势能等于

$$\Pi_p=\frac{EJ}{2}\int_0^l\left(\frac{d^2w}{dx^2}\right)^2dx+\frac{N}{2}\int_0^l\left(\frac{dw}{dx}\right)^2dx+\frac{k}{2}\int_0^l w^2dx-M_0\left(\frac{dw}{dx}\right)_{x=0} \qquad (A1.6)$$

假设　$$w(x)=\sum_{m=1,2}^{\infty}a_m\sin\frac{m\pi}{l}x \qquad (A1.7)$$

并应用最小势能原理，我们求得

$$w(x)=\sum_{m=1}^{\infty}\frac{\frac{m\pi}{l}M_0}{\frac{EJ\pi^4}{2l^3}\left[m^4+2\eta m^2\frac{l^2}{\pi^2}+\rho^2\frac{l^4}{\pi^4}\right]}\sin\frac{m\pi}{l}x \tag{A1.8}$$

比较(A1.5)和(A1.8)，我们得到

$$\sum_{m=1,2}^{\infty}\frac{m\sin\frac{m\pi x}{l}}{m^4+2\eta m^2\frac{l^2}{\pi^2}+\rho^2\frac{l^4}{\pi^4}}=\frac{\pi^3}{2l^2(\alpha^2-\beta^2)}\left[-\frac{\text{sh}\alpha(l-x)}{\text{sh}\alpha l}+\frac{\text{sh}\beta(l-x)}{\text{sh}\beta l}\right] \tag{A1.9}$$

式(A1.9)是基本转换式。一系列其他转换关系可由它导出。

取(A1.9)对x的二次积分并利用端部条件，我们得到

$$\sum_{m=1,2}^{\infty}\frac{\sin\frac{m\pi}{l}x}{m\left[m^4+2\eta m^2\frac{l^2}{\pi^2}+\rho^2\frac{l^4}{\pi^4}\right]}$$
$$=\frac{\pi^5}{2l^4}\left\{\frac{1}{\alpha^2-\beta^2}\left[\frac{\text{sh}\alpha(l-x)}{\alpha^2\text{sh}2l}-\frac{\text{sh}\beta(l-x)}{\beta^2\text{sh}\beta l}\right]+\frac{l-x}{\alpha^2\beta^2 l}\right\} \tag{A1.10}$$

对(A1.9)取x的二阶导数，我们得到

$$\sum_{m=1,2}^{\infty}\frac{m^3\sin\frac{m\pi}{l}x}{m^4+2\eta m^2\frac{l^2}{\pi^2}+\rho^2\frac{l^4}{\pi^4}}=\frac{\pi}{2(\alpha^2-\beta^2)}\left[\frac{\alpha^2\text{sh}\alpha(l-x)}{\text{sh}\alpha l}-\frac{\beta^2\text{sh}\beta(l-x)}{\text{sh}\beta l}\right] \tag{A1.11}$$

对(A1.10)求导，我们得到

$$\sum_{m=1,2}^{\infty}\frac{\cos\frac{m\pi}{l}x}{m^4+2\eta m^2\frac{l^2}{\pi^2}+\rho^2\frac{l^4}{\pi^4}}=\frac{\pi^4}{2l^3}\left\{\frac{1}{\alpha^2-\beta^2}\left[-\frac{\text{ch}\alpha(l-x)}{\alpha\text{sh}\alpha l}\right.\right.$$
$$\left.\left.+\frac{\text{ch}\beta(l-x)}{\beta\text{sh}\beta l}\right]-\frac{1}{\alpha^2\beta^2 l}\right\} \tag{A1.12}$$

将$x=l-x$代入(A1.9)，我们得到

$$\sum_{m=1,2}^{\infty}\frac{-m\cos m\pi\sin\frac{m\pi}{l}x}{m^4+2\eta m^2\frac{l^2}{\pi^2}+\rho^2\frac{l^4}{\pi^4}}=\frac{\pi^3}{2l^2(\alpha^2-\beta^2)}\left[-\frac{\text{sh}\alpha x}{\text{sh}\alpha l}+\frac{\text{sh}\beta x}{\text{sh}\beta l}\right] \tag{A1.13}$$

将$x=l-x$代入(A1.10)，我们得到

$$\sum_{m=1,2}^{\infty}\frac{-\cos m\pi\sin\frac{m\pi}{l}x}{m\left[m^4+2\eta m^2\frac{l^2}{\pi^2}+\rho^2\frac{l^4}{\pi^4}\right]}=\frac{\pi^5}{2l^4}\left\{\frac{1}{\alpha^2-\beta^2}\left[\frac{\text{sh}\alpha x}{\alpha^2\text{sh}\alpha l}\right.\right.$$
$$\left.\left.-\frac{\text{sh}\beta x}{\beta^2\text{sh}\beta l}\right]+\frac{x}{\alpha^2\beta^2 l}\right\} \tag{A1.14}$$

将$x=l-x$代入(A1.11)，我们得到

$$\sum_{m=1,2}^{\infty}\frac{-m^3\cos m\pi\sin\frac{m\pi}{l}x}{m^4+2\eta m^2\frac{l^2}{\pi^2}+\rho^2\frac{l^4}{\pi^4}}=\frac{\pi}{2(\alpha^2-\beta^2)}\left[\frac{\alpha^2\text{sh}\alpha x}{\text{sh}\alpha l}-\frac{\beta^2\text{sh}\beta x}{\text{sh}\beta l}\right] \tag{A1.15}$$

将(A1.9)与(A1.13)相加，我们得到

$$\sum_{m=1,3}^{\infty}\frac{m\sin\frac{m\pi}{l}x}{m^4+2\eta m^2\frac{l^2}{\pi^2}+\rho^2\frac{l^4}{\pi^4}}$$

$$=\frac{\pi^3}{4l^2(\alpha^2-\beta^2)}\left[-\frac{\text{ch}\alpha\left(\frac{l}{2}-x\right)}{\text{ch}\alpha\frac{l}{2}}+\frac{\text{ch}\beta\left(\frac{l}{2}-x\right)}{\text{ch}\beta\frac{l}{2}}\right] \tag{A1.16}$$

将(A1.10)与(A1.14)相加，我们得到

$$\sum_{m=1,3}^{\infty}\frac{\sin\frac{m\pi}{l}x}{m\left(m^4+2\eta m^2\frac{l^2}{\pi^2}+\rho^2\frac{l^4}{\pi^4}\right)}=\frac{\pi^5}{4l^4}\left\{\frac{1}{\alpha^2-\beta^2}\left[\frac{\text{ch}\alpha\left(\frac{l}{2}-x\right)}{\alpha^2\text{ch}\alpha\frac{l}{2}}-\frac{\text{ch}\beta\left(\frac{l}{2}-x\right)}{\beta^2\text{ch}\beta\frac{l}{2}}\right]+\frac{1}{\alpha^2\beta^2}\right\} \tag{A1.17}$$

将(A1.11)与(A1.15)相加，我们得到

$$\sum_{m=1,3}^{\infty}\frac{m^3\sin\frac{m\pi}{l}x}{m^4+2\eta m^2\frac{l^2}{\pi^2}+\rho^2\frac{l^4}{\pi^4}}=\frac{\pi}{4(\alpha^2-\beta^2)}\left[\frac{\alpha^2\text{ch}\alpha\left(\frac{l}{2}-x\right)}{\text{ch}\alpha\frac{l}{2}}-\frac{\beta^2\text{ch}\beta\left(\frac{l}{2}-x\right)}{\text{ch}\beta\frac{l}{2}}\right] \tag{A1.18}$$

附 录 Ⅱ

当我们转换三角级数为双曲函数时，必须处理$\lambda<k^2_m$和$\lambda<k^2_n$的情况。为此，让我们考虑图19 所示梁的受迫振动。

假设$M=M_0\sin\omega t$并且忽略阻尼，我们得到$\bar{w}(x, t)=w(x)\sin\omega t$和微分方程

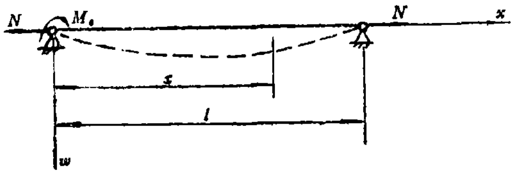

图 19

$$\frac{d^4w}{dx^4}-\frac{N}{EJ}\frac{d^2w}{dx^2}-\frac{\omega^2\rho'}{EJ}w=0 \tag{A2.1}$$

令
$$2\eta_1=\frac{N}{EJ},\quad \rho_1^2=\frac{\omega^2\rho'}{EJ}$$

则(A2.1)成为

$$\frac{d^4w}{dx^4}-2\eta\frac{d^2w}{dx^2}-\rho^2w=0 \tag{A2.2}$$

其解为 $w(x)=A_1\text{sh}\alpha_1x+B_1\text{ch}\alpha_1x+C_1\sin\beta_1x+D_1\cos\beta_1x$ (A2.3)

这里 $\alpha_1=\sqrt{\eta_1+\sqrt{\eta_1^2+\rho_1^2}}$，$\beta_1=\sqrt{\sqrt{\eta_1^2+\rho_1^2}-\eta_1}$

(A2.3)必须满足边界条件

$$w(0)=w(l)=w''(l)=0 \quad 和 w''(0)=-\frac{M_0}{EJ} \tag{A2.4}$$

执行边界条件(A2.4)，我们得到

$$w(x)=\frac{M_0}{(\alpha_1^2+\beta_1^2)EJ}\left(-\frac{\text{sh}\alpha_1(l-x)}{\text{sh}\alpha_1l}+\frac{\sin\beta_1(l-x)}{\sin\beta_1l}\right) \tag{A2.5}$$

我们还可以应用能量法解此问题。总势能为

$$\Pi_p=\frac{EJ}{2}\int_0^l\left(\frac{d^2w}{dx^2}\right)^2dx-\frac{N}{2}\int_0^l\left(\frac{dw}{dx}\right)^2dx-\frac{\omega^2\rho'}{2}\int_0^lw^2dx-M_0\left(\frac{dw}{dx}\right)_{x=0} \tag{A2.6}$$

应用最小势能原理，我们得

$$w(x)=\sum_{m=1,2}^{\infty}\frac{\frac{m\pi}{l}M_0}{\frac{EJ\pi^4}{2l^2}\left(m^4+2\eta_1 m^2\frac{l^2}{\pi^2}-\rho_1^2\frac{l^4}{\pi^4}\right)}\sin\frac{m\pi}{l}x \tag{A2.7}$$

让(A2.5)等于(A2.7)，我们得

$$\sum_{m=1,2}^{\infty}\frac{m\sin\frac{m\pi}{l}x}{m^4+2\eta_1 m^2\frac{l^2}{\pi^2}-\rho_1^2\frac{l^4}{\pi^4}}=\frac{\pi^3}{2l^2(\alpha_1^2+\beta_1^2)}\left(-\frac{\mathrm{sh}\alpha_1(l-x)}{\mathrm{sh}\alpha_1 l}+\frac{\sin\beta_1(l-x)}{\sin\beta_1 l}\right) \tag{A2.8}$$

应用与附录Ⅰ相同的方法，我们可得下面一系列转换关系

$$\sum_{m=1,2}^{\infty}\frac{\sin\frac{m\pi}{l}x}{m\left(m^4+2\eta_1 m^2\frac{l^2}{\pi^2}-\rho_1^2\frac{l^4}{\pi^4}\right)}=\frac{\pi^5}{2l^4}\left\{\frac{1}{\alpha_1^2+\beta_1^2}\left(\frac{\mathrm{sh}\alpha_1(l-x)}{\alpha_1^2\mathrm{sh}\alpha_1 l}+\frac{\sin\beta_1(l-x)}{\beta_1^2\sin\beta_1 l}\right)-\frac{l-x}{\alpha_1^2\beta_1^2 l}\right\} \tag{A2.9}$$

$$\sum_{m=1,2}^{\infty}\frac{m^3\sin\frac{m\pi}{l}x}{m^4+2\eta_1 m^2\frac{l^2}{\pi^2}-\rho_1^2\frac{l^4}{\pi^4}}=\frac{\pi}{2(\alpha_1^2+\beta_1^2)}\left(\frac{\alpha_1^2\mathrm{sh}\alpha_1(l-x)}{\mathrm{sh}\alpha_1 l}+\frac{\beta_1^2\sin\beta_1(l-x)}{\sin\beta_1 l}\right) \tag{A2.10}$$

$$\sum_{m=1,2}^{\infty}\frac{\cos\frac{m\pi}{l}x}{m^4+2\eta_1 m^2\frac{l^2}{\pi^2}-\rho_1^2\frac{l^4}{\pi^4}}=\frac{\pi^4}{2l^3}\left\{\frac{1}{\alpha_1^2+\beta_1^2}\left(-\frac{\mathrm{ch}\alpha_1(l-x)}{\alpha_1\mathrm{sh}\alpha_1 l}-\frac{\cos\beta_1(l-x)}{\beta_1\sin\beta_1 l}\right)+\frac{1}{\alpha_1^2\beta_1^2 l}\right\} \tag{A2.11}$$

$$\sum_{m=1,2}^{\infty}\frac{-m\cos m\pi\sin\frac{m\pi}{l}x}{m^4+2\eta_1 m^2\frac{l^2}{\pi^2}-\rho_1^2\frac{l^4}{\pi^4}}=\frac{\pi^3}{2l(\alpha_1^2+\beta_1^2)}\left(-\frac{\mathrm{sh}\alpha_1 x}{\mathrm{sh}\alpha_1 l}+\frac{\sin\beta_1 x}{\sin\beta_1 l}\right) \tag{A2.12}$$

$$\sum_{m=1,2}^{\infty}\frac{-\cos m\pi\sin\frac{m\pi}{l}x}{m\left(m^4+2\eta_1 m^2\frac{l^2}{\pi^2}-\rho_1^2\frac{l^4}{\pi^4}\right)}=\frac{\pi^5}{2l^4}\left[\frac{1}{\alpha_1^2+\beta_1^2}\left(\frac{\mathrm{sh}\alpha_1 x}{\alpha_1^2\mathrm{sh}\alpha_1 l}+\frac{\sin\beta_1 x}{\beta_1^2\sin\beta_1 l}\right)-\frac{x}{\alpha_1^2\beta_1^2 l}\right] \tag{A2.13}$$

$$\sum_{m=1,2}^{\infty}\frac{-m^3\cos m\pi\sin\frac{m\pi}{l}x}{m^4+2\eta_1 m^2\frac{l^2}{\pi^2}-\rho_1^2\frac{l^4}{\pi^4}}=\frac{\pi}{2(\alpha_1^2+\beta_1^2)}\left(\frac{\alpha_1^2\mathrm{sh}\alpha_1 x}{\mathrm{sh}\alpha_1 l}+\frac{\beta_1^2\sin\beta_1 x}{\sin\beta_1 l}\right) \tag{A2.14}$$

$$\sum_{m=1,3}^{\infty}\frac{m\sin\frac{m\pi}{l}x}{m^4+2\eta_1 m^2\frac{l^2}{\pi^2}-\rho_1^2\frac{l^4}{\pi^4}}=\frac{\pi^3}{4l^2(\alpha_1^2+\beta_1^2)}\left[-\frac{\mathrm{ch}\alpha_1\left(\frac{l}{2}-x\right)}{\mathrm{ch}\alpha_1\frac{l}{2}}+\frac{\cos\beta_1\left(\frac{l}{2}-x\right)}{\cos\beta_1\frac{l}{2}}\right] \tag{A2.15}$$

$$\sum_{m=1,3}^{\infty}\frac{\sin\frac{m\pi}{l}x}{m\left(m^4+2\eta_1 m^2\frac{l^2}{\pi^2}-\rho_1^2\frac{l^4}{\pi^4}\right)}$$

$$=\frac{\pi^5}{4l^4}\left\{\frac{1}{\alpha_1^2+\beta_1^2}\left[\frac{\mathrm{ch}\alpha_1\left(\frac{l}{2}-x\right)}{\alpha_1^2\mathrm{ch}\alpha_1\frac{l}{2}}+\frac{\cos\beta_1\left(\frac{l}{2}-x\right)}{\beta_1^2\cos\beta_1\frac{l}{2}}\right]-\frac{1}{\alpha_1^2\beta_1^2}\right\} \tag{A2.16}$$

$$\sum_{m=1,3}^{\infty}\frac{m^3\sin\frac{m\pi}{l}x}{m^4+2\eta_1 m^2\frac{l^2}{\pi^2}-\rho_1^2\frac{l^4}{\pi^4}}$$

$$=\frac{\pi}{4(\alpha_1^2+\beta_1^2)}\left[\frac{\alpha_1^2\mathrm{ch}\alpha_1\left(\frac{l}{2}-x\right)}{\mathrm{ch}\alpha_1\frac{l}{2}}+\frac{\beta_1^2\cos\beta_1\left(\frac{l}{2}-x\right)}{\cos\beta_1\frac{l}{2}}\right] \tag{A2.17}$$

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The Method of the Reciprocal Theorem of Forced Vibration for the Elastic Thin Rectangular Plates (Ⅲ) —Cantilever Rectangular Plates

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Abstract

In this paper, applying the method of the reciprocal theorem, we give the stationary solutions of the forced vibration of cantilever rectangular plates under uniformly distributed harmonic load and concentrated harmonic load acting at any point of the plates, the figures and tables of number value of bending moment and the deflection amplitudes as well.

Key words the method of reciprocal theorem, forced vibration, cantilever rectangular plate, the amplitude of bending moment, the amplitude of deflection