奇摄动向量问题的边界层和内层现象*

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摘 要

本文考虑非线性向量边值问题：

$$\varepsilon y''=f(x,\ y,\ z,\ y',\ \varepsilon),\quad y(0)=A_1,\quad y(1)=B_1$$
$$\varepsilon z''=g(x,\ y,\ z,\ z',\ \varepsilon),\quad z(0)=A_2,\quad z(1)=B_2$$

其中 ε 是正的小参数，$0\leqslant x\leqslant 1$，f，g是 R^4 中的连续函数．在适当的假设下，利用微分不等式理论，我们证明了上述问题的解的存在性，并得到包括边界层和内层在内的解的估计．

关键词 奇摄动 微分不等式 边界层 内层

至今，标量问题和向量问题已被不少作者[1~6]给以不同程度的研究．本文利用文[7]的方法，借助微分不等式[8]、[9]的理论来考虑如下形式的边值问题：

$$\varepsilon y''=f(x,y,z,y',\varepsilon)\qquad (0<x<1) \tag{1}$$

$$\varepsilon z''=g(x,y,z,z',\varepsilon) \tag{2}$$

$$y(0,\varepsilon)=A_1,\ y(1,\varepsilon)=B_1 \tag{3}$$

$$z(0,\varepsilon)=A_2,\ z(1,\varepsilon)=B_2 \tag{4}$$

其中 ε是正的小参数，f,g是R^4中的连续函数，且f关于$(y,z,y')\in R^3$和g关于$(y,z,z')\in R^3$是C^1类的．对于上述问题，本文所用的方法可以类似地推广到高维系统．为了研究边值问题(1)～(4)，需要引入退化问题：

$$f(x,u,v,u',\ 0)=0 \tag{5}$$

$$g(x,u,v,v',0)=0 \tag{6}$$

首先考虑两个分量都在$x=1$出现边界层的情形．

定理1 假设：[Ⅰ] 退化问题(5)，(6)有满足 $u(0)=A_1$，$v(0)=A_2$的$C^2[0,1]$类解偶$(u(x),v(x))$；[Ⅱ] $f(x,y,z,y',\varepsilon)$在区域D上连续，且关于(y,z,y')是C^1类的，而$g(x,y,z,z',\varepsilon)$在区域E上连续，关于(y,z,z')是C^1类的，其中

$$D=\{(x,y,z,y',\varepsilon):\ 0\leqslant x\leqslant 1,\ |y-u(x)|\leqslant c(x),\ |z-v(x)|\leqslant d(x),\ |y'|<\infty,\ 0\leqslant\varepsilon\leqslant\varepsilon_1\};$$

$$E=\{(x,y,z,z',\varepsilon):0\leqslant x\leqslant 1,\ |y-u(x)|\leqslant c(x),\ |z-v(x)|\leqslant d(x),\ |z'|<\infty,\ 0\leqslant\varepsilon\leqslant\varepsilon_1\},$$

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ε_1是正的小参数，$c(x)$，$d(x)$是光滑的正函数，且满足：

$$c(x)=|B_1-u(1)|+\delta\quad(1-\frac{\delta}{2}\leqslant x\leqslant 1);\quad c(x)=\delta\quad(0\leqslant x\leqslant 1-\delta),$$

和
$$d(x)=|B_2-v(1)|+\delta\quad(1-\frac{\delta}{2}\leqslant x\leqslant 1);\quad d(x)=\delta\quad(0\leqslant x\leqslant 1-\delta);$$

［Ⅲ］ 在区域D，E中，$f_{y'}\geqslant k_1>0$，$g_{z'}\geqslant k_2>0$ （其中k_1，k_2是正常数）；

［Ⅳ］ $f(x,u,v,u',\varepsilon)=O(\varepsilon)$，$g(x,u,v,v',\varepsilon)=O(\varepsilon)$

［Ⅴ］ 在区域D和E中，恒有

$$|f_y|\leqslant N_1^1,\quad |f_z|\leqslant N_1^2;\quad |g_y|\leqslant N_2^1,\quad |g_z|\leqslant N_2^2,$$

N_1^1，N_1^2，N_2^1和N_2^2是给定正常数；

［Ⅵ］ $f(x,y,z,y',\varepsilon)$和$g(x,y,z,z',\varepsilon)$都满足 Nagumo 条件，

则对$0<\varepsilon\ll 1$，边值问题(1)～(4)在[0,1]上至少存在一组解$(y(x,\varepsilon)$，$z(x,\varepsilon))$，且满足

$$|y(x,\varepsilon)-u(x)|\leqslant\eta\left(\frac{1-x}{\varepsilon}\right)+O(\varepsilon)\tag{7}$$

$$(0\leqslant x\leqslant 1)$$

$$|z(x,\varepsilon)-v(x)|\leqslant\Gamma\left(\frac{1-x}{\varepsilon}\right)+O(\varepsilon)\tag{8}$$

其中 η，Γ是在下文确定的边界层函数。

证明 不妨设$B_1>u(1)$，$B_2>v(1)$，为了利用微分不等式理论，先对y构造函数偶(α,β)：

$$\alpha(x,\varepsilon)=u(x)-\eta\left(\frac{1-x}{\varepsilon}\right)-\varphi\left(\frac{x}{\varepsilon}\right)-c\varepsilon\exp[\delta(x-1)/\varepsilon]$$
$$-E\varepsilon\exp[\lambda(x-1)]\tag{9}$$

$$\beta(x,\varepsilon)=u(x)+\eta\left(\frac{1-x}{\varepsilon}\right)+\varphi\left(\frac{x}{\varepsilon}\right)+c\varepsilon\exp[\delta(x-1)/\varepsilon]$$
$$+E\varepsilon\exp[\lambda(x-1)]\tag{10}$$

对于z，我们假设：

$$v(x)-\Gamma\left(\frac{1-x}{\varepsilon}\right)-D\varepsilon\exp[\lambda(x-1)]-O(\varepsilon)\leqslant z$$
$$\leqslant v(x)+\Gamma\left(\frac{1-x}{\varepsilon}\right)+D\varepsilon\exp[\lambda(x-1)]+O(\varepsilon)\tag{11}$$

其中 c，E，D，δ和λ是待定正常数，φ，ψ是正的$O(\varepsilon)$函数，η，Γ是满足方程：

$$\varepsilon\eta''-k_1\eta'=-\gamma\eta',\qquad \eta(0)=B_1-u(1)\tag{12}$$

和
$$\varepsilon\Gamma''-k_2\Gamma'=-\rho\Gamma',\qquad \Gamma(0)=B_2-v(1)\tag{13}$$

的单调增的正的边界层函数，γ，ρ是小的正常数．下证α，β是方程(1)的下解和上解．

$$\varepsilon\beta''-f(x,\beta,z,\beta',\varepsilon)=\varepsilon u''+\varepsilon\eta''+\delta^2 c\exp[\delta(x-1)/\varepsilon]$$
$$+\varepsilon\varphi''+\varepsilon^2\lambda^2E\exp[\lambda(x-1)]-f(x,u,v,u',\varepsilon)$$
$$-f_y[x](\beta-u)-f_z[x](z-v)-f_{y'}[x](\beta'-u'),\tag{14}$$

其中 $[x]=(x,u+\theta(\beta-u),\ v+\theta(z-v),\ u'+\theta(\beta'-u'),\ \varepsilon)\quad(0<\theta<1)$

从假设［Ⅰ］，［Ⅳ］知，存在$M_1>0$，$\delta_1>0$，使得：

$$|u''|\leqslant M_1,\quad |f(x,u,v,u',\varepsilon)|\leqslant\delta_1\varepsilon。$$

再从假设[Ⅲ]，[Ⅴ]可得(14)右端小于等于：

$$\varepsilon(M_1+\delta_1)+O(\varepsilon)+N_1^1\varphi+\varepsilon\eta''-k_1\eta'+N_1^1\eta$$
$$+c\exp[\delta(x-1)/\varepsilon](\delta^2+N_1^1\varepsilon-k_1\delta)+\varepsilon\varphi''-k_1\varphi'+N_1^2\Gamma$$
$$+\varepsilon\exp[\lambda(x-1)](E\lambda^2\varepsilon+EN_1^1+N_1^2D-k_1E\lambda) \tag{15}$$

其中φ定义为：

$$\varphi\left(\frac{x}{\varepsilon}\right)=\varepsilon\int_0^{x/\varepsilon}\exp[k_1\tau]\int_\tau^\infty\exp[-k_1s]N_1^2\Gamma(s)dsd\tau \tag{16}$$

通过直接计算可知，φ 满足方程 $\varepsilon\varphi''-k_1\varphi'+N_1^2\Gamma=0$，且$\varphi$，$\varphi'>0$，$\varphi x/\varepsilon=O(\varepsilon)$。取 $\delta=k_1/2$，利用(12)式，恒有(15)式不大于

$$-\gamma\eta'+N_1^1\eta+\varepsilon(M_1+\delta_1)+O(\varepsilon)+N_1^1\varphi$$
$$+\varepsilon\exp[\lambda(x-1)](E\lambda^2\varepsilon+EN_1^1+N_1^2D-kE\lambda) \tag{17}$$

由于在边界层内部，$-\gamma\eta'<0$是主要项(因为$\eta'>0$，$\eta=O(\varepsilon\eta')$)，而在外部层，只要取D，E为适当的正数，λ充分大，就有

$$\varepsilon(M_1+\delta_1)+O(\varepsilon)+N_1^1\varphi+\varepsilon\exp[\lambda(x-1)](E\lambda^2\varepsilon$$
$$+EN_1^1+N_1^2D-k_1E\lambda)\leqslant 0\qquad(0\leqslant x\leqslant 1).$$

因此，存在充分大的$\lambda_0>0$以及足够小的$\varepsilon_0>0$，使得当$\lambda>\lambda_0$时，恒有(17)式小于等于零，即β是方程(1)的上解；类似地可证α是方程(1)的下解。又通过直接计算可知：

$$\alpha(0)\leqslant A_1\leqslant\beta(0),\quad \alpha(1)\leqslant B_1\leqslant\beta(1).$$

现在假设

$$u(x)-\eta\left(\frac{1-x}{\varepsilon}\right)-c\varepsilon\exp[\delta(x-1)/\varepsilon]-\varphi(x/\varepsilon)-E\varepsilon\exp[\lambda(x-1)]\leqslant y$$

$$\leqslant u(x)+\eta\left(\frac{1-x}{\varepsilon}\right)+c\varepsilon\exp[\delta(x-1)/\varepsilon]+\varphi(x/\varepsilon)+E\varepsilon\exp[\lambda(x-1)],$$

对z构造函数偶$(\bar{\alpha}(x,\varepsilon),\ \bar{\beta}(x,\varepsilon))$：

$$\bar{\alpha}(x,\varepsilon)=v(x)-\Gamma\left(\frac{1-x}{\varepsilon}\right)-D\varepsilon\exp[\lambda(x-1)]-\psi\left(\frac{x}{\varepsilon}\right)$$
$$-F\varepsilon\exp[\delta(x-1)/\varepsilon],$$

$$\bar{\beta}(x,\varepsilon)=v(x)+\Gamma\left(\frac{1-x}{\varepsilon}\right)+D\varepsilon\exp[\lambda(x-1)]+\psi\left(\frac{x}{\varepsilon}\right)$$
$$+F\varepsilon\exp[\delta(x-1)/\varepsilon],$$

其中

$$\psi\left(\frac{x}{\varepsilon}\right)=\varepsilon\int_0^{x/\varepsilon}\exp[k_2\tau]\int_\tau^\infty\exp[-k_2s]N_2^2\eta(s)dsd\tau \tag{18}$$

可以证明$\bar{\alpha}$，$\bar{\beta}$是方程(2)的下解和上解，借助微分不等式[8],[9]的理论知，定理结论成立。

定理1讨论了两个分量在同一端点出现边界层的情形，下述定理给出两个分量在不同点出现边界层的问题。

定理 2 假设：[Ⅰ] 退化问题(5)，(6)有满足$u(0)=A_1$，$v(1)=B_2$ 的$C^2[0,1]$类解偶$(u(x),\ v(x))$；

[Ⅱ] f，g满足定理1假设[Ⅱ]的条件，这里取$d(x)$为：

$$d(x)=|A_2-v(0)|+\delta \qquad (0\leqslant x\leqslant \frac{\delta}{2});\quad d(x)=\delta \qquad (\delta\leqslant x\leqslant 1);$$

[Ⅱ] $f_{y'}\geqslant k_1>0 \qquad (x,y,z,y')\in D,$

$g_{z'}\leqslant -k_2<0 \qquad (x,y,z,z')\in E,$

则在定理1假设条件[Ⅳ]～[Ⅵ]下，对$0<\varepsilon\ll 1$，边值问题(1)～(4)在[0,1]上存在一组满足：

$$|y(x,\varepsilon)-u(x)|\leqslant \eta\left(\frac{1-x}{\varepsilon}\right)+O(\varepsilon),$$
$$(0\leqslant x\leqslant 1)$$
$$|z(x,\varepsilon)-v(x)|\leqslant \Gamma\left(\frac{x}{\varepsilon}\right)+O(\varepsilon)$$

的解$(y(x,\varepsilon),\ z(x,\varepsilon))$。$\eta$，$\Gamma$是待定的边界层函数。

证明．我们只考虑$A_2>v(0)$，$B_1>u(1)$情形，对y构造函数偶$(\alpha(x,\varepsilon),\ \beta(x,\varepsilon))$：

$$\alpha(x,\varepsilon)=u(x)-\eta\left(\frac{1-x}{\varepsilon}\right)-\varphi\left(\frac{x}{\varepsilon}\right)-E\varepsilon\exp[\lambda(x-1)] \tag{19}$$

$$\beta(x,\varepsilon)=u(x)+\eta\left(\frac{1-x}{\varepsilon}\right)+\varphi\left(\frac{x}{\varepsilon}\right)+E\varepsilon\exp[\lambda(x-1)] \tag{20}$$

为了证明α，β是方程(1)的下解和上解，我们假设：

$$v(x)-\Gamma\left(\frac{x}{\varepsilon}\right)-D\varepsilon\exp[\lambda(1-x)]-O(\varepsilon)\leqslant z$$

$$\leqslant v(x)+\Gamma\left(\frac{x}{\varepsilon}\right)+D\varepsilon\exp[\lambda(1-x)]+O(\varepsilon), \tag{21}$$

其中 D，E，λ是待定正常数，η，Γ是边界层函数，且满足方程：

$$\varepsilon\eta''-k_1\eta'=-\gamma\eta',\quad \eta(0)=B_1-u(1) \tag{22}$$

$$\varepsilon\Gamma''+k_2\Gamma'=\rho\Gamma',\quad \Gamma(0)=A_2-v(0) \tag{23}$$

γ，ρ是正的小常数，$\varphi(x/\varepsilon)$满足(16)式．现在我们计算：

$$\begin{aligned}\varepsilon\alpha''-f(x,\alpha,z,\alpha',\varepsilon)=&\varepsilon u''-\varepsilon\eta''-\varepsilon\varphi''-E\lambda^2\varepsilon^2\exp[\lambda(x-1)]\\&-f(x,u,v,u',\varepsilon)-f_y[\bar{x}](\alpha-u)\\&-f_z[\bar{x}](z-v)-f_{y'}[\bar{x}](\alpha'-u'),\end{aligned} \tag{24}$$

其中 $[\bar{x}]=(x,u+\theta(\alpha-u),\ v+\theta(z-v),\ u'+\theta(\alpha'-u'),\ \varepsilon)(0<\theta<1)$

利用定理1的记号，从假设[Ⅰ]，[Ⅱ]～[Ⅴ]可证，(24)右端不小于

$$\begin{aligned}&-\varepsilon(M_1+\delta_1)-N_1^2\varphi+O(\varepsilon)-\varepsilon\eta''+k_1\eta'-N_1^1\eta+\varepsilon\exp[\lambda(x-1)]\\&\cdot(-E\lambda^2\varepsilon-EN_1^1+k_1\lambda E-DN_1^2\exp[2\lambda(1-x)])\end{aligned} \tag{25}$$

类似定理1的分析，由假设知，可取某正数D和适当大的正数λ，E，使得：

$$\varepsilon\alpha''-f(x,\alpha,z,\alpha',\varepsilon)\geqslant 0,$$

同理可证

$$\varepsilon\beta''-f(x,\beta,z,\beta',\varepsilon)\leqslant 0.$$

类似地，对z构造函数$\bar{\alpha}$，$\bar{\beta}$：

$$\bar{\alpha}(x,\varepsilon)=v(x)-\Gamma\left(\frac{x}{\varepsilon}\right)-\psi\left(\frac{1-x}{\varepsilon}\right)-D\varepsilon\exp[\lambda(1-x)],$$

$$\bar{\beta}(x,\varepsilon)=v(x)+\Gamma\left(\frac{x}{\varepsilon}\right)+\psi\left(\frac{1-x}{\varepsilon}\right)+D\varepsilon\exp[\lambda(1-x)],$$

其中ψ满足(18)式．我们通过假设

$$u(x)-\eta\left(\frac{1-x}{\varepsilon}\right)-\varphi\left(\frac{x}{\varepsilon}\right)-E\varepsilon\exp[\lambda(1-x)]\leqslant y$$

$$\leqslant u(x)+\eta\left(\frac{1-x}{\varepsilon}\right)+\varphi\left(\frac{x}{\varepsilon}\right)+E\varepsilon\exp[\lambda(1-x)],$$

可以证明$\bar{\alpha}$，$\bar{\beta}$分别是方程(2)的下解和上解．因此，即得定理结论成立．

定理1和2研究了两种形式的边界层问题，至于分量y，z分别在x_1，x_2点（$0<x_1,x_2<1$）关于退化解出现角层现象的情形，我们有下面的结果．

定理 3　假设：

［Ⅰ］　退化问题(5),(6)有一组解$(u(x),\ v(x))$：$u(x)\in C[0,1]\cap C^2([0,1]\backslash\{x_1\})$，$v(x)\in C[0,1]\cap C^2([0,1]\backslash\{x_2\})$满足：$u(1)=B_1$，$v(0)=A_2$，且$u_L(x_1)<u_R(x_1)$，$v_L(x_2)>v_R(x_2)$；

［Ⅱ］　f,g 满足定理1假设［Ⅱ］的条件，而$c(x)$取为：

$$c(x)=|A_1-u(0)|+\delta\ \left(0\leqslant x\leqslant\frac{\delta}{2}\right),\ c(x)=\delta\ \left(\delta\leqslant x\leqslant 1\right);$$

［Ⅲ］　在区域D，E中，有

$$f_{y'}\leqslant -k_1<0,\ \ g_{z'}\geqslant k_2>0;$$

则在定理1假设条件［Ⅳ］～［Ⅵ］下，对$0<\varepsilon\ll 1$，边值问题(1)～(4)存在一组解$(y(x,\varepsilon),z(x,\varepsilon))$，满足：

$$\left.\begin{aligned}&|y(x,\varepsilon)-u(x)|\leqslant\Gamma\left(\frac{x}{\varepsilon}\right)+\xi\left(\frac{x-x_1}{\varepsilon}\right)+O(\varepsilon)\\&|z(x,\varepsilon)-v(x)|\leqslant\eta\left(\frac{1-x}{\varepsilon}\right)+\zeta\left(\frac{x-x_2}{\varepsilon}\right)+O(\varepsilon)\end{aligned}\right\}\ (0\leqslant x\leqslant 1)\qquad(26)$$

其中　Γ，η和ξ，ζ是分别在下文确定的正的边界层函数和角层函数．

证明　考虑$A_1>u(0)$，$B_2>v(1)$情形，我们在假设：

$$v(x)-\eta\left(\frac{1-x}{\varepsilon}\right)-\zeta\left(\frac{x-x_2}{\varepsilon}\right)-\varphi\left(\frac{x}{\varepsilon}\right)-E\varepsilon\exp[\lambda x]\leqslant z$$

$$\leqslant v(x)+\eta\left(\frac{1-x}{\varepsilon}\right)+\zeta\left(\frac{x-x_2}{\varepsilon}\right)+\varphi\left(\frac{x}{\varepsilon}\right)+E\varepsilon\exp[\lambda x]\qquad(27)$$

的条件下，对y构造函数偶(α,β)：

$$\alpha(x,\varepsilon)=u(x)-\Gamma\left(\frac{x}{\varepsilon}\right)-\xi\left(\frac{x-x_1}{\varepsilon}\right)-\psi\left(\frac{1-x}{\varepsilon}\right)-D\varepsilon\exp[-\lambda x]\qquad(28)$$

$$\beta(x,\varepsilon)=u(x)+\Gamma\left(\frac{x}{\varepsilon}\right)+\xi\left(\frac{x-x_1}{\varepsilon}\right)+\psi\left(\frac{1-x}{\varepsilon}\right)+D\varepsilon\exp[-\lambda x]\qquad(29)$$

其中　D，E，λ是待定正常数，Γ，η是边界层型函数，且满足方程：

$$\varepsilon\Gamma''+k_1\Gamma'=\sigma_1\Gamma',\ \ \Gamma(0)=A_1-u(0)\qquad(30)$$

$$\varepsilon\eta''-k_2\eta'=-\sigma_3\eta',\ \ \eta(0)=B_2-v(1)\qquad(31)$$

σ_1，σ_3是小的正常数，而ξ，ζ是满足方程：

$$\varepsilon\xi''+k_1\xi'=\sigma_2\xi' \quad (x_1<x\leqslant 1);\ \xi'(0)=u_L(x_1)-u_R(x_1),$$

$$\xi\left(\frac{x-x_1}{\varepsilon}\right)=\varepsilon\frac{u_L(x_1)-u_R(x_1)}{\sigma_2-k_1} \qquad (0\leqslant x\leqslant x_1) \tag{32}$$

和

$$\varepsilon\zeta''-k_2\zeta'=-\sigma_4\zeta' \qquad (0\leqslant x<x_2),\ \zeta'(0)=v_R(x_2)-v_L(x_2),$$

$$\zeta\left(\frac{x-x_2}{\varepsilon}\right)=\varepsilon\frac{v_R(x_2)-v_L(x_2)}{k_2-\sigma_4} \qquad (x_2\leqslant x\leqslant 1) \tag{33}$$

的角层性质函数，σ_2，σ_4是小的正数，且φ，ψ定义如下：

$$\varphi\left(\frac{x}{\varepsilon}\right)=\varepsilon\int_0^{x/\varepsilon}\exp[k_2\tau]\int_\tau^\infty\exp[-k_2 s]N_2^2\Gamma(s)dsd\tau \tag{34}$$

$$\psi\left(\frac{1-x}{\varepsilon}\right)=\varepsilon\int_0^{\frac{1-x}{\varepsilon}}\exp[k_1\tau]\int_\tau^\infty\exp[-k_1 s]N_1^2\eta(s)dsd\tau \tag{35}$$

从上述函数构造，不难验证：

$$\Gamma>0,\ \Gamma'<0;\ \eta>0,\ \eta'>0,$$

而 $\xi>0$，$\xi'\leqslant 0$，$\xi=O(\varepsilon)$；$\zeta>0$，$\zeta'\geqslant 0$，$\zeta=O(\varepsilon)$，

和 $\varphi>0$，$\varphi'>0$，$\varphi=O(\varepsilon)$；$\psi>0$，$\psi'<0$，$\psi=O(\varepsilon)$，

且φ，ψ分别满足方程：

$$\varepsilon\varphi''-k_2\varphi'+N_2^2\Gamma=0,\quad \varepsilon\psi''+k_1\psi'+N_1^2\eta=0.$$

因而，借助定理1的记号，由假设[Ⅰ]，[Ⅲ]～[Ⅴ]知，恒有

$$\begin{aligned}
&\varepsilon\alpha''-f(x,\alpha,z,\alpha',\varepsilon)=\varepsilon\alpha''-f(x,u,v,u',\varepsilon)\\
&\quad -f_y[\bar{x}](\alpha-u)-f_z[\bar{x}](z-v)-f_{y'}[\bar{x}](\alpha'-u')\\
&\geqslant -\varepsilon M_1-\varepsilon\delta_1-N_1^2(\zeta+\varphi)-N_1^2(\psi+\xi)\\
&\quad -\varepsilon\Gamma''-k_1\Gamma'-N_1^2\Gamma-\varepsilon\xi''-k_1\xi'-\varepsilon\psi''-k_1\psi'-N_1^2\eta\\
&\quad +\varepsilon\exp[-\lambda x](-D\varepsilon\lambda^2-N_1^2D+k_1D\lambda-N_1^2E\exp[2\lambda x]\\
&=-\sigma_1\Gamma'-N_1^2\Gamma-\sigma_2\xi'+\varepsilon\exp[-\lambda x](-D\varepsilon\lambda^2-N_1^2D\\
&\quad +k_1D\lambda-N_1^2E\exp[2\lambda x])+O(\varepsilon)\\
&\geqslant -2\sigma_1\Gamma'+\varepsilon\exp[-\lambda x](-D\varepsilon\lambda^2-N_1^2D\\
&\quad +k_1D\lambda-N_1^2E\exp[2\lambda x])+O(\varepsilon)
\end{aligned} \tag{36}$$

上述表达式中，$O(\varepsilon)$与D，E，λ无关．由于在边界层内部，$-\sigma_1\Gamma'>0$是主要项($\Gamma=O(\varepsilon\Gamma')$)，而在外部层，取$E$为某正数，则当$D$，$\lambda$取得适当大时，就有

$$\exp[-\lambda x](-D\varepsilon\lambda^2-N_1^2D+k_1D\lambda-N_1^2E\exp[2\lambda x])+O(\varepsilon)\geqslant 0 \tag{37}$$

从而，恒有(36)式右端不小于零，即α是方程(1)的下解．类似地，可以证明β是方程(1)的上解．

现在假设

$$u(x)-\Gamma\left(\frac{x}{\varepsilon}\right)-\xi\left(\frac{x-x_1}{\varepsilon}\right)-\psi\left(\frac{1-x}{\varepsilon}\right)-D\varepsilon\exp[-\lambda x]\leqslant y$$

$$\leqslant u(x)+\Gamma\left(\frac{x}{\varepsilon}\right)+\xi\left(\frac{x-x_1}{\varepsilon}\right)+\psi\left(\frac{1-x}{\varepsilon}\right)+D\varepsilon\exp[-\lambda x],$$

同样可以证明$(\bar{\alpha}(x,\varepsilon),\ \bar{\beta}(x,\varepsilon))$：

$$\bar{\alpha}(x,\varepsilon)=v(x)-\eta\left(\frac{1-x}{\varepsilon}\right)-\zeta\left(\frac{x-x_2}{\varepsilon}\right)-\varphi\left(\frac{x}{\varepsilon}\right)-E\varepsilon\exp[\lambda x],$$

$$\bar{\beta}(x,\varepsilon)=v(x)+\eta\left(\frac{1-x}{\varepsilon}\right)+\zeta\left(\frac{x-x_2}{\varepsilon}\right)+\varphi\left(\frac{x}{\varepsilon}\right)+E\varepsilon\exp[\lambda x],$$

分别是方程(2)的下解和上解，又易见

$$\alpha(x,\varepsilon)\leqslant\beta(x,\varepsilon),\quad \bar{\alpha}(x,\varepsilon)\leqslant\bar{\beta}(x,\varepsilon),\quad 0\leqslant x\leqslant 1$$

和

$$\alpha(0,\varepsilon)\leqslant A_1\leqslant\beta(0,\varepsilon),\quad \alpha(1,\varepsilon)\leqslant B_1\leqslant\beta(1,\varepsilon),$$

$$\bar{\alpha}(0,\varepsilon)\leqslant A_2\leqslant\bar{\beta}(0,\varepsilon),\quad \bar{\alpha}(1,\varepsilon)\leqslant B_2\leqslant\bar{\beta}(1,\varepsilon),$$

综上所述，借助微分不等式[8],[9]的结果知，(1)～(4)存在一组解$(y(x,\varepsilon),\ z(x,\varepsilon))$，$0\leqslant x\leqslant 1$，且(26)式成立。

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Boundary and Interior Layer Behavior for Singularly Perturbed Vector Problem

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Abstract

In this paper, we consider the vector nonlinear boundary value problem:

$$\varepsilon y''=f(x,y,z,y',\varepsilon),\quad y'(0)=A_1,\quad y(1)=B_1,$$

$$\varepsilon z''=g(x,y,z,z',\varepsilon),\quad z(0)=A_2,\quad z(1)=B_2,$$

where $\varepsilon>0$ is a small parameter, $0\leqslant x\leqslant 1$, f and g are continuous functions in R^4. Under appropriate assumptions, by means of the differential inequalities, we demonstrate the existence and estimation, involving boundary and interior layers, of the solutions to the above problem.

Key words: singular perturbation, differential inequality, boundary layer