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复合材料加筋薄壁圆锥壳体有限变形的混合型理论*

王 虎 王俊奎

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摘 要

本文利用变分原理和平均筋条刚度法，建立了在任意载荷作用下纵向和环向密加筋复合材料圆锥壳体有限变形的Donnell型理论．考虑了面板最一般的弯曲拉伸耦合关系和加筋筋条的偏心效应的影响．导出了平衡条件、边界条件和变形协调方程．给出了以应力函数和挠度函数表示的耦合形式的非性性变系数偏微分方程组．对于一些特殊情况，给出了相应的简化方程．

关键词 复合材料 圆锥壳 加筋壳 薄壳 有限变形 混合型理论

一、引 言

复合材料加筋薄壁结构是航空航天等现代工程中广泛采用的一种结构形式．这种结构具有比强度高、比刚度大、重量轻、抗疲劳性能好和很强的可设计性等优点．采用这种结构可以显著地提高面板局部失稳的临界载荷，增加结构稳定性，并且可以根据外载荷调整加筋筋条的参数．因此，对复合材料加筋薄壁结构在任意载荷作用下的应力、应变、稳定性和振动等力学问题的研究，具有重要意义．

圆锥形壳体是一种工程领域中常用的薄壁承载结构．自1937年 Pflüger[1] 最早研究圆锥形壳体稳定性以来，已对圆锥形壳体的力学性能进行了大量分析并建立了各种相应的理论[2~12]．文[8]利用变分原理所建立的位移型方程组，由于略去了中面应变和曲率表达式中的某些项，仅适用于半锥角小于30°的情况．文[9]保留了文[8]中略去的项，用同样方法导出了适用于任意锥角的圆锥形壳体弯曲和稳定性问题的Donnell型方程组．文[10]将文[9]结果推广到正交各向异性材料．文[11]在建立圆锥壳体稳定性问题的位移型方程时，考虑了壳体微元在中面内的变形和绕各个轴的转动．通过引入位移函数和广义载荷，文[12]建立了两种类型夹层锥壳的位移型统一理论．由此可见，对各向同性材料圆锥壳体的力学性能的分析已经做了大量工作，而对各向异性材料，特别是复合材料加筋圆锥壳体的力学性能的研究则要少得多．

本文利用变分原理和平均筋条刚度法，建立了在任意载荷作用下纵向和环向密加筋复合材料圆锥壳体有限变形的Donnell型理论．本文所建立的混合型理论可以作为研究在任意载

* 钱伟长推荐．

荷作用下复合材料加筋圆锥壳体许多具体力学问题的理论基础。对于一些特殊情况，给出了相应的简化方程。

二、基本假设和基本关系式

在本文理论推导过程中，作如下基本假设：

（1） 壳体很薄：$\frac{h}{R_1}\ll 1$，$\frac{h}{R_2}\ll 1$，$\frac{h}{L}\ll 1$；

（2） 应变很小：$\varepsilon\ll 1$；

（3） Kirchhoff-Love 假设和 Donnell 扁壳理论。

对于筋条再作如下简化假设：

（1） 筋条为弹性梁；

（2） 筋条与面板在径向和筋条方向位移连续；

（3） 略去筋条侧向弯曲刚度；

（4） 不考虑纵向筋条和环向筋条间变形的相互影响；

（5） 筋条等距分布在面板上，不计筋条宽度；

（6） 筋条的扭转近似为自由扭转。

考虑图 1 所示的复合材料加筋圆锥壳体，其中面应变和曲率表达式为

$$\left.\begin{aligned}
\varepsilon_s&=\frac{\partial u}{\partial s}+\frac{1}{2}\left(\frac{\partial w}{\partial s}\right)^2\\
\varepsilon_\theta&=\frac{u-w\mathrm{ctg}\alpha}{s}+\frac{1}{s\sin\alpha}\frac{\partial v}{\partial\theta}+\frac{1}{2}\left(\frac{1}{s\sin\alpha}\frac{\partial w}{\partial\theta}\right)^2\\
\gamma_{s\theta}&=\frac{\partial v}{\partial s}-\frac{v}{s}+\frac{1}{s\sin\alpha}\frac{\partial u}{\partial\theta}+\frac{1}{s\sin\alpha}\frac{\partial w}{\partial s}\frac{\partial w}{\partial\theta}
\end{aligned}\right\}\tag{2.1}$$

$$\left.\begin{aligned}
\varkappa_s&=-\frac{\partial^2 w}{\partial s^2}\\
\varkappa_\theta&=-\frac{1}{s}\frac{\partial w}{\partial s}-\frac{1}{s^2\sin^2\alpha}\cdot\frac{\partial^2 w}{\partial\theta^2}\\
\varkappa_{s\theta}&=-\frac{1}{\sin\alpha}\frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial w}{\partial\theta}\right)
\end{aligned}\right\}\tag{2.2}$$

式中 u,v,w 是圆锥壳体中面的位移分量，s 是自锥顶量起的距离，θ是环向角度。

面板应变位移关系：

$$\varepsilon_s^f=\varepsilon_s+z\varkappa_s,\quad \varepsilon_\theta^f=\varepsilon_\theta+z\varkappa_\theta,\quad \gamma_{s\theta}^f=\gamma_{s\theta}+2z\varkappa_{s\theta}\tag{2.3}$$

纵筋应变位移关系：

$$\varepsilon_s^s=\varepsilon_s+z\varkappa_s\tag{2.4}$$

横筋应变位移关系：

$$\varepsilon_\theta^\theta=\varepsilon_\theta+z\varkappa_\theta\tag{2.5}$$

面板本构方程为：

$$\begin{Bmatrix} N_s^f \\ N_\theta^f \\ N_{s\theta}^f \\ \hdashline M_s^f \\ M_\theta^f \\ M_{s\theta}^f \end{Bmatrix} = \left[\begin{array}{ccc:ccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ \hdashline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{array}\right] \begin{Bmatrix} \varepsilon_s \\ \varepsilon_\theta \\ \gamma_{s\theta} \\ \hdashline \varkappa_s \\ \varkappa_\theta \\ 2\varkappa_{s\theta} \end{Bmatrix} \tag{2.6}$$

式中各内力素定义为:

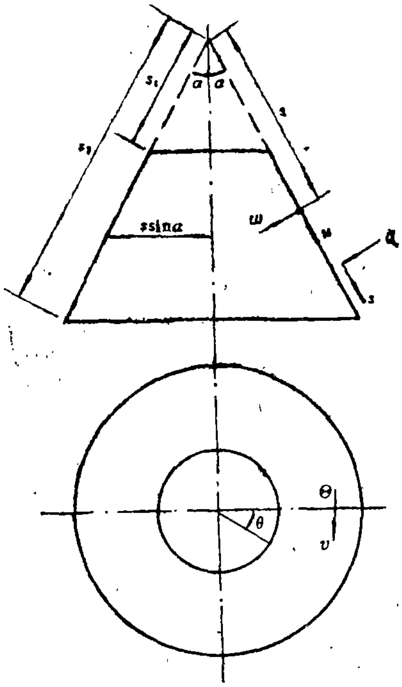

图1　几何尺寸和载荷情况

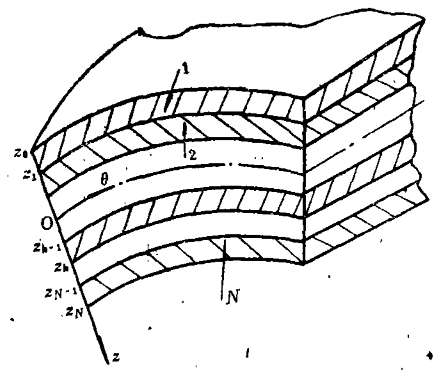

图2　N层叠层壳的单元体

$$\left.\begin{aligned} \{N_s^f,\ N_\theta^f,\ N_{s\theta}^f\} &= \int_{z_0}^{z_N} \{\sigma_s,\ \sigma_\theta,\ \tau_{s\theta}\}dz \\ \{M_s^f,\ M_\theta^f,\ M_{s\theta}^f\} &= \int_{z_0}^{z_N} \{\sigma_s,\ \sigma_\theta,\ \tau_{s\theta}\}z\,dz \end{aligned}\right\} \tag{2.7}$$

各刚度系数为:

$$\left.\begin{aligned} A_{ij} &= \sum_{k=1}^{N} (\bar{Q}_{ij})_k (z_k - z_{k-1}) \\ B_{ij} &= \sum_{k=1}^{N} (\bar{Q}_{ij})_k \frac{(z_k^2 - z_{k-1}^2)}{2} \\ D_{ij} &= \sum_{k=1}^{N} (\bar{Q}_{ij})_k \frac{(z_k^3 - z_{k-1}^3)}{3} \end{aligned}\right\} \tag{2.8}$$

式中　z_k 和 z_{k-1} 分别为第 k 层单层壳体的底坐标和顶坐标，如图 2 所示，$(\bar{Q}_{ij})_k$ 为第 k 层铺层的转换弹性刚度系数：

$$\begin{Bmatrix} \bar{Q}_{11} \\ \bar{Q}_{12} \\ \bar{Q}_{22} \\ \bar{Q}_{16} \\ \bar{Q}_{26} \\ \bar{Q}_{66} \end{Bmatrix}_k = \begin{Bmatrix} Q_{11}m^4+2(Q_{12}+2Q_{66})m^2n^2+Q_{22}n^4 \\ (Q_{11}+Q_{22}-4Q_{66})m^2n^2+Q_{12}(m^4+n^4) \\ Q_{11}n^4+2(Q_{12}+2Q_{66})m^2n^2+Q_{22}m^4 \\ (Q_{11}-Q_{12}-2Q_{66})m^3n+(Q_{12}-Q_2^2+2Q_{66})mn^3 \\ (Q_{11}-Q_{12}-2Q_{66})mn^3+(Q_{12}-Q_{22}+2Q_{66})m^3n \\ (Q_{11}+Q_{22}-2Q_{12}-2Q_{66})m^2n^2+Q_{66}(m^4+n^4) \end{Bmatrix}_k \tag{2.9}$$

式中 $m=\cos\theta, n=\sin\theta$；$\theta$是纤维与$s$轴所成夹角，$(Q_{ij})_k$是第$k$层铺层的弹性刚度系数[13]。

面板势能：

$$u_0=\frac{1}{2}\int_{s_1}^{s_2}\int_{\theta_1}^{\theta_2}(N_s^f\varepsilon_s+N_\theta^f\varepsilon_\theta+N_{s\theta}^f\gamma_{s\theta}+M_s^f\chi_s +M_\theta^f\chi_\theta+2M_{s\theta}^f\chi_{s\theta})s\sin\alpha ds d\theta \tag{2.10}$$

纵筋势能：

$$u_s=\frac{1}{2d_s}\int_{s_1}^{s_2}\int_{\theta_1}^{\theta_2}\left[\iint_{A_s}E_s\varepsilon_s^{2s}dA+G_sJ_s\chi_{s\theta}^2\right]s\sin\alpha ds d\theta$$

$$=\frac{1}{2}\int_{s_1}^{s_2}\int_{\theta_1}^{\theta_2}(N_s^s\varepsilon_s+2e_sN_s^s\chi_s+M_s^s\chi_s+M_{s\theta}^s\chi_{s\theta})s\sin\alpha ds d\theta$$

$$=\frac{1}{2}\int_{s_1}^{s_2}\int_{\theta_1}^{\theta_2}\left[\left(N_s^s+\frac{S_s}{I_s}M_s^s\right)\varepsilon_s+(M_s^s+e_sN_s^s)\chi_s +M_{s\theta}^s\chi_{s\theta}\right]s\sin\alpha ds d\theta \tag{2.11}$$

式中 d_s、E_s、A_s、I_s和e_s分别为纵筋间距、弹性模量、横截面积、对面板中面的惯性矩和偏心距，且

$$N_s^s=\frac{E_sA_s}{d_s}\varepsilon_s,\quad M_s^s=\frac{E_sI_s}{d_s}\chi_s,\quad M_{s\theta}^s=\frac{G_sJ_s}{d_s}\chi_{s\theta} \tag{2.12}$$

$$I_s=\int_{A_s}z^2dA,\quad e_s=\frac{1}{A_s}\int_{A_s}zdA=\frac{S_s}{A_s} \tag{2.13}$$

式(2.13)中S_s为纵筋对于面板中面的静矩。

横筋势能：

$$u_\theta=\frac{1}{2d_\theta}\int_{s_1}^{s_2}\int_{\theta_1}^{\theta_2}\left[\iint_{A_\theta}E_\theta\varepsilon_\theta^{2\theta}dA+G_\theta J_\theta\chi_{s\theta}^2\right]s\sin\alpha ds d\theta$$

$$=\frac{1}{2}\int_{s_1}^{s_2}\int_{\theta_1}^{\theta_2}\left[\left(N_\theta^\theta+\frac{S_\theta}{I_\theta}M_\theta^\theta\right)\varepsilon_\theta+(M_\theta^\theta+e_\theta N_\theta^\theta)\chi_\theta +M_{s\theta}^\theta\chi_{s\theta}\right]s\sin\alpha ds d\theta \tag{2.14}$$

式中

$$N_\theta^\theta=\frac{E_\theta A_\theta}{d_\theta}\varepsilon_\theta,\quad M_\theta^\theta=\frac{E_\theta I_\theta}{d_\theta}\chi_\theta,\quad M_{s\theta}^\theta=\frac{G_\theta J_\theta}{d_\theta}\chi_{s\theta} \tag{2.15}$$

式(2.14)～(2.15)中，E_θ和A_θ分别为横筋弹性模量和横截面积，G_θ和J_θ分别是横筋剪切模量和扭转惯性矩，I_θ，S_θ和e_θ分别为横筋对于面板中面的惯性矩、静矩和偏心矩，且

$$I_\theta=\int_{A_\theta}z^2dA,\quad S_\theta=\int_{A_\theta}zdA=e_\theta A_\theta \tag{2.16}$$

外载荷势能：

$$u_L=-\int_{s_1}^{s_2}\int_{\theta_1}^{\theta_2}(Su+\Theta v+qw)s\sin\alpha dsd\theta$$
$$-\int_{\theta_1}^{\theta_2}\left[\left(N_s^f u+N_{s\theta}^f v-M_s^f\frac{\partial w}{\partial s}-M_{s\theta}^f\frac{1}{s\sin\alpha}\frac{\partial w}{\partial\theta}\right.\right.$$
$$\left.\left.+\bar{Q}_s^f w\right)s\right]\Big|_{s=s_1}^{s=s_2}\sin\alpha d\theta-\int_{s_1}^{s_2}\left[N_{s\theta}^f u+N_\theta^s v-M_\theta^f\frac{1}{s\sin\alpha}\frac{\partial w}{\partial\theta}\right.$$
$$\left.-M_{s\theta}^f\frac{\partial w}{\partial s}+\bar{Q}_\theta^f w\right]\Big|_{\theta=\theta_1}^{\theta=\theta_2}ds-\int_{\theta_1}^{\theta_2}\left\{\left[\left(N_s^s+\frac{S_s}{I_s}M_s^s\right)u-(M_s^s\right.\right.$$
$$\left.\left.+e_sN_s^s)\frac{\partial w}{\partial s}-\frac{M_{s\theta}^s+M_{s\theta}^\theta}{2}\frac{1}{s\sin\alpha}\frac{\partial w}{\partial\theta}\right]s\right\}\Big|_{s=s_1}^{s=s_2}\sin\alpha d\theta$$
$$-\int_{s_1}^{s_2}\left[\left(N_\theta^\theta+\frac{S_\theta}{I_\theta}M_\theta^\theta\right)v-(M_\theta^\theta+e_\theta M_\theta^\theta)\frac{1}{s\sin\alpha}\frac{\partial w}{\partial\theta}\right.$$
$$\left.-\frac{M_{s\theta}^s+M_{s\theta}^\theta}{2}\frac{\partial w}{\partial s}\right]\Big|_{\theta=\theta_1}^{\theta=\theta_2}ds \qquad (2.17)$$

三、平衡条件、边界条件和变形协调方程

由总势能的一阶变分为零，即

$$\delta u=\delta(u_0+u_s+u_\theta+u_L)=0 \qquad (3.1)$$

得到平衡条件

$$\frac{1}{s}\frac{\partial}{\partial s}(sN_s)-\frac{N_\theta}{s}+\frac{1}{s\sin\alpha}\frac{\partial N_{s\theta}}{\partial\theta}+S=0$$
$$\frac{1}{s\sin\alpha}\frac{\partial N_\theta}{\partial\theta}+\frac{1}{s^2}\frac{\partial}{\partial s}(s^2N_{s\theta})+\Theta=0$$
$$\frac{1}{s}\frac{\partial^2(sM_s)}{\partial s}-\frac{1}{s}\frac{\partial M_\theta}{\partial s}+\frac{1}{s^2\sin^2\alpha}\frac{\partial^2M_\theta}{\partial\theta^2}+\frac{2}{s\sin\alpha}\frac{\partial^2M_{s\theta}}{\partial s\partial\theta}$$
$$+\frac{2}{s^2\sin\alpha}\frac{\partial M_{s\theta}}{\partial\theta}+\frac{1}{s}\frac{\partial}{\partial s}\left(sN_s\frac{\partial w}{\partial s}\right)$$
$$+\frac{1}{s^2\sin^2\alpha}\frac{\partial}{\partial\theta}\left(N_\theta\frac{\partial w}{\partial\theta}\right)+\frac{1}{s\sin\alpha}\frac{\partial}{\partial\theta}\left(N_{s\theta}\frac{\partial w}{\partial s}\right)$$
$$+\frac{1}{s\sin\alpha}\frac{\partial}{\partial s}\left(N_{s\theta}\frac{\partial w}{\partial\theta}\right)+\frac{N_\theta}{s}\mathrm{ctg}\alpha+q=0 \qquad (3.2)$$

式中

$$\left.\begin{aligned}&N_s=N_s^f+N_s^s+\frac{S_s}{I_s}M_s^s,\quad N_\theta=N_\theta^f+N_\theta^\theta+\frac{S_\theta}{I_\theta}M_\theta^\theta\\&N_{s\theta}=N_{s\theta}^f,\quad M_s=M_s^f+M_s^s+e_sN_s^s\\&M_\theta=M_\theta^f+M_\theta^\theta+e_\theta N_\theta^\theta,\quad M_{s\theta}=M_{s\theta}^f+\frac{1}{2}(M_{s\theta}^s+M_{s\theta}^\theta)\end{aligned}\right\} \qquad (3.3)$$

和边界条件

在 $s=s_1$ 和 $s=s_2$

$$\left.\begin{aligned}
&N_s=\bar{N}_s \quad \text{或} \quad \delta u=0\\
&N_{s\theta}=\bar{N}_{s\theta} \quad \text{或} \quad \delta v=0\\
&M_s=\bar{M}_s \quad \text{或} \quad \delta\frac{\partial w}{\partial s}=0\\
&sN_s\frac{\partial w}{\partial s}+\frac{N_{s\theta}}{\sin\alpha}\frac{\partial w}{\partial\theta}+\frac{\partial}{\partial s}(sM_s)-M_\theta+\frac{2}{\sin\alpha}\frac{\partial M_{s\theta}}{\partial\theta}\\
&\quad=s\bar{Q}_s+\frac{1}{\sin\alpha}\frac{\partial\bar{M}_{s\theta}}{\partial\theta} \quad \text{或} \quad \delta w=0
\end{aligned}\right\} \tag{3.4}$$

在 $\theta=\theta_1$ 和 $\theta=\theta_2$

$$\left.\begin{aligned}
&N_\theta=\bar{N}_\theta \quad \text{或} \quad \delta v=0\\
&N_{s\theta}=\bar{N}_{s\theta} \quad \text{或} \quad \delta u=0\\
&M_\theta=\bar{M}_\theta \quad \text{或} \quad \delta\frac{\partial w}{\partial\theta}=0\\
&\frac{N_\theta}{s\sin\alpha}\frac{\partial w}{\partial\theta}+N_{s\theta}\frac{\partial w}{\partial s}+\frac{1}{s\sin\alpha}\frac{\partial M_\theta}{\partial\theta}+\frac{2}{s}\frac{\partial(sM_{s\theta})}{\partial s}\\
&\quad=\bar{Q}_\theta+\frac{\partial\bar{M}_{s\theta}}{\partial s} \quad \text{或} \quad \delta w=0
\end{aligned}\right\} \tag{3.5}$$

在 $s=s_1$ 和 $s=s_2$ 以及 $\theta=\theta_1$ 和 $\theta=\theta_2$

$$M_{s\theta}=\bar{M}_{s\theta} \tag{3.6}$$

平衡条件(3.2)式中的第三式中的部分项可作某些简化:

$$\begin{aligned}
&\frac{1}{s}\frac{\partial}{\partial s}\left(sN_s\frac{\partial w}{\partial s}\right)+\frac{1}{s^2\sin^2\alpha}\frac{\partial}{\partial\theta}\left(N_\theta\frac{\partial w}{\partial\theta}\right)+\frac{1}{s\sin\alpha}\frac{\partial}{\partial\theta}\left(N_{s\theta}\frac{\partial w}{\partial s}\right)\\
&\quad+\frac{1}{s\sin\alpha}\frac{\partial}{\partial s}\left(N_{s\theta}\frac{\partial w}{\partial\theta}\right)\\
&=\frac{\partial w}{\partial s}\left[\frac{1}{s}\frac{\partial}{\partial s}(sN_s)+\frac{1}{s\sin\alpha}\frac{\partial N_{s\theta}}{\partial\theta}-\frac{N_{s\theta}}{s}\right]\\
&\quad+\frac{1}{s\sin\alpha}\frac{\partial w}{\partial\theta}\left[\frac{1}{s\sin\alpha}\frac{\partial N_\theta}{\partial\theta}+\frac{\partial N_{s\theta}}{\partial s}+\frac{2}{s}N_{s\theta}\right]\\
&\quad+N_s\frac{\partial^2 w}{\partial s^2}+N_\theta\left(\frac{1}{s^2\sin^2\alpha}\frac{\partial^2 w}{\partial\theta^2}+\frac{1}{s}\frac{\partial w}{\partial s}\right)\\
&\quad+2N_{s\theta}\left(\frac{1}{s\sin\alpha}\frac{\partial^2 w}{\partial s\partial\theta}-\frac{1}{s^2\sin\alpha}\frac{\partial w}{\partial\theta}\right)
\end{aligned} \tag{3.7}$$

利用平衡条件(3.2)式中的第一式和第二式，则式(3.7)的右端可表示为:

$$\begin{aligned}
&-\frac{\partial w}{\partial s}S-\frac{1}{s\sin\alpha}\frac{\partial w}{\partial\theta}\Theta+N_s\frac{\partial^2 w}{\partial s^2}+N_\theta\left(\frac{1}{s^2\sin^2\alpha}\frac{\partial^2 w}{\partial\theta^2}\right.\\
&\left.\quad+\frac{1}{s}\frac{\partial w}{\partial s}\right)+2N_{s\theta}\frac{1}{\sin\alpha}\frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial w}{\partial\theta}\right)
\end{aligned} \tag{3.8}$$

于是平衡条件(3.2)式中的第三式可以化简为:

$$\frac{1}{s}\frac{\partial^2(sM_s)}{\partial s^2}-\frac{1}{s}\frac{\partial M_\theta}{\partial s}+\frac{1}{s^2\sin^2\alpha}\frac{\partial^2 M_\theta}{\partial\theta^2}$$

$$+\frac{2}{s\sin\alpha}\frac{\partial}{\partial\theta}\left[\frac{1}{s}\frac{\partial(sM_{s\theta})}{\partial s}\right]+\frac{N_\theta}{s}\mathrm{ctg}\alpha+N_s\frac{\partial^2 w}{\partial s^2}$$

$$+\frac{N_\theta}{s^2\sin^2\alpha}\frac{\partial^2 w}{\partial\theta^2}+\frac{N_\theta}{s}\frac{\partial w}{\partial s}+\frac{2N_{s\theta}}{\sin\alpha}\frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial w}{\partial\theta}\right)$$

$$-\frac{\partial w}{\partial s}S-\frac{\Theta}{s\sin\alpha}\frac{\partial w}{\partial\theta}+q=0 \tag{3.9}$$

在中面应变表达式(2.1)中消去位移分量u、v，经过一系列复杂的推导，可以得到变形协调方程

$$\frac{1}{s^2\sin\alpha}\frac{\partial}{\partial s}\left(s\frac{\partial\gamma_{s\theta}}{\partial\theta}\right)-\frac{1}{s^2\sin^2\alpha}\frac{\partial^2\varepsilon_s}{\partial\theta^2}+\frac{1}{s}\frac{\partial\varepsilon_s}{\partial s}$$

$$-\frac{1}{s^2}\frac{\partial}{\partial s}\left(s^2\frac{\partial\varepsilon_\theta}{\partial s}\right)=\left(\mathrm{ctg}\alpha+\frac{\partial w}{\partial s}+\frac{1}{s\sin^2\alpha}\frac{\partial^2 w}{\partial\theta^2}\right)\frac{1}{s}\frac{\partial^2 w}{\partial s^2}$$

$$+\frac{1}{s^2\sin^2\alpha}\left(\frac{2}{s}\frac{\partial w}{\partial\theta}-\frac{\partial^2 w}{\partial\theta\partial s}\right)\frac{\partial^2 w}{\partial\theta\partial s}-\frac{1}{s^4\sin^2\alpha}\left(\frac{\partial w}{\partial\theta}\right)^2 \tag{3.10}$$

四、混合型方程

引入符号

$$\left.\begin{aligned}
&N=\{N_s,\ N_\theta,\ N_{s\theta}\}^T,\quad M=\{M_s,\ M_\theta,\ M_{s\theta}\}^T\\
&\varepsilon=\{\varepsilon_s,\ \varepsilon_\theta,\ \gamma_{s\theta}\}^T,\quad K=\{\varkappa_s,\ \varkappa_\theta,\ 2\varkappa_{s\theta}\}^T\\
&A=\begin{bmatrix}A_{11}&A_{12}&A_{16}\\A_{12}&A_{22}&A_{26}\\A_{16}&A_{26}&A_{66}\end{bmatrix}_f+\begin{bmatrix}\frac{E_sA_s}{d_s}&0&0\\0&\frac{E_\theta A_\theta}{d_\theta}&0\\0&0&0\end{bmatrix}\\
&B=\begin{bmatrix}B_{11}&B_{12}&B_{16}\\B_{12}&B_{22}&B_{26}\\B_{16}&B_{26}&B_{66}\end{bmatrix}_f+\begin{bmatrix}\frac{E_sA_se_s}{d_s}&0&0\\0&\frac{E_\theta A_\theta e_\theta}{d_\theta}&0\\0&0&0\end{bmatrix}\\
&D=\begin{bmatrix}D_{11}&D_{12}&D_{16}\\D_{12}&D_{22}&D_{26}\\D_{16}&D_{26}&D_{66}\end{bmatrix}_f+\begin{bmatrix}\frac{E_sI_s}{d_s}&0&0\\0&\frac{E_\theta I_\theta}{d_\theta}&0\\0&0&\frac{1}{4}\left(\frac{G_sJ_s}{d_s}+\frac{G_\theta J_\theta}{d_\theta}\right)\end{bmatrix}
\end{aligned}\right\}$$

(4.1)

于是式(3.3)可以改写为

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon \\ K \end{Bmatrix} \tag{4.2}$$

这就是复合材料加筋圆锥壳体广义力和广义位移之间的关系。

式(4.2)可以表示为

$$\begin{Bmatrix} \varepsilon \\ M \end{Bmatrix} = \begin{bmatrix} a & b \\ -b^T & d \end{bmatrix} \begin{Bmatrix} N \\ K \end{Bmatrix} \tag{4.3}$$

式中

$$a = A^{-1},\ b = -A^{-1}B,\ d = D - BA^{-1}B \tag{4.4}$$

矩阵a和d是对称的，b不一定是对称的。

引入应力函数F，其定义为

$$\left.\begin{aligned} N_s &= \frac{1}{s}\frac{\partial F}{\partial s} + \frac{1}{s^2\sin^2\alpha}\frac{\partial^2 F}{\partial\theta^2} - \frac{1}{s}\int sSds \\ N_\theta &= \frac{\partial^2 F}{\partial s^2} + \frac{1}{\sin\alpha}\frac{\partial\Theta_p}{\partial\theta} \\ N_{s\theta} &= -\frac{1}{\sin\alpha}\frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial F}{\partial\theta}\right) + \Theta_p \end{aligned}\right\} \tag{4.5}$$

式中Θ_p是满足

$$\frac{1}{s\sin^2\alpha}\frac{\partial^2(s^2N_{s\theta})}{\partial\theta^2} + \frac{\partial(s^2N_{s\theta})}{\partial s} = -\Theta s^2 \tag{4.6}$$

的特解，则纵向和横向的平衡条件，即式(3.2)中的第一式和第二式自然得到满足。利用中面曲率表达式(2.2)和应力函数(4.5)，可以将径向平衡条件，即式(3.2)中的第三式表示为

$$\begin{aligned} &L_{11}(w) + L_{12}(F) - \left(\mathrm{ctg}\alpha + \frac{\partial w}{\partial s} + \frac{1}{s\sin^2\alpha}\frac{\partial^2 w}{\partial\theta^2}\right)\frac{1}{s}\left(\frac{\partial^2 F}{\partial s^2}\right. \\ &\quad \left. + \frac{1}{\sin\alpha}\frac{\partial\Theta_p}{\partial\theta}\right) - \frac{\partial^2 w}{\partial s^2}\left(\frac{1}{s}\frac{\partial F}{\partial s} + \frac{1}{s^2\sin^2\alpha}\frac{\partial^2 F}{\partial\theta^2} - \frac{1}{s}\int sSds\right) \\ &\quad - \frac{2}{\sin\alpha}\frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial w}{\partial\theta}\right)\left[-\frac{1}{\sin\alpha}\frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial F}{\partial\theta}\right) + \Theta_p\right] \\ &\quad + \frac{\partial w}{\partial s}S + \frac{\Theta}{s\sin\alpha}\frac{\partial w}{\partial\theta} = q_b \end{aligned} \tag{4.7}$$

式中

$$\begin{aligned} L_{11} &= d_{11}l_{11} + d_{12}l_{12} + d_{16}l_{16} + d_{22}l_{22} + d_{26}l_{26} + d_{66}l_{66} \\ L_{12} &= b_{11}\bar{l}_{11} + b_{21}\bar{l}_{21} - b_{61}\bar{l}_{61} + b_{12}\bar{l}_{12} + b_{22}\bar{l}_{22} \\ &\quad - b_{62}\bar{l}_{62} + b_{16}\bar{l}_{16} + b_{26}\bar{l}_{26} - b_{66}\bar{l}_{66} \end{aligned} \tag{4.8}$$

式(4.8)中l_{ij}和$\bar{l}_{ij}$是微分算子，定义为

$$
\left.
\begin{aligned}
&l_{11}=\frac{\partial^4}{\partial s^4}+\frac{2}{s}\frac{\partial^3}{\partial s^3}\\
&l_{12}=\frac{2}{s^2\sin^2\alpha}\left(\frac{\partial^4}{\partial s^2\partial\theta^2}-\frac{1}{s}\frac{\partial^3}{\partial s\partial\theta^2}+\frac{1}{s^2}\frac{\partial^2}{\partial\theta^2}\right)\\
&l_{16}=\frac{4}{s\sin\alpha}\left(\frac{\partial^4}{\partial s^3\partial\theta}+\frac{1}{s^2}\frac{\partial^2}{\partial s\partial\theta}-\frac{1}{s^3}\frac{\partial}{\partial\theta}\right)\\
&l_{22}=\frac{1}{s^4\sin^4\alpha}\frac{\partial^4}{\partial\theta^4}-\frac{1}{s}\frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial}{\partial s}\right)+\frac{2}{s^4\sin^2\alpha}\frac{\partial^2}{\partial\theta^2}\\
&l_{26}=\frac{4}{s\sin\alpha}\left[\frac{1}{s\sin\alpha}\frac{\partial}{\partial s}\left(\frac{1}{s\sin\alpha}\frac{\partial^3}{\partial\theta^3}\right)+\frac{1}{s^2}\frac{\partial^2}{\partial s\partial\theta}-\frac{1}{s^3}\frac{\partial}{\partial\theta}\right]\\
&l_{66}=\frac{4}{s^2\sin\alpha}\frac{\partial}{\partial s}\left[\frac{s}{\sin\alpha}\frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial^2}{\partial\theta^2}\right)\right]\\
&\bar{l}_{11}=\frac{1}{s}\frac{\partial^3}{\partial s^3}+\frac{1}{s^2\sin^2\alpha}\frac{\partial^4}{\partial s^2\partial\theta^2}-\frac{2}{s^3\sin^2\alpha}\frac{\partial^3}{\partial s\partial\theta^2}+\frac{2}{s^4\sin^2\alpha}\frac{\partial^2}{\partial\theta^2}\\
&\bar{l}_{21}=\bar{l}_{11},\quad \bar{l}_{61}=\frac{1}{s\sin\alpha}\frac{\partial^2}{\partial s^2}\left[s\frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial}{\partial\theta}\right)\right]\\
&\bar{l}_{12}=\bar{l}_{22},\quad \bar{l}_{22}=\frac{1}{s^2\sin^2\alpha}\frac{\partial^4}{\partial s^2\partial\theta^2}-\frac{1}{s}\frac{\partial^3}{\partial s^3}\\
&\bar{l}_{62}=\frac{1}{s^2\sin^2\alpha}\frac{\partial}{\partial s}\left(\frac{1}{s\sin\alpha}\frac{\partial^3}{\partial\theta^3}\right)-\frac{1}{s\sin\alpha}\frac{\partial^2}{\partial s^2}\left(\frac{1}{s}\frac{\partial}{\partial\theta}\right)\\
&\bar{l}_{16}=\frac{2}{s^2\sin\alpha}\left[\frac{\partial^3}{\partial s^2\partial\theta}+\frac{\partial}{\partial s}\left(\frac{1}{s\sin^2\alpha}\frac{\partial^3}{\partial\theta^3}\right)\right]\\
&\bar{l}_{26}=\frac{2}{s^2\sin\alpha}\frac{\partial}{\partial s}\left(s\frac{\partial^3}{\partial s^2\partial\theta}\right)\\
&\bar{l}_{66}=\frac{2}{s^2\sin\alpha}\frac{\partial}{\partial s}\left[\frac{s}{\sin\alpha}\frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial^2}{\partial\theta^2}\right)\right]
\end{aligned}
\right\}
\tag{4.9}
$$

在式(4.9)中，常数项q_b为

$$q_b=q+L_s\left(\frac{1}{s}\int sSds\right)-\frac{1}{\sin\alpha}L_\theta\left(\frac{\partial\Theta_r}{\partial\theta}\right)-L_{s\theta}(\Theta_r) \tag{4.10}$$

式中L_s，L_θ和$S_{s\theta}$为微分算子

$$\begin{Bmatrix}L_s\\L_\theta\\L_{s\theta}\end{Bmatrix}=\begin{bmatrix}b_{11}&b_{12}&b_{16}\\b_{21}&b_{22}&b_{26}\\b_{61}&b_{62}&b_{66}\end{bmatrix}\begin{Bmatrix}l_s\\l_\theta\\l_{s\theta}\end{Bmatrix} \tag{4.11}$$

其中

$$l_s=\frac{\partial^2}{\partial s^2}+\frac{2}{s}\frac{\partial}{\partial s},\quad l_\theta=\frac{1}{s^2\sin^2\alpha}\frac{\partial^2}{\partial\theta^2}-\frac{1}{s}\frac{\partial}{\partial s}$$

$$l_{s\theta}=\frac{2}{s\sin\alpha}\left(\frac{\partial^2}{\partial s\partial\theta}+\frac{1}{s}\frac{\partial}{\partial\theta}\right) \tag{4.12}$$

再将中面曲率表达式(2.2)和应力函数(4.5)代入变形协调方程(3.10)，并利用广义力和广义位移关系式(4,3)，可以得到

$$L_{21}(w)+L_{22}(F)-\left(\mathrm{ctg}\alpha+\frac{\partial w}{\partial s}+\frac{1}{s\sin^2\alpha}\frac{\partial^2 w}{\partial\theta^2}\right)\frac{1}{s}\frac{\partial^2 w}{\partial s^2}$$
$$-\frac{1}{s^2\sin^2\alpha}\left(\frac{2}{s}\frac{\partial w}{\partial\theta}-\frac{\partial^2 w}{\partial s\partial\theta}\right)\frac{\partial^2 w}{\partial s\partial\theta}+\frac{1}{s^4\sin^2\alpha}\left(\frac{\partial w}{\partial\theta}\right)^2=q_0 \tag{4.13}$$

式中 L_{21}和L_{22}是微分算子，定义为：

$$\left.\begin{aligned}L_{21}&=b_{11}l_{22}+b_{12}l_{12}+2b_{16}l_{62}+b_{21}l_{21}+b_{22}l_{11}\\&\quad+2b_{26}l_{61}-\frac{b_{61}}{2}l_{26}-\frac{b_{62}}{2}l_{16}-b_{66}l_{66}\\L_{22}&=-a_{11}l_{22}-a_{12}l_{12}+\frac{a_{16}}{2}l_{26}-a_{22}l_{11}+\frac{a_{26}}{2}l_{16}-\frac{a_{66}}{4}l_{66}\end{aligned}\right\} \tag{4.14}$$

式(4.13)中的常数项q_0为

$$q_0=L_s\left(\frac{1}{s}\int sS\,ds\right)-\frac{1}{\sin\alpha}L_\theta\left(\frac{\partial\Theta_r}{\partial\theta}\right)-L_{s\theta}(\Theta_r) \tag{4.15}$$

式中L_s，L_θ和$L_{s\theta}$为微分算子

$$\begin{Bmatrix}L_s\\L_\theta\\L_{s\theta}\end{Bmatrix}=\begin{bmatrix}-a_{11}&-a_{12}&a_{16}/2\\-a_{12}&-a_{22}&a_{26}/2\\-a_{16}&-a_{26}&a_{66}/2\end{bmatrix}\begin{Bmatrix}l_\theta\\l_s\\l_{s\theta}\end{Bmatrix} \tag{4.16}$$

式(4.7)和(4.13)就是复合材料加筋圆锥壳体有限变形的混合型理论的一般方程式。

五、一般方程式的特例

用应力函数F和挠度函数w表示的耦合形式的一般方程式(4.7)和(4.13)，非常复杂，求解非常困难。对于一些特殊情况，一般方程式可以得到某种程度的简化。

1. 对于仅承受径向载荷的情况，即$q\neq 0$，$s=0$和$\Theta=0$时，式(4.7)中$q_b=q$，式(4.13)中$q_0=0$。

2. 对于小变形情况，式(4.7)和(4.13)分别简化为：

$$L_{11}(w)+L_{12}(F)-\frac{\mathrm{ctg}\alpha}{s}\left(\frac{\partial^2 F}{\partial s^2}+\frac{1}{\sin\alpha}\frac{\partial\Theta_r}{\partial\theta}\right)=q_b \tag{5.1}$$

$$L_{21}(w)+L_{22}(F)-\frac{\mathrm{ctg}\alpha}{s}\frac{\partial^2 w}{\partial s^2}=q_0 \tag{5.2}$$

3. 对于线性稳定问题，一般方程式(4.7)和(4.13)可以表示为

$$L_{11}(w)+L_{12}(F)-N_{s_0}\frac{\partial^2 w}{\partial s^2}-N_{\theta_0}\left(\frac{1}{s}\frac{\partial w}{\partial s}+\frac{1}{s^2\sin^2\alpha}\frac{\partial^2 w}{\partial\theta^2}\right)$$
$$-\frac{2N_{s\theta_0}}{\sin\alpha}\frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial w}{\partial s}\right)+S\frac{\partial w}{\partial s}+\frac{\Theta}{s\sin\alpha}\frac{\partial w}{\partial\theta}-\frac{\mathrm{ctg}\alpha}{s}\frac{\partial^2 F}{\partial s}=0 \tag{5.3}$$

$$L_{21}(w)+L_{22}(F)-\frac{\mathrm{ctg}\alpha}{s}\frac{\partial^2 w}{\partial s^2}=0 \tag{5.4}$$

式(5.3)和(5.4)中的位移分量是从失稳前的平衡位置到失稳后的平衡位置的增量，且失稳前处于薄膜应力状态。N_{s_0}，N_{θ_0}和$N_{s\theta_0}$为失稳时的内力。

4. 在式(4.7)和(4.13)中取$\alpha=90°$，则得到复合材料加筋圆薄板有限变形的基本方程

式。

5. 在式(4.7)和(4.13)中取$\alpha=0^\circ$，$s=+\infty$，$s\sin\alpha=R$，则可以得到半径为R的复合材料加筋圆柱壳体有限变形的混合型方程组

$$L_{11}(w)+L_{12}(F)-\frac{1}{R}\frac{\partial^2 F}{\partial x^2}-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 F}{\partial y^2}-\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 F}{\partial x^2}+2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2 F}{\partial x\partial y}=q \tag{5.5}$$

$$L_{21}(w)+L_{22}(F)-\frac{1}{R}\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2=0 \tag{5.6}$$

式(5.5)和(5.6)中，微分算子L_{ij}定义为：

$$\begin{aligned}L_{11}=&d_{11}\frac{\partial^4}{\partial x^4}+4d_{16}\frac{\partial^4}{\partial x^3\partial y}+2(d_{12}+2d_{66})\frac{\partial^4}{\partial x^2\partial y^2}\\&+4d_{26}\frac{\partial^4}{\partial x\partial y^3}+d_{66}\frac{\partial^4}{\partial y^4}\\L_{12}=L_{21}=&b_{21}\frac{\partial^4}{\partial x^4}+(2b_{26}-b_{61})\frac{\partial^4}{\partial x^3\partial y}+(b_{11}+b_{22}-2b_{66})\frac{\partial^4}{\partial x^2\partial y^2}\\&+(2b_{16}-b_{62})\frac{\partial^4}{\partial x\partial y^3}+b_{12}\frac{\partial^4}{\partial y^4}\\L_{22}=&-a_{22}\frac{\partial^4}{\partial x^4}+2a_{26}\frac{\partial^4}{\partial x^3\partial y}-(2a_{12}+a_{66})\frac{\partial^4}{\partial x^2\partial y^2}\\&+a_{16}\frac{\partial^4}{\partial x\partial y^3}-a_{11}\frac{\partial^4}{\partial y^4}\end{aligned} \tag{5.7}$$

其中变量$x=s$，$y=R\theta=s\sin\alpha\theta$。若不计筋条刚度的影响，式(5.5)和(5.6)与文献[14]相似。

6. 对于面板刚度可以化简的情况，可以参阅文献[13]。

六、结　束　语

本文利用变分原理和平均筋条刚度法，建立了复合材料加筋薄壁圆锥壳体有限变形的混合型理论。考虑了面板最一般的弯曲拉伸耦合关系和筋条的偏心效应。本文所建立的新理论可以作为研究复合材料加筋圆锥壳体许多力学问题的理论基础。

参 考 文 献

[1] Pflüger, A., Stabilität dünner Kegelschalen, *Ing. Archiv.*, 8, 3 (1937), 151—172.

[2] Seide, P., A survey of buckling theory and experiment for circular conical shells of constant thickness, NASA TN D-1510, 401—426.

[3] Fung, Y. C., and E. E. Sechler, Instability of thin elastic shells, *Structural Mechanics*, (1960), 115—168.

[4] Amiro, I. Ya and V. A. Zarutskü, Stability of ribbed shells, *Soviet Applied Mechanics*, May (1984), 925—940.

[5] Tong Li-yong and Wang Tsun-kuei, Axisymmetric Buckling of laminated composites conical shells under axial compression, *Proceedings of International Conference on Composite Materials and Structures*, Madras, India, Jan. 6—9 (1988).

[6] 贺益波、王俊奎，在外压作用下复合材料圆锥壳失稳的初步研究，第五届全国复合材料学术会议论文集(下册)，西安 (1988, 11), 940—947.

[7] Wang Hu and Wang Tsun-kuei, Snap-buckling of thin shallow conical shells, *Proceedings of International Conference on Applied Mechanics*, Beijing, China (1989).

[8] Hoff, N. J., The circular conical shells under arbitrary loads, *Journal of Applied Mechanics*, 77 (Dec. 1955), 557—562.

[9] Seide, P., A Donnell-type theory for asymmetrical bending and buckling of thin conical shells, *Journal of Applied Mechanics*, 24, 4 (Dec. 1957), 547—552.

[10] Singer, J., Donnell-type equations for bending and buckling of orthotropic conical shells, *Journal of Applied Mechanics*, 30, 2 (June 1963), 303—305.

[11] 赵金德，锥形壳体的弹性稳定性，航空学报，5, 1 (1984), 37—45.

[12] 孙博华，两种类型夹层锥壳的位移型统一理论及应用(上)，应用力学学报，5, 1 (1987), 79—89.

[13] Jones, R. M., *Mechanics of Composite Materials*, Scripta Book Company (1975).

[14] 蒋咏秋，纤维增强复合材料层合圆柱壳的优化设计，复合材料学报，2, 3 (1985), 13—22.

A Donnell Type Theory for Finite Deflection of Stiffened Thin Conical Shells Composed of Composite Materials

Wang Hu Wang Tsun-kuei

(*Beijing University of Aeronautics and Astronautics, Beijing*)

Abstract

A Donnell type theory is developed for finite deflection of closely stiffened truncated laminated composite conical shells under arbitrary loads by using the variational calculus and smeared-stiffener theory. The most general bending-stretching coupling and the effect of eccentricity of stiffeners are considered. The equilibrium equations, boundary conditions and the equation of compatibility are derived. The new equations of the mixed type of stiffened laminated composite conical shells are obtained in terms of the transverse deflection and stress function. The simplified equations are also given for some commonly encountered cases.

Key words composite materials, circular conical shells, stiffened shells, thin shells, finite deflection, mixed-type theory