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球壳轴对称弯曲问题精确的挠度微分方程及其奇异摄动解*

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摘　要

本文提出了球壳轴对称弯曲问题精确的挠度(w)微分方程和精确的转角($dw/d\alpha$)微分方程。

本文重点研究了挠度微分方程的精度，基本思路是：首先假设边缘效应时经线中面位移$u=0$，从而建立挠度微分方程，然后再精确地证明挠度微分方程与原来微分方程内力解答完全相同.再精确地证明边缘效应时经线中面位移$u=0$是精确解.

本文给出了挠度微分方程的奇异摄动解，最后验算了平衡条件，证明摄动解求出的内力和外荷载是完全平衡的．这一方面表明摄动解的计算是正确的；另一方面也再一次表明挠度微分方程是精确的微分方程.

新微分方程的优点是：1.新微分方程和原来微分方程精度完全相同；2.新微分方程满足的边界条件非常简单；3.新微分方程便于使用摄动解；4.新微分方程可以得到挠度(w)和转角($dw/d\alpha$)的表达式.

新微分方程使球壳的计算得到很大的简化.本文采用的符号与徐芝纶《弹性力学》第二版下册相同[1].

关键词　球壳　挠度微分方程　奇异摄动解

一、序　言

球壳轴对称弯曲问题平衡方程和弹性方程

$$d(N_1\sin\alpha)/d\alpha-N_2\cos\alpha+Q_1\sin\alpha+XR\sin\alpha=0 \tag{1.1a}$$

$$d(Q_1\sin\alpha)/d\alpha-(N_1+N_2)\sin\alpha+ZR\sin\alpha=0 \tag{1.1b}$$

$$d(M_1\sin\alpha)/d\alpha-M_2\cos\alpha-Q_1R\sin\alpha=0 \tag{1.1c}$$

和

$$du/d\alpha+w=(R/Et)(N_1-\mu N_2) \tag{1.2a}$$

$$u\,\mathrm{ctg}\alpha+w=(R/Et)(N_2-\mu N_1) \tag{1.2b}$$

$$M_1=\frac{D}{R^2}\left[\frac{d}{d\alpha}\left(u-\frac{dw}{d\alpha}\right)+\mu\left(u-\frac{dw}{d\alpha}\right)\mathrm{ctg}\alpha\right] \tag{1.2c}$$

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$$M_2=\frac{D}{R^2}\left[\left(u-\frac{dw}{d\alpha}\right)\text{ctg}\alpha+\mu\frac{d}{d\alpha}\left(u-\frac{dw}{d\alpha}\right)\right] \tag{1.2d}$$

$$N_1=Q_1\text{ctg}\alpha+\frac{R}{\sin^2\alpha}\left[\int_{\alpha'}^{\alpha}(z\cos\alpha-X\sin\alpha)\sin\alpha d\alpha+\frac{P}{2\pi R^2}\right]^{[6]}=Q_1\text{ctg}\alpha+N_1^* \tag{A}$$

$$N_2=\frac{dQ_1}{d\alpha}+ZR-\frac{R}{\sin^2\alpha}\left[\int_{\alpha''}^{\alpha}(Z\cos\alpha-X\sin\alpha)\sin\alpha d\alpha+\frac{P}{2\pi R^2}\right]^{[6]}=\frac{dQ_1}{d\alpha}+N_2^* \tag{B}$$

二、无矩状态经线中面位移u*的公式

本文在右上角打上*号代表无矩状态的位移和内力．(1.2a)式和(1.2b)式适用于无矩状态变为

$$du^*/d\alpha+w^*=(R/Et)(N_1^*-\mu N_2^*)$$

$$u^*\text{ctg}\alpha+w^*=(R/Et)(N_2^*-\mu N_1^*)$$

两式相减，消除w^*，可得

$$\frac{du^*}{d\alpha}-u^*\text{ctg}\alpha=\frac{(1+\mu)R}{Et}(N_1^*-N_2^*) \tag{a}$$

引入代换变量 $u^*=\eta^*\sin\alpha,\quad du^*/d\alpha=(d\eta^*/d\alpha)\sin\alpha+\eta^*\cos\alpha$

(a)式变为 $\dfrac{d\eta^*}{d\alpha}=\dfrac{(1+\mu)R}{Et\sin\alpha}(N_1^*-N_2^*)$

积分一次 $\eta^*=\dfrac{(1+\mu)R}{Et}\displaystyle\int_0^{\alpha}\frac{N_1^*-N_2^*}{\sin\alpha}d\alpha+C$

代回原变量 $\dfrac{u^*}{\sin\alpha}=\dfrac{(1+\mu)R}{Et}\displaystyle\int_0^{\alpha}\frac{N_1^*-N_2^*}{\sin\alpha}d\alpha+C$

利用壳顶对称性条件

$$(u^*)_{\alpha=0}=0,\quad (du^*/d\alpha)_{\alpha=0}=0$$

当$\alpha=0$时 $C=\dfrac{\lim\limits_{\alpha\to 0}u^*}{\lim\limits_{\alpha\to 0}\sin\alpha}=\dfrac{\lim\limits_{\alpha\to 0}(du^*/d\alpha)}{\lim\limits_{\alpha\to 0}\cos\alpha}=0$

$$u^*=\frac{(1+\mu)R}{Et}\sin\alpha\int_0^{\alpha}\frac{N_1^*-N_2^*}{\sin\alpha}d\alpha \tag{2.1}$$

三、挠度w和转角dw/dα微分方程

本文首先假设边缘效应时经线中面位移$u=0$，从而建立挠度微分方程，然后再精确地证明挠度微分方程与原来微分方程内力解答完全相同．再精确地证明边缘效应时经线中面位移$u=0$是精确解．

由(1.2c)式和(1.2d)式注意到边缘效应时$u=0$，并令$\theta=dw/d\alpha$

$$M_1=-\frac{D}{R^2}\left(\frac{d^2w}{d\alpha^2}+\mu\frac{dw}{d\alpha}\text{ctg}\alpha\right)=-\frac{D}{R^2}\left(\frac{d\theta}{d\alpha}+\mu\theta\text{ctg}\alpha\right)$$

$$M_2=-\frac{D}{R^2}\left(\frac{dw}{d\alpha}\text{ctg}\alpha+\mu\frac{d^2w}{d\alpha^2}\right)=-\frac{D}{R^2}\left(\theta\text{ctg}\alpha+\mu\frac{d\theta}{d\alpha}\right)$$

代入(1.1c)式可得

$$\frac{d^2\theta}{d\alpha^2}+\frac{d\theta}{d\alpha}\mathrm{ctg}\alpha-\theta(\mathrm{ctg}^2\alpha+\mu)=-\frac{R^3}{D}Q_1 \tag{3.1a}$$

由(1.2a)式和(1.2b)式注意到边缘效应时$u=0$，可得

$$du^*/d\alpha+w=(R/Et)(N_1-\mu N_2) \tag{b}$$

$$\mathrm{ctg}\alpha u^*+w=(R/Et)(N_2-\mu N_1) \tag{c}$$

(b)式－(c)式

$$0=(1+\mu)(N_1-N_2)+\frac{Et}{R}\mathrm{ctg}\alpha u^*-\frac{Et}{R}\frac{du^*}{d\alpha} \tag{d}$$

(c)式对α求一次导数

$$\frac{Et}{R}\theta=\frac{d}{d\alpha}(N_2-\mu N_1)-\frac{Et}{R}\frac{d}{d\alpha}(\mathrm{ctg}\alpha u^*) \tag{e}$$

(e)式－ctgα×(d)式，可得

$$\frac{Et}{R}\theta=\frac{d}{d\alpha}(N_2-\mu N_1)+(1+\mu)\mathrm{ctg}\alpha(N_2-N_1)+\frac{Et}{R}u^*$$

将(A)式和(B)式代入上式，可得

$$\frac{d^2Q_1}{d\alpha^2}+\frac{dQ_1}{d\alpha}\mathrm{ctg}\alpha-Q_1(\mathrm{ctg}^2\alpha-\mu)=\frac{Et}{R}\theta-\frac{d}{d\alpha}(N_2^*-\mu N_1^*)$$

$$-(1+\mu)\mathrm{ctg}\alpha(N_2^*-N_1^*)-\frac{Et}{R}u^* \tag{3.1b}$$

$$\frac{d^3w}{d\alpha^3}+\frac{d^2w}{d\alpha^2}\mathrm{ctg}\alpha-\frac{dw}{d\alpha}(\mathrm{ctg}^2\alpha+\mu)=-\frac{R^3}{D}Q_1 \tag{3.2a}$$

$$\frac{d^2Q}{d\alpha^2}+\frac{dQ_1}{d\alpha}\mathrm{ctg}\alpha-Q_1(\mathrm{ctg}^2\alpha-\mu)=\frac{Et}{R}\frac{dw}{d\alpha}-\frac{d}{d\alpha}(N_2^*-\mu N_1^*)$$

$$-(1+\mu)\mathrm{ctg}\alpha(N_2^*-N_1^*)-\frac{Et}{R}u^* \tag{3.2b}$$

(3.1)式和(3.2)式消除Q_1可得转角和挠度微分方程

$$\left[\frac{d^2}{d\alpha^2}+\mathrm{ctg}\alpha\frac{d}{d\alpha}-\mathrm{ctg}^2\alpha\right]^2\theta+\left(\frac{R^2Et}{D}-\mu^2\right)\theta=\frac{R^3}{D}\frac{d}{d\alpha}(N_2^*-\mu N_1^*)$$

$$+(1+\mu)\frac{R^3}{D}\mathrm{ctg}\alpha(N_2^*-N_1^*)+\frac{R^2Et}{D}u^* \tag{3.3}$$

$$\left[\frac{d^2}{d\alpha^2}+\mathrm{ctg}\alpha\frac{d}{d\alpha}\mathrm{ctg}^2\alpha\right]^2\frac{dw}{d\alpha}+\left(\frac{R^2Et}{D}-\mu^2\right)\frac{dw}{d\alpha}=\frac{R^3}{D}\frac{d}{d\alpha}(N_2^*-\mu N_1^*)$$

$$+(1+\mu)\frac{R^3}{D}\mathrm{ctg}\alpha(N_2^*-N_1^*)+\frac{R^2Et}{D}u^* \tag{3.4}$$

四、精确的挠度微分方程

适用于边缘效应的挠度微分方程

$$\left.\begin{aligned}
&\frac{d^3w}{d\alpha^3}+\frac{d^2w}{d\alpha^2}\mathrm{ctg}\alpha-\frac{dw}{d\alpha}(\mathrm{ctg}^2\alpha+\mu)=-\frac{R^3}{D}Q_1\\
&\frac{d^2Q_1}{d\alpha^2}+\frac{dQ_1}{d\alpha}\mathrm{ctg}\alpha-Q_1(\mathrm{ctg}^2\alpha-\mu)=\frac{Et}{R}\frac{dw}{d\alpha}\\
&\text{固定边：}\left(\frac{dw}{d\alpha}\right)_{\alpha=\alpha''}=-\left(\frac{dw^*}{d\alpha}\right)_{\alpha=\alpha''},\ (\varepsilon_2)_{\alpha=\alpha''}=-(\varepsilon_2^*)_{\alpha=\alpha''}\\
&\text{铰支边：}(M_1)_{\alpha=\alpha''}=-\frac{D}{R^2}\left(\frac{d^2w}{d\alpha^2}+\mu\frac{dw}{d\alpha}\mathrm{ctg}\alpha\right)_{\alpha=\alpha''}=0\\
&\qquad(\delta)_{\alpha=\alpha''}=-(\delta^*)_{\alpha=\alpha''}\\
&\text{自由边：}(M_1)_{\alpha=\alpha''}=-\frac{D}{R^2}\left(\frac{d^2w}{d\alpha^2}+\mu\frac{dw}{d\alpha}\mathrm{ctg}\alpha\right)_{\alpha=\alpha''}=0,\quad (Q_1)_{\alpha=\alpha''}=0\\
&M_1=-\frac{D}{R^2}\left(\frac{d^2w}{d\alpha^2}+\mu\frac{dw}{d\alpha}\mathrm{ctg}\alpha\right),\qquad M_2=-\frac{D}{R^2}\left(\frac{dw}{d\alpha}\mathrm{ctg}\alpha+\mu\frac{d^2w}{d\alpha^2}\right)\\
&N_1=Q_1\mathrm{ctg}\alpha+N_1^*,\qquad N_2=dQ_1/d\alpha+N_2^*
\end{aligned}\right\}(\mathrm{I})$$

由文[1]p.307中(b)式和(c)式，可得适用于边缘效应的原微分方程（V和Q_1微分方程）

$$\left.\begin{aligned}
&\frac{d^2V}{d\alpha^2}+\frac{dV}{d\alpha}\mathrm{ctg}\alpha-V(\mathrm{ctg}^2\alpha+\mu)=\frac{R^2}{D}Q_1\\
&\frac{d^2Q_1}{d\alpha^2}+\frac{dQ_1}{d\alpha}\mathrm{ctg}\alpha-Q_1(\mathrm{ctg}^2\alpha-\mu)=-EtV\\
&\text{固定边：}(V)_{\alpha=\alpha''}=-(V^*)_{\alpha=\alpha''}=-\frac{1}{R}\left(u^*-\frac{dw^*}{d\alpha}\right)_{\alpha=\alpha''}=\frac{1}{R}\left(\frac{dw^*}{d\alpha}\right)_{\alpha=\alpha''}\\
&\qquad(\varepsilon_2)_{\alpha=\alpha''}=-(\varepsilon_2^*)_{\alpha=\alpha''}\\
&\text{铰支边：}(M_1)_{\alpha=\alpha''}=\frac{D}{R}\left(\frac{dV}{d\alpha}+\mu V\mathrm{ctg}\alpha\right)_{\alpha=\alpha''}=0,\ (\delta)_{\alpha=\alpha''}=-(\delta^*)_{\alpha=\alpha''}\\
&\text{自由边：}(M_1)_{\alpha=\alpha''}=\frac{D}{R}\left(\frac{dV}{d\alpha}+\mu V\mathrm{ctg}\alpha\right)_{\alpha=\alpha''}=0,\ (Q_1)_{\alpha=\alpha''}=0\\
&M_1=\frac{D}{R}\left(\frac{dV}{d\alpha}+\mu V\mathrm{ctg}\alpha\right),\qquad M_2=\frac{D}{R}\left(V\mathrm{ctg}\alpha+\mu\frac{dV}{d\alpha}\right)\\
&N_1=Q_1\mathrm{ctg}\alpha+N_1^*,\qquad N_2=dQ_1/d\alpha+N_2^*
\end{aligned}\right\}(\mathrm{II})$$

由文[1]p.298中(21-36)式

$$u^*=\frac{(1+\mu)q_0R^2}{Et}\left[\ln\frac{1+\cos\alpha}{1+\cos\alpha''}+\frac{1}{1+\cos\alpha''}-\frac{1}{1+\cos\alpha}\right]\sin\alpha$$

显然$(u^*)_{\alpha=\alpha''}=0$。固定边和铰支边都有边界条件$(u^*)_{\alpha=\alpha''}=0$。

方程组(Ⅰ)是根据边缘效应时$u=0$推导出来的，方程组(Ⅱ)是考虑边缘效应时$u\neq0$的原方程组。事实上作变量代换

$$\phi=-\frac{1}{R}\frac{dw}{d\alpha}$$

方程组(Ⅰ)变为

$$\left.\begin{aligned}
&\frac{d^2\phi}{da^2}+\frac{d\phi}{da}\operatorname{ctg}a-\phi(\operatorname{ctg}^2a+\mu)=\frac{R^2}{D}Q_1\\
&\frac{d^2Q_1}{da^2}+\frac{dQ_1}{da}\operatorname{ctg}a-Q_1(\operatorname{ctg}^2a-\mu)=-Et\phi\\
&\text{固定边：}(\phi)_{a=a''}=-(\phi^*)_{a=a''}=-\left(-\frac{1}{R}\frac{dw^*}{da}\right)_{a=a''}=\frac{1}{R}\left(\frac{dw^*}{da}\right)_{a=a''}\\
&\qquad(\varepsilon_2)_{a=a''}=-(\varepsilon_2^*)_{a=a''}\\
&\text{铰支边：}(M_1)_{a=a''}=\frac{D}{R}\left(\frac{d\phi}{da}+\mu\phi\operatorname{ctg}a\right)_{a=a''}=0,\quad(\delta)_{a=a''}=-(\delta^*)_{a=a''}\\
&\text{自由边：}(M_1)_{a=a''}=\frac{D}{R}\left(\frac{d\phi}{da}+\mu\phi\operatorname{ctg}a\right)_{a=a''}=0,\quad(Q_1)_{a=a''}=0\\
&M_1=\frac{D}{R}\left(\frac{d\phi}{da}+\mu\phi\operatorname{ctg}a\right),\qquad M_2=\frac{D}{R}\left(\phi\operatorname{ctg}a+\mu\frac{d\phi}{da}\right)\\
&N_1=Q_1\operatorname{ctg}a+N_1^*,\qquad N_2=dQ_1/da+N_2^*
\end{aligned}\right\}\quad(\text{Ⅱ})$$

方程组(Ⅱ)和方程组(Ⅰ)完全相同，仅仅是数学符号V换成了ϕ，微分方程相同，边界条件相同，内力公式相同．也就是说方程组(Ⅰ)与方程组(Ⅱ)内力的解答是完全相同的．受力状态相同，根据虎克定律，力和位移成正比，它们的位移状态是必然相同的，即原来微分方程边缘效应时u也应等于0，故边缘效应时$u=0$是原微分方程的精确解．或者由方程组(Ⅰ)和方程组(Ⅱ)可得$V=\phi$，即

$$\frac{1}{R}\left(u-\frac{dw}{da}\right)=-\frac{1}{R}\frac{dw}{da}$$

故$u=0$，我们精确地得到了边缘效应时$u=0$的精确解．

五、奇异摄动解

固定边球壳在任意轴对称荷载作用下．

适用于边缘效应的挠度w微分方程为

$$\left.\begin{aligned}
&\frac{d^3w}{da^3}+\frac{d^2w}{da^2}\operatorname{ctg}a-(\operatorname{ctg}^2a+\mu)\frac{dw}{da}=-\frac{R^3}{D}Q_1\\
&\frac{d^2Q_1}{da^2}+\frac{dQ_1}{da}\operatorname{ctg}a-(\operatorname{ctg}^2a-\mu)Q_1=\frac{Et}{R}\frac{dw}{da}
\end{aligned}\right\}\quad(5.1)$$

$$(w)_{a=a''}=-(w^*)_{a=a''}=\alpha,\ (dw/da)_{a=a''}=-(dw^*/da)_{a=a''}=\beta$$

$$\lim_{a\to0}w=\lim_{a\to0}Q_1=\lim_{a\to0}\frac{dw}{da}=0$$

$$\operatorname{tg}a\frac{d^3w}{da^3}+\frac{d^2w}{da^2}-\operatorname{tg}a(\operatorname{ctg}^2a+\mu)\frac{dw}{da}=-\frac{R^3}{D}\operatorname{tg}aQ_1$$

$$\frac{d^2\operatorname{tg}aQ_1}{da^2}+\left(\operatorname{ctg}a-\frac{2}{\sin a\cos a}\right)\frac{d\operatorname{tg}aQ_1}{da}+\left(\frac{1}{\sin^2a}-\operatorname{ctg}^2a+\mu\right)\operatorname{tg}aQ_1=\frac{Et}{R}\operatorname{tg}a\frac{dw}{da}$$

$$(w)_{a=a''}=\alpha,\ (dw/da)_{a=a''}=\beta,\ \lim_{a\to0}w=\lim_{a\to0}Q_1=\lim_{a\to0}(dw/da)=0$$

令 $\psi=a''-a,\ d\psi=-da$

$$\mathrm{tg}(\alpha''-\psi)\frac{d^3w}{d\psi^3}-\frac{d^2w}{d\psi^2}-[\mathrm{ctg}(\alpha''-\psi)+\mu\,\mathrm{tg}(\alpha''-\psi)]\frac{dw}{d\psi}=\frac{R^3}{D}\mathrm{tg}\alpha Q_1$$

$$\frac{d^2\mathrm{tg}\alpha Q_1}{d\psi^2}-\left[\mathrm{ctg}(\alpha''-\psi)-\frac{2}{\sin(\alpha''-\psi)\cos(\alpha''-\psi)}\right]\frac{d\mathrm{tg}\alpha Q_1}{d\psi}+\left[\frac{1}{\sin^2(\alpha''-\psi)}-\mathrm{ctg}^2(\alpha''-\psi)+\mu\right]\mathrm{tg}\alpha Q_1=-\frac{Et}{R}\mathrm{tg}(\alpha''-\psi)\frac{dw}{d\psi}$$

$$(w)_{\psi=0}=\alpha,\qquad -\left(\frac{dw}{d\psi}\right)_{\psi=0}=\beta,\qquad \lim_{\psi\to\alpha''}w=\lim_{\psi\to\alpha''}Q_1=\lim_{\psi\to\alpha''}\frac{dw}{d\psi}=0$$

代换变量 $\qquad x=\dfrac{\sin(\alpha''-\psi)}{\sin\alpha''},\qquad a=\dfrac{1}{\sin\alpha''}\qquad a>1$

$$x^2(x^2-a^2)\frac{d^3w}{dx^3}+x(4x^2-a^2)\frac{d^2w}{dx^2}+[(1+\mu)x^2+a^2]\frac{dw}{dx}=\frac{R^3}{D}x\,\mathrm{tg}\alpha Q_1$$

$$x(x^2-a^2)\frac{d^2\mathrm{tg}\alpha Q_1}{dx^2}+(2x^2+a^2)\frac{d\mathrm{tg}\alpha Q_1}{dx}-(1+\mu)x\,\mathrm{tg}\alpha Q_1=-\frac{Et}{R}x^2\frac{dw}{dx}$$

$$(w)_{x=1}=\alpha,\quad \frac{\cos\alpha''}{\sin\alpha''}\left(\frac{dw}{dx}\right)_{x=1}=\beta,\qquad \lim_{x\to0}w=\lim_{x\to0}Q_1=\lim_{x\to0}\frac{dw}{dx}=0$$

无量纲化

$$Q=\sqrt{\frac{\sqrt{12(1-\mu^2)}\,R}{t}}\;\frac{R\,\mathrm{tg}\alpha Q_1}{Et^2},\qquad W=\frac{w}{t},\qquad \varepsilon^2=\frac{t}{\sqrt{12(1-\mu^2)}\,R}$$

$$\left.\begin{aligned}&\varepsilon^3x^2(x^2-a^2)\frac{d^3W}{dx^3}+\varepsilon^3x(4x^2-a^2)\frac{d^2W}{dx^2}+\varepsilon^3[(1+\mu)x^2+a^2]\frac{dW}{dx}=xQ\\&\varepsilon x(x^2-a^2)\frac{d^2Q}{dx^2}+\varepsilon(2x^2+a^2)\frac{dQ}{dx}-\varepsilon(1+\mu)xQ=-x^2\frac{dW}{dx}\end{aligned}\right\}\qquad(5.2)$$

$$(W)_{x=1}=\frac{\alpha}{t}=\alpha_0,\qquad \left(\frac{dW}{dx}\right)_{x=1}=\frac{\sin\alpha''}{\cos\alpha''}\frac{\beta}{t}$$

$$\lim_{x\to0}W=\lim_{x\to0}Q=\lim_{x\to0}\frac{dW}{dx}=0$$

应用合成展开法,

$$W(x,\ \varepsilon)=E(x,\ \varepsilon)+G(\eta,\ \varepsilon),\qquad Q(x,\ \varepsilon)=F(x,\ \varepsilon)+\Omega(\eta,\ \varepsilon)$$

$$\eta=\frac{x-1}{\varepsilon},\qquad d\eta=\varepsilon^{-1}dx,\qquad x=\varepsilon\eta+1$$

将$W(x,\ \varepsilon)$和$Q(x,\ \varepsilon)$代入(5.2)式，可分解为两组方程

$$\varepsilon^3x^2(x^2-a^2)\frac{d^3E}{dx^3}+\varepsilon^3x(4x^2-a^2)\frac{d^2E}{dx^2}+\varepsilon^3[(1+\mu)x^2+a^2]\frac{dE}{dx}=xF$$

$$\varepsilon x(x^2-a^2)\frac{d^2F}{dx^2}+\varepsilon(2x^2+a^2)\frac{dF}{dx}-\varepsilon(1+\mu)xF=-x^2\frac{dE}{dx}$$

$$E(0)=0,\ F(0)=0$$

边缘效应函数E和F离开边界较远处应为0，将外解 $E=E_0+E_1\varepsilon+E_2\varepsilon^2+\cdots$，$F=F_0+F_1\varepsilon+F_2\varepsilon^2+\cdots$代入，显然可得，$E=F=0$，边缘效应方程没有外解。

令 $\quad 1/2b^2=a^2-1\qquad b>0$

$$(\varepsilon\eta+1)^2[(\varepsilon\eta+1)^2-a^2]\frac{d^3G}{d\eta^3}+\varepsilon(\varepsilon\eta+1)[4(\varepsilon\eta+1)^2-a^2]\frac{d^2G}{d\eta^2}$$
$$+\varepsilon^2[(1+\mu)(\varepsilon\eta+1)^2+a^2]\frac{dG}{d\eta}=(\varepsilon\eta+1)\Omega$$
$$(\varepsilon\eta+1)[(\varepsilon\eta+1)^2-a^2]\frac{d^2\Omega}{d\eta^2}+\varepsilon[2(\varepsilon\eta+1)^2+a^2]\frac{d\Omega}{d\eta}-\varepsilon^2(1+\mu)(\varepsilon\eta+1)\Omega$$
$$=-(\varepsilon\eta+1)^2\frac{dG}{d\eta}$$

$$(G)_{\eta=0}=\alpha_0,\ \left(\frac{dG}{d\eta}\right)_{\eta=0}=\sqrt{\frac{t}{\sqrt{12(1-\mu^2)}R}}\ \frac{\sin\alpha''}{\cos\alpha''}\ \frac{\beta}{t}=\beta_0$$

$$\lim_{\eta\to\infty}G=\lim_{\eta\to\infty}\Omega=\lim_{\eta\to\infty}\frac{dG}{d\eta}=0$$

$$\left[\varepsilon^4\eta^4+4\varepsilon^3\eta^3+\left(5-\frac{1}{2b^2}\right)\varepsilon^2\eta^2+\left(2-\frac{1}{b^2}\right)\varepsilon\eta-\frac{1}{2b^2}\right]\frac{d^3G}{d\eta^3}+\left[4\varepsilon^4\eta^3+12\varepsilon^3\eta^2\right.$$
$$\left.+\left(11-\frac{1}{2b^2}\right)\varepsilon^2\eta+\left(3-\frac{1}{2b^2}\right)\varepsilon\right]\frac{d^2G}{d\eta^2}+\left[(1+\mu)\varepsilon^4\eta^2+2(1+\mu)\varepsilon^3\eta\right.$$
$$\left.+\left(2+\mu+\frac{1}{2b^2}\right)\varepsilon^2\right]\frac{dG}{d\eta}=(\varepsilon\eta+1)\Omega$$

$$\left[\varepsilon^3\bar\eta^3+3\varepsilon^2\eta^2+\left(2-\frac{1}{2b^2}\right)\varepsilon\eta-\frac{1}{2b^2}\right]\frac{d^2\Omega}{d\bar\eta^2}+\left[2\varepsilon^3\bar\eta^2+4\varepsilon^2\bar\eta+\left(3+\frac{1}{2b^2}\right)\varepsilon\right]\frac{d\Omega}{d\bar\eta}$$
$$+[-(1+\mu)\varepsilon^3\bar\eta-(1+\mu)\varepsilon^2]\Omega=[-\varepsilon^2\bar\eta^2-2\varepsilon\bar\eta-1]dG/d\bar\eta$$
$$(G)_{\eta=0}=\alpha_0,\ (dG/d\eta)_{\eta=0}=\beta_0,\quad \lim_{\eta\to\infty}G=\lim_{\eta\to\infty}\Omega=\lim_{\eta\to\infty}(dG/d\bar\eta)=0$$

将内解$G=G_0+G_1\varepsilon+G_2\varepsilon^2+\cdots$，$\Omega=\Omega_0+\Omega_1\varepsilon+\Omega_2\varepsilon^2+\cdots$代入

ε^0: $\quad d^3G_0/d\bar\eta^3=-2b^2\Omega_0,\quad d^2\Omega_0/d\bar\eta^2=2b^2dG_0/d\bar\eta$

$$(G_0)_{\eta=0}=\alpha_0,\ (dG_0/d\bar\eta)_{\eta=0}=\beta_0,\ \lim_{\eta\to\infty}G_0=\lim_{\eta\to\infty}G_0=\lim_{\eta\to\infty}(dG_0/d\bar\eta)=0$$

通解为

$$G_0=\exp[-b\bar\eta](A_1\cos b\bar\eta+A_2\sin b\bar\eta)+\exp[b\eta](A_3\cos b\bar\eta+A_4\sin b\bar\eta)+A_5$$
$$A_3=A_4=A_5=0,\ \ A_1=\alpha_0,\ \ A_2=\alpha_0+\beta_0/b$$
$$G_0=\exp[-b\bar\eta](\alpha_0\cos b\bar\eta+A_2\sin b\bar\eta)$$
$$dG_0/d\bar\eta=b\exp[-b\bar\eta][(-\alpha_0+A_2)\cos b\bar\eta-(\alpha_0+A_2)\sin b\bar\eta]$$
$$d^2G_0/d\bar\eta^2=2b^2\exp[-b\bar\eta](-A_2\cos b\bar\eta+\alpha_0\sin b\bar\eta)$$
$$d^3G_0/d\bar\eta^3=2b^3\exp[-b\bar\eta][(\alpha_0+A_2)\cos b\bar\eta+(-\alpha_0+A_2)\sin b\bar\eta]$$
$$\Omega_0=b\exp[-b\bar\eta][-(\alpha_0+A_2)\cos b\bar\eta+(\alpha_0+A_2)\sin b\bar\eta]$$
$$d\Omega_0/d\bar\eta=2b^2\exp[-b\bar\eta](\alpha_0\cos b\bar\eta+A_2\sin b\bar\eta)$$
$$d^2\Omega_0/d\bar\eta^2=2b^3\exp[-b\bar\eta][(-\alpha_0+A_2)\cos b\bar\eta-(\alpha_0+A_2)\sin b\bar\eta]$$

ε^1: $\quad d^3G_1/d\bar\eta^3=-2b^2\Omega_1+[8b^5(\alpha_0+A_2)\bar\eta-2b^3(\alpha_0+A_2)\bar\eta-12b^4A_2$

$$+2b^2A_2]\exp[-b\bar\eta]\cos b\bar\eta+[8b^5(-\alpha_0+A_2)\bar\eta$$
$$-2b^3(-\alpha_0+A_2)\bar\eta+12b^4\alpha_0-2b^2\alpha_0]\exp[-b\bar\eta]\sin b\bar\eta$$
$$d^2\Omega_1/d\bar\eta^2=2b^2dG_1/d\bar\eta+[8b^5(-\alpha_0+A_2)\bar\eta+2b^3(-\alpha_0+A_2)\bar\eta+12b^4\alpha_0$$

$$+2b^2a_0]\exp[-b\bar{\eta}]\cos b\bar{\eta}+[-8b^5(a_0+A_2)\bar{\eta}-2b^3(a_0+A_2)\bar{\eta}$$
$$+12b^4A_2+2b^2A_2]\exp[-b\bar{\eta}]\sin b\bar{\eta}$$

$$(G_1)_{\bar{\eta}=0}=0,\ (dG_1/d\bar{\eta})_{\bar{\eta}=0}=0,\ \lim_{\bar{\eta}\to\infty}G_1=\lim_{\bar{\eta}\to\infty}\Omega_1=\lim_{\bar{\eta}\to\infty}(dG_1/d\bar{\eta})=0$$

其解为

$$G_1=[-b^3(a_0-A_2)\bar{\eta}^2-a_0\bar{\eta}/2]\exp[-b\bar{\eta}]\cos b\bar{\eta}+[B_2-b^3(a_0+A_2)\bar{\eta}^2$$
$$-A_2\bar{\eta}/2]\exp[-b\bar{\eta}]\sin b\bar{\eta}$$

$$dG_1/d\bar{\eta}=[bB_2-2b^4A_2\bar{\eta}^2-2b^3(a_0-A_2)\bar{\eta}+b(a_0-A_2)\bar{\eta}/2-a_0/2]\exp[-b\bar{\eta}]\cos b\bar{\eta}$$
$$+[-bB_2+2b^4a_0\bar{\eta}^2-2b^3(a_0+A_2)\bar{\eta}+b(a_0+A_2)\bar{\eta}/2-A_2/2]\exp[-b\bar{\eta}]\sin b\bar{\eta}$$

$$d^2G_1/d\bar{\eta}^2=[-2b^2B_2+2b^5(a_0+A_2)\bar{\eta}^2-8b^4A_2\bar{\eta}+b^2A_2\bar{\eta}-2b^3(a_0-A_2)$$
$$+b(a_0-A_2)]\exp[-b\bar{\eta}]\cos b\bar{\eta}+[2b^5(-a_0+A_2)\bar{\eta}^2+8b^4a_0\bar{\eta}$$
$$-b^2a_0\bar{\eta}-2b^3(a_0+A_2)+b(a_0+A_2)]\exp[-b\bar{\eta}]\sin b\bar{\eta}$$

$$d^3G_1/d\bar{\eta}^3=[2b^3B_2-4b^6a_0\bar{\eta}^2+12b^5(a_0+A_2)\bar{\eta}-b^3(a_0+A_2)\bar{\eta}-12b^4A_2$$
$$+3b^2A_2]\exp[-b\bar{\eta}]\cos b\bar{\eta}+[2b^3B_2-4b^6A_2\bar{\eta}^2+12b^5(-a_0+A_2)\bar{\eta}$$
$$+b^3(a_0-A_2)\bar{\eta}+12b^4a_0-3b^2a_0]\exp[-b\bar{\eta}]\sin b\bar{\eta}$$

$$\Omega_1=[-bB_2+2b^4a_0\bar{\eta}^2-2b^3(a_0+A_2)\bar{\eta}-b(a_0+A_2)\bar{\eta}/2-A_2/2]\exp[-b\bar{\eta}]\cos b\bar{\eta}$$
$$+[-bB_2+2b^4A_2\bar{\eta}^2-2b^3(-a_0+A_2)\bar{\eta}-b(-a_0+A_2)\bar{\eta}/2+a_0/2]$$
$$\cdot\exp[-b\bar{\eta}]\sin b\bar{\eta}$$

式中 $B_2=a_0/2b$

$$d\Omega_1/d\bar{\eta}=[2b^5(-a_0+A_2)\bar{\eta}^2+8b^4a_0\bar{\eta}+b^2a_0\bar{\eta}-2b^3(a_0+A_2)]\exp[-b\bar{\eta}]\cos b\bar{\eta}$$
$$+[2b^2B_2-2b^5(a_0+A_2)\bar{\eta}^2+8b^4A_2\bar{\eta}+b^2A_2\bar{\eta}-2b^3(-a_0+A_2)]\exp[-b\bar{\eta}]\sin b\bar{\eta}$$

$$d^2\Omega_1/d\bar{\eta}^2=[2b^3B_2-4b^6A_2\bar{\eta}^2+12b^5(-a_0+A_2)\bar{\eta}+b^3(-a_0+A_2)\bar{\eta}+12b^4a_0+b^2a_0]$$
$$\cdot\exp[-b\bar{\eta}]\cos b\bar{\eta}+[-2b^3B_2+4b^6a_0\bar{\eta}^2-12b^5(a_0+A_2)\bar{\eta}$$
$$-b^3(a_0+A_2)\bar{\eta}+12b^4A_2+b^2A_2]\exp[-b\bar{\eta}]\sin b\bar{\eta}$$

微分方程(5.2)的解为

$$W=G_0+\varepsilon G_1+O(\varepsilon^2),\qquad Q=\Omega_0+\varepsilon\Omega_1+O(\varepsilon^2)$$

微分方程(5.1)的解为

$$w=t(G_0+\varepsilon G_1),\qquad Q_1=\sqrt{\frac{t}{\sqrt{12(1-\mu^2)}\,R}}\frac{Et^2}{R\operatorname{tg}a}(\Omega_0+\varepsilon\Omega_1)$$

内力公式

$$M_1=-\frac{D}{R^2}\frac{\cos^2(a''-\psi)}{\varepsilon^2\sin^2a''}t\left(\frac{d^2G_0}{d\bar{\eta}^2}+\varepsilon\frac{d^2G_1}{d\bar{\eta}^2}\right)+\frac{D}{R^2}\frac{\sin(a''-\psi)}{\varepsilon\sin a''}t\left(\frac{dG_0}{d\bar{\eta}}+\varepsilon\frac{dG_1}{d\bar{\eta}}\right)$$
$$-\mu\frac{D}{R^2}\operatorname{ctg}(a''-\psi)\frac{\cos(a''-\psi)}{\varepsilon\sin a''}t\left(\frac{dG_0}{d\bar{\eta}}+\varepsilon\frac{dG_1}{d\bar{\eta}}\right)$$

$$M_2=-\frac{D}{R^2}\operatorname{ctg}(a''-\psi)\frac{\cos(a''-\psi)}{\varepsilon\sin a''}t\left(\frac{dG_0}{d\bar{\eta}}+\varepsilon\frac{dG_1}{d\bar{\eta}}\right)-\mu\frac{D^2}{R^2}\frac{\cos^2(a''-\psi)}{\varepsilon^2\sin^2a''}t\left(\frac{d^2G_0}{d\bar{\eta}^2}\right.$$
$$\left.+\varepsilon\frac{d^2G_1}{d\bar{\eta}^2}\right)+\mu\frac{D}{R^2}\frac{\sin(a''-\psi)}{\varepsilon\sin a''}t\left(\frac{dG_0}{d\bar{\eta}}+\varepsilon\frac{dG_1}{d\bar{\eta}}\right)$$

$$N_1=Q_1\operatorname{ctg}a+N_1^*$$

$$N_2=\frac{\cos(a''-\psi)}{\varepsilon\sin a''}\sqrt{\frac{t}{\sqrt{12(1-\mu^2)}\,R}}\frac{Et^2}{R\operatorname{tg}(a''-\psi)}\left(\frac{d\Omega_0}{d\bar{\eta}}+\varepsilon\frac{d\Omega_1}{d\bar{\eta}}\right)$$

$$-\sqrt{\frac{t}{\sqrt{12(1-\mu^2)}\,R}}\frac{Et^2}{R\sin^2(\alpha''-\psi)}(\Omega_0+\varepsilon\Omega_1)+N_2^*$$

算例 固定边球壳受法向均布荷载作用，求边界处($\psi=0$或$\bar{\eta}=0$)的内力。

令 $\alpha''=50^\circ$，$t=0.2\text{m}$，$R=12\text{m}$

混凝土 $E=2.85\times10^7\text{kN/m}^2$，$\mu=1/6$，$a=1/\sin\alpha''=1.3054072$

$b=\sqrt{1/2(a^2-1)}=0.842697$， $\varepsilon=\sqrt{t/\sqrt{12(1-\mu^2)}\,R}=0.0698534$

$$D=\frac{Et^3}{12(1-\mu^2)}=19542.857,\qquad N_1^*=N_2^*=-\frac{q_0R}{2}=-6q_0$$

$$w^*=-\frac{(1-\mu)q_0R^2}{2Et}$$

$$\alpha_0=-\frac{1}{t}(w^*)_{\alpha=\alpha''}=\frac{(1-\mu)q_0R^2}{2Et^2}=526.31578\times10^{-7}q_0$$

$$\beta=-\left(\frac{dw^*}{d\alpha}\right)_{\alpha=\alpha''}=0,\qquad 故\beta_0=0$$

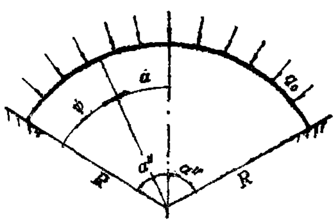

图 1

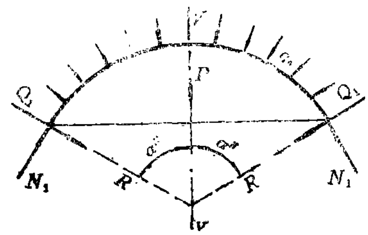

图 2

表1 法向均布荷载q_0作用下边界处内力值

内力名称 内力值	M_1	M_2	Q_1	N_1	N_2
摄 动 解	$0.6150488q_0$	$0.1025081q_0$	$-0.51441q_0$	$-6.4316412q_0$	$-0.5439619q_0$

校核平衡条件

$$P=\int_0^{2\pi}\int_0^{\alpha''}q_0\cos\alpha Rd\alpha R\sin\alpha d\theta=\pi q_0R^2\sin^2\alpha''$$

N_1的投影 $N_1 2\pi R\sin^2\alpha''$

Q_1的投影 $Q_1 2\pi R\sin\alpha''\cos\alpha''$

$$\Sigma V=\pi q_0R^2\sin^2\alpha''+Q_1 2\pi R\sin\alpha''\cos\alpha''-N_1 2\pi R\sin^2\alpha''$$
$$=284.571129q_0-284.57113q_0=0$$

摄动解求出的内力满足平衡条件。

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Refined Differential Equations of Deflections in Axial Symmetrical Bending Problems of Spherical Shell and Their Singular Perturbation Solutions

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Abstract

This paper deals with the research of accuracy of differential equations of deflections. The basic idea is as follows. Firstly, considering the boundary effect the meridian midsurface displacement $u=0$, thus we derive the deflection differential equations; secondly we accurately prove that by use of the deflection differential equations or the original differential equations the same inner forces solutions are obtained; finally, we accurately prove that considering the boundary effect the meridian surface displacement $u=0$ is an exact solation. In this paper we give the singular perturbation solution of the deflection differential equations. Finally we check the equilibrium condition and prove the inner forces solved by perturbation method and the outer load are fully equilibrated. It shows that perturbation solution is accurate. On the other hand, it shows again that the deflection differential equation is an exact equation.

The features of the new differential equations are as follows.

1. The accuracies of the new differential equations and the original differential equations are the same.

2. The new differential equations can satisfy the boundary conditions simply.

3. It is advantageous to use perturbation method with the new differential equations.

4. We may obtain the deflection expression (w) and slope expression (dw/da) by using the new differential equations.

The new differential equations greatly simplify the calculation of spherical shell. The notation adopted in this paper is the same as that in Ref. [1].

Key words spherical shell, differential equation of deflections, singular perturbation solution