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具有零阶退化方程的二阶双曲型方程奇异摄动问题的一致差分格式

苏煜诚　林　平

（南京大学，1989年2月2日收到）

摘　要

本文讨论了一个二阶双曲型奇异摄动问题，它的一阶导数项含有小参数ε。首先给出该问题解的能量估计及渐近解的余项估计，然后在均匀网格上构造了一个指数型拟合差分格式，最后证明了差分解在离散的能量范数意义下一致收敛于问题的精确解。

一、引　言

本文考虑如下的双曲型方程奇异摄动问题

$$L_\varepsilon u \equiv \varepsilon^2\left(\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}\right)+\varepsilon a(x,t)\frac{\partial u}{\partial t}+b(x,t)u=f(x,t)$$

$$(x,t)\in G \tag{1.1}$$

$$u(x,0)=\varphi(x),\quad \frac{\partial u}{\partial t}(x,0)=\psi(x)\qquad x\in[0,l] \tag{1.2}$$

$$u(0,t)=0,\quad u(l,t)=0\qquad t\in[0,T] \tag{1.3}$$

其中区域$G=\{(x,t)|0<x<l,\ 0<t\leqslant T\}$，函数$a(x,t)$，$b(x,t)$，$f(x,t)$，$\varphi(x)$，$\psi(x)$分别在区域$\bar{G}$和$[0,l]$上充分光滑，$a(x,t)>0$，$b(x,t)>0$对一切$(x,t)\in\bar{G}$成立，并且$\varphi(x)$，$\psi(x)$，$f(x,t)$满足下面的相容性条件：

C1　　$\varphi(0)=0$，$\psi(0)=0$，$\varphi(l)=0$，$\psi(l)=0$

C2　　$\varphi''(0)=0$，$f(0,0)=0$，$\varphi''(l)=0$，$f(l,0)=0$

在这个问题中二阶导数项和一阶导数项都含有小参数ε。[1]讨论了它的渐近解。在[2]中讨论了一阶导数项$\partial u/\partial t$不带小参数ε的情形的差分解法，利用非均匀网格得到差分解在能量范数下的一致收敛性。

本文首先给出问题(1.1)，(1.2)，(1.3)的能量不等式，并利用它获得渐近解在最大模意义下的余项估计。然后在均匀网格上构造了一个指数型拟合差分格式并作出离散的能量估计。最后证明了差分解离散的能量范数意义下一致收敛于问题(1.1)，(1.2)，(1.3)的解。

二、能量不等式

下面的定理给出问题(1.1)，(1.2)，(1.3)解的能量估计。

定理2.1 设$u(x,t)$是问题(1.1)，(1.2)，(1.3)的解且对一切$(x,t)\in\bar{G}$有$a(x,t)>0$，$b(x,t)>0$。那么存在与ε无关的正常数C，使

$$\|u\|_\varepsilon\leqslant CK(G,\varepsilon) \tag{2.1}$$

其中

$$\|u\|_\varepsilon=\left\{\int_0^l\left[u^2+\varepsilon^2\left(\frac{\partial u}{\partial t}\right)^2+\varepsilon^2\left(\frac{\partial u}{\partial x}\right)^2\right]dx\right\}^{1/2}$$

$$K(G,\varepsilon)=\varepsilon^{-\frac{1}{2}}\|f\|_G+\|\varphi\|_{[0,l]}+\varepsilon\|\psi\|_{[0,l]}+\varepsilon\|\varphi'\|_{[0,l]}$$

$$\|f\|_G=\left[\iint_G f^2dxdt\right]^{1/2}$$

及

$$\|v\|_{[0,l]}=\left[\int_0^l v^2dx\right]^{1/2}$$

证 用$2\varepsilon a\,\partial u/\partial t$乘方程两端，得到

$$\frac{\partial}{\partial t}\left[\varepsilon^3a\left(\frac{\partial u}{\partial t}\right)^2+\varepsilon^3a\left(\frac{\partial u}{\partial x}\right)^2+\varepsilon abu^2\right]+\frac{\partial}{\partial x}\left[-2\varepsilon^3a\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right]$$

$$=\varepsilon^3\frac{\partial a}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2+\varepsilon^3\frac{\partial a}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2+\varepsilon\frac{\partial(ab)}{\partial t}u^2$$

$$-2\varepsilon^3\frac{\partial a}{\partial x}\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}-2\varepsilon^2a^2\left(\frac{\partial u}{\partial t}\right)^2+2\varepsilon a\frac{\partial u}{\partial t}f$$

设

$$\left|\frac{\partial a}{\partial t}\right|,\ \left|\frac{\partial a}{\partial x}\right|,\ \left|\frac{\partial(ab)}{\partial t}\right|\leqslant M$$

$$0<a_0\leqslant a\leqslant a_1,\quad 0<b_0\leqslant b\leqslant b_1$$

则上式的右端

$$\leqslant\varepsilon^3\frac{\partial a}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2+\varepsilon^3\frac{\partial a}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2+\varepsilon\frac{\partial(ab)}{\partial t}u^2+\varepsilon^3\left|\frac{\partial a}{\partial x}\right|\left(\frac{\partial u}{\partial t}\right)^2$$

$$+\varepsilon^3\left|\frac{\partial a}{\partial x}\right|\left(\frac{\partial u}{\partial x}\right)^2-2\varepsilon^2a^2\left(\frac{\partial u}{\partial t}\right)^2+2\varepsilon^2a^2\left(\frac{\partial u}{\partial t}\right)^2+\frac{1}{2}f^2$$

$$\leqslant M\left[\varepsilon^3\left(\frac{\partial u}{\partial t}\right)^2+\varepsilon^3\left(\frac{\partial u}{\partial x}\right)^2+\varepsilon u^2\right]+\frac{1}{2}f^2$$

对上面的不等式两端按区域$G_t=\{(x,s)\,|\,0\leqslant x\leqslant l,\ 0<s\leqslant t\}$积分，则有

$$\varepsilon m\int_0^l\left[u^2+\varepsilon^2\left(\frac{\partial u}{\partial t}\right)^2+\varepsilon^2\left(\frac{\partial u}{\partial x}\right)^2\right]dx$$

$$\leqslant\varepsilon M\int_0^l\int_0^t\left[u^2+\varepsilon^2\left(\frac{\partial u}{\partial t}\right)^2+\varepsilon^2\left(\frac{\partial u}{\partial x}\right)^2\right]dxdt$$

$$+\frac{1}{2}\int_0^l\int_0^t f^2dxdt+\varepsilon M\left\{\|\varphi\|^2_{[0,l]}+\varepsilon^2\|\psi\|^2_{[0,l]}+\varepsilon^2\|\varphi'\|^2_{[0,l]}\right\}$$

所以

$$\int_0^l\left[u^2+\varepsilon^2\left(\frac{\partial u}{\partial t}\right)^2+\varepsilon^2\left(\frac{\partial u}{\partial x}\right)^2\right]dx\leqslant\frac{M}{m}\iint_{G_t}\left[u^2+\varepsilon^2\left(\frac{\partial u}{\partial t}\right)^2\right.$$

$$\left.+\varepsilon^2\left(\frac{\partial u}{\partial x}\right)^2\right]dxdt+\frac{1}{2m}\iint_{G_t}\left(\frac{f}{\sqrt{\varepsilon}}\right)^2dxdt$$

$$+\frac{M}{m}\left\{\|\varphi\|^2_{[0,l]}+\varepsilon^2\|\psi\|^2_{[0,l]}+\varepsilon^2\|\varphi'\|^2_{[0,l]}\right\}$$

由 Gronwall 不等式得到

$$\int_0^l\left[u^2+\varepsilon^2\left(\frac{\partial u}{\partial t}\right)^2+\varepsilon^2\left(\frac{\partial u}{\partial x}\right)^2\right]dx\leqslant\frac{M}{m}\exp\left(\frac{MT}{m}\right)K^2(G,\varepsilon)$$

这就是定理的结论.

三、形式渐近解及其余项估计

我们把问题(1.1)，(1.2)，(1.3)的渐近解及其余项估计叙述在下面的定理中.

定理3.1 设问题(1.1)，(1.2)，(1.3)的零阶渐近展开式为

$$U_0(x,t)=u_0(x,t)+\Pi_0(x,\tau)+Q_0(\xi,t)+\tilde{Q}_0(\tilde{\xi},t) \tag{3.1}$$

其中$\tau=t/\varepsilon$，$\xi=x/\varepsilon$，$\tilde{\xi}=(l-x)/\varepsilon$，$u_0(x,t)$，$\Pi_0(x,\tau)$，$Q_0(\xi,t)$，$\tilde{Q}_0(\tilde{\xi},t)$分别满足

$$u_0(x,t)=f(x,t)/b(x,t) \tag{3.2}$$

$$\left.\begin{aligned}&L_1\Pi_0\equiv\frac{\partial^2\Pi_0}{\partial\tau^2}+a(x,0)\frac{\partial\Pi_0}{\partial\tau}+b(x,0)\Pi_0=0\\&\Pi_0(x,0)=\varphi(x)-u_0(x,0),\quad\frac{\partial\Pi_0}{\partial\tau}(x,0)=0\end{aligned}\right\} \tag{3.3}$$

$$\left.\begin{aligned}&L_2Q_0\equiv-\frac{\partial^2Q_0}{\partial\xi^2}+b(0,t)Q_0=0\\&Q_0(0,t)=-u_0(0,t),\ Q_0(\xi,t)\to0\quad(\xi\to\infty)\end{aligned}\right\} \tag{3.4}$$

$$\left.\begin{aligned}&L_3\tilde{Q}_0\equiv-\frac{\partial^2\tilde{Q}_0}{\partial\tilde{\xi}^2}+b(l,t)\tilde{Q}_0=0\\&\tilde{Q}_0(0,t)=-u_0(l,t),\ \tilde{Q}_0(\tilde{\xi},t)\to0\quad(\tilde{\xi}\to\infty)\end{aligned}\right\} \tag{3.5}$$

那么有

$$|u(x,t)-U_0(x,t)|\leqslant C\varepsilon \tag{3.6}$$

其中C是与ε无关的正数.

证 形式渐近解的构造参看[1]. 为了进行余项估计我们在表达式(3.1)中还需加进含有ε一次幂的项，此时将引入角层函数，即渐近表达式有如下形式：

$$U_1(x,t)=\sum_{i=0}^{1}\varepsilon^i[u_i(x,t)+\Pi_i(x,\tau)+Q_i(\xi,t)+\tilde{Q}_i(\tilde{\xi},t)$$

$$+p_i(\xi,\tau)+\tilde{p}_i(\tilde{\xi},\tau)]$$

其中 $$u_1(x,t)=-\frac{a(x,t)}{b(x,t)}\frac{\partial u_0}{\partial t}$$

$\Pi_1(x,\tau)$满足：

$$\frac{\partial^2\Pi_1}{\partial\tau^2}+a(x,0)\frac{\partial\Pi_1}{\partial\tau}+b(x,0)\Pi_1=\frac{\partial a}{\partial t}(x,0)\tau\frac{\partial\Pi_0}{\partial\tau}$$

$$\frac{\partial\Pi_1}{\partial\tau}(x,0)=\psi(x)-\frac{\partial u_0}{\partial t}(x,0),\ \Pi_1(x,\tau)\to 0\qquad(\tau\to\infty)$$

$Q_1(\xi,t)$，$\tilde{Q}_1(\tilde{\xi},t)$也满足相应的方程。$p_i(\xi,\tau)$，$\tilde{p}_i(\tilde{\xi},\tau)$分别是点$(0,0)$和$(l,0)$附近的角层函数，$p_i(\xi,\tau)$满足

$$\frac{\partial^2 p_i}{\partial\tau^2}-\frac{\partial^2 p_i}{\partial\xi^2}+a(0,0)\frac{\partial p_i}{\partial\tau}+b(0,0)p_i=0$$

$$p_i(\xi,0)=-Q_i(\xi,0),\ \frac{\partial p_i}{\partial\tau}(\xi,0)=-\frac{\partial Q_{i-1}}{\partial\tau}(\xi,0)$$

$$p_i(0,\tau)=-\Pi_i(0,\tau),\qquad i=0,1$$

可以证明$p_0(\xi,\tau)=0$。$\tilde{p}_i(\tilde{\xi},\tau)$由类似的问题确定。

设$w(x,t)=u(x,t)-U_1(x,t)$则$w(x,t)$满足

$$L_\varepsilon w\equiv\varepsilon^2\left(\frac{\partial^2 w}{\partial t^2}-\frac{\partial^2 w}{\partial x^2}\right)+\varepsilon a(x,t)\frac{\partial w}{\partial t}+b(x,t)w=h(x,t,\varepsilon)$$

$$w(x,0)=0,\quad\frac{\partial w}{\partial t}(x,0)=O(\varepsilon),\ w(0,t)=w(l,t)=0$$

其中$h(x,t,\varepsilon)=O(\varepsilon^2)$。利用定理2.1的能量估计有

$$\int_0^l\left[w^2+\varepsilon^2\left(\frac{\partial w}{\partial t}\right)^2+\varepsilon^2\left(\frac{\partial w}{\partial x}\right)^2\right]dx\leqslant C\varepsilon^3$$

又

$$w^2(x,t)-w^2(0,t)=2\int_0^x w(s,t)\frac{\partial w}{\partial x}(s,t)ds$$

$$\leqslant 2\left(\int_0^l w^2 ds\right)^{1/2}\left(\int_0^l\left(\frac{\partial w}{\partial x}\right)^2 ds\right)^{1/2}\leqslant C\varepsilon^{3/2}\cdot\varepsilon^{1/2}=C\varepsilon^2$$

所以 $$|w(x,t)|\leqslant C\varepsilon$$

但 $$w(x,t)=u(x,t)-U_1(x,t)=u(x,t)-U_0(x,t)+O(\varepsilon)$$

故最后得到$|u(x,t)-U_0(x,t)|\leqslant C\varepsilon$，定理证毕。

关于渐近解中的项$\Pi_0(x,\tau)$，$Q_0(\xi,t)$，$\tilde{Q}_0(\tilde{\xi},t)$，我们分别有（参见[1]）：

$$|\Pi_0(x,\tau)|\leqslant M(\tau)\exp(-\lambda(x)\tau)\tag{3.7}$$

其中

$$\lambda(x)=\begin{cases}\dfrac{a(x,0)}{2}-\sqrt{\dfrac{a^2(x,0)}{4}-b(x,0)}\ , & 当\ \dfrac{a^2(x,0)}{4}-b(x,0)>0\\ \dfrac{a(x,0)}{2}\ , & 当\ \dfrac{a^2(x,0)}{4}-b(x,0)\leqslant 0\end{cases}$$

$M(\tau)$为τ的具有正系数的一次多项式。

$$Q_0(\xi,t)=-u_0(0,t)\exp\left(-\sqrt{b(0,t)}\,\xi\right) \tag{3.8}$$

$$\tilde{Q}_0(\tilde{\xi},t)=-u_0(l,t)\exp\left(-\sqrt{b(l,t)}\,\tilde{\xi}\right) \tag{3.9}$$

从解的渐近表达式$U_0(x,t)$及(3.7)，(3.8)，(3.9)可知问题(1.1)，(1.2)，(1.3)的解及其导数有如下的估计：

$$\left|\frac{\partial^k u(x,t)}{\partial x^i\partial t^{k-i}}\right|\leqslant M\varepsilon^{-k},\qquad 0\leqslant i\leqslant k,\ 0\leqslant k\leqslant m \tag{3.10}$$

这里假设$u(x,t)\in C^m(G)$。

四、差 分 格 式

我们沿t方向作完全指数型拟合，沿x方向作通常的指数型拟合来构造问题(1.1)，(1.2)，(1.3)的差分格式。为使完全指数型拟合有意义，我们假设

$$a^2(x,0)/4-b(x,0)>0,\qquad x\in[0,l] \tag{4.1}$$

方程(1.1)的差分近似为

$$L_\varepsilon^d u^d(x,t)\equiv\varepsilon^2\sigma_1\,u^d_{\bar{t}t}(x,t)+\varepsilon\sigma_2 a(x,t)\,u^d_{\bar{t}}(x,t)$$

$$-\varepsilon^2\sigma_3 u^d_{x\bar{x}}(x,t)+b(x,t)u^d(x,t)=f(x,t) \tag{4.2}$$

其中$(x,t)=(x_i,t_j)$，$x_i=i\Delta x$，$i=1,\cdots,N-1$，$x_0=0$，$x_N=l$，$t_j=j\Delta t$，$j=1,\cdots,J$，$t_0=0$，Δx和Δt分别为x方向和t方向的网格步长，并假定

$$c_1\Delta t\leqslant\Delta x\leqslant c_2\Delta t \tag{4.3}$$

其中$0<c_1<c_2$。拟合因子σ_1，σ_2，σ_3分别定义为

$$\sigma_1=\frac{b(x,0)\Delta t^2}{4\varepsilon^2}\cdot\frac{\exp((\lambda_1+\lambda_2)\Delta t/2)}{\sinh(\lambda_1\Delta t/2)\sinh(\lambda_2\Delta t/2)}$$

$$\sigma_2=-\frac{b(x,0)\Delta t}{2\varepsilon a(x,0)}\left[\frac{\exp(\lambda_1\Delta t/2)}{\sinh(\lambda_1\Delta t/2)}+\frac{\exp(\lambda_2\Delta t/2)}{\sinh(\lambda_2\Delta t/2)}\right]$$

$$\sigma_3=-\frac{b(x,t)\Delta x^2}{4\varepsilon^2}\sinh^{-2}\left(\frac{\sqrt{b(x,t)}\,\Delta x}{2\varepsilon}\right)$$

这里 $$\lambda_{1,2}=\frac{1}{\varepsilon}\left[-\frac{a(x,0)}{2}\pm\sqrt{\frac{1}{4}a^2(x,0)-b(x,0)}\right]$$

关于初始条件和边界条件(1.2)，(1.3)有如下近似：

$$u^d(x,0)=\varphi(x),\quad u^d(x,\Delta t)=\varphi(x)+\Delta t\psi(x) \tag{4.4}$$

$$u^d(0,t)=0,\quad u^d(l,t)=0 \tag{4.5}$$

五、离散的能量不等式

差分问题(4.2)，(4.4)，(4.5)的解有如下的能量估计.

定理5.1 设Δt与Δx满足不等式(4.3). 则当$\Delta x/\varepsilon\leqslant \max(1,c_0)$时，有

$$\|u^d\|_s^2+\varepsilon^2\|\sqrt{\sigma_1}u_{\bar{x}}^d\|_s^2+\varepsilon^2\|u_{\bar{x}}^d\|_s^2$$

$$\leqslant C\left[\frac{\Delta x\Delta t}{\max(\varepsilon,\Delta x)}\sum_{j=2}^{s}\sum_{i=1}^{N}f^2+\varepsilon^2\|u_{\bar{x}}^d\|_1^2+\varepsilon^2\|u_{\bar{x}}^d\|_1^2+\|u^d\|_1^2\right] \tag{5.1}$$

当$\Delta x/\varepsilon\geqslant \max(1,c_0)$时，有

$$\|u^d\|_s^{*2}+\varepsilon^2\|\sqrt{\sigma_1}u_{\bar{x}}^d\|_s^{*2}+\varepsilon^2\|\sqrt{\sigma_3}u_{\bar{x}}^d\|_s^{*2}$$

$$\leqslant C\left[\frac{\Delta x\Delta t}{\max(\varepsilon,\Delta x)}\sum_{j=2}^{s}\sum_{i=1}^{N-1}f^2+\varepsilon^2\|u_{\bar{x}}^d\|_1^{*2}+\varepsilon^3\|u_{\bar{x}}^d\|_1^{*2}+\|u^d\|_1^{*2}\right] \tag{5.2}$$

其中 $s=2,\cdots,J$，$\|u^d\|_s^2=\Delta x\sum_{i=1}^{N}[u^d(i\Delta x,s\Delta t)]^2$

$$\|u^d\|_s^{*2}=\Delta x\sum_{i=1}^{N-1}[u^d(i\Delta x,s\Delta t)]^2$$

c_0为某一适当大的正数，c_0与正数C均与$\varepsilon,\Delta x,\Delta t$无关.

证 由[3]，不难有（这里为方便起见用u表示u^d）

$$buu_{\bar{t}}=\frac{1}{2}b(u^2)_{\bar{t}}+\frac{\Delta t}{2}bu_{\bar{t}}^2$$

$$=\frac{1}{2}(bu^2)_{\bar{t}}+\frac{\Delta t}{2}bu_{\bar{t}}^2-\frac{1}{2}b_{\bar{t}}u^2(x,t-\Delta t) \tag{5.3}$$

$$\sigma_1u_{\bar{x}\bar{t}}u_{\bar{x}}=\frac{1}{2}\sigma_1(u_{\bar{x}}^2)_{\bar{t}}+\frac{\Delta t}{2}\sigma_1u_{\bar{x}\bar{t}}^2$$

$$=\frac{1}{2}(\sigma_1u_{\bar{x}}^2)_{\bar{t}}+\frac{\Delta t}{2}\sigma_1u_{\bar{x}\bar{t}}^2-\frac{1}{2}(\sigma_1)_{\bar{t}}u_{\bar{x}}^2(x,t-\Delta t) \tag{5.4}$$

$$\sigma_3u_{x\bar{x}}u_{\bar{t}}=\sigma_3(u_{\bar{t}}u_x)_{\bar{x}}-\frac{1}{2}\sigma_3(u_{\bar{x}}^2)_{\bar{t}}-\frac{\Delta t}{2}\sigma_3u_{\bar{x}\bar{t}}^2$$

$$=(\sigma_3u_{\bar{t}}u_x)_{\bar{x}}-\frac{1}{2}(\sigma_3u_{\bar{x}}^2)_{\bar{t}}-\frac{\Delta t}{2}\sigma_3u_{\bar{x}\bar{t}}^2$$

$$-(\sigma_3)_{\bar{x}}u_{\bar{t}}(x-\Delta x,t)u_{\bar{x}}+\frac{1}{2}(\sigma_3)_{\bar{t}}u_{\bar{x}}^2(x,t-\Delta t) \tag{5.5}$$

在(4.2)两边作用$u_{\bar{t}}$，注意到$(\sigma_1)_{\bar{t}}=0$，得

$$\frac{1}{2}\varepsilon^2(\sigma_1 u_{\bar{t}}^2)_{\bar{t}}+\frac{1}{2}\varepsilon^2\Delta t\sigma_1 u_{\bar{t}\bar{t}}^2+\varepsilon\sigma_2 a u_{\bar{t}}^2-\varepsilon^2(\sigma_3 u_{\bar{t}}u_x)_{\bar{x}}$$

$$+\frac{1}{2}\varepsilon^2\sigma_3(u_{\bar{x}}^2)_{\bar{t}}+\frac{1}{2}\varepsilon^2\Delta t\sigma_3 u_{\bar{x}\bar{t}}^2+\frac{1}{2}(bu^2)_{\bar{t}}+\frac{\Delta t}{2}bu_{\bar{t}}^2$$

$$=fu_{\bar{t}}-\frac{1}{2}\varepsilon^2(\sigma_3)_x u_{\bar{t}}(x-\Delta x,t)u_{\bar{x}}+\frac{1}{2}\varepsilon^2(\sigma_3)_{\bar{t}}u_{\bar{x}}^2(x,t-\Delta t)$$

$$+\frac{1}{2}b_{\bar{t}}u^2(x,t-\Delta t) \tag{5.6}$$

关于拟合因子σ_1，σ_2，σ_3，我们有下面的估计：

$$\left.\begin{array}{l}0<\sigma_1\leqslant C,\ c_1\leqslant\sigma_2\leqslant c_2\Delta t/\varepsilon,\ 0<\sigma_3\leqslant C\\ |(\sigma_3)_x|\leqslant C,\ |(\sigma_3)_{\bar{t}}|\leqslant C\end{array}\right\} \tag{5.7}$$

下面分两种情况讨论：

(i) $\Delta x/\varepsilon\leqslant\max(1,c_0)$，常数$c_0$在下面确定。这时，拟合因子$\sigma_3$满足

$$\sigma_3\geqslant C>0 \tag{5.8}$$

对(5.6)两边按$\Delta x\Delta t\sum\limits_{j=2}^{s}\sum\limits_{i=1}^{N}$求和，利用不等式：

$$ab\leqslant a^2/2\delta+\delta b^2/2,\ \delta>0$$

边界条件(4.5)及估计式(5.7)，可得

$$\frac{1}{2}\varepsilon^2\|\sqrt{\sigma_1}u_{\bar{t}}\|_s^2+\frac{\Delta t}{2}\varepsilon^2\cdot\Delta x\Delta t\sum_{j=2}^{s}\sum_{i=1}^{N}\sigma_1 u_{\bar{t}\bar{t}}^2$$

$$+\Delta x\Delta t\sum_{j=2}^{s}\sum_{i=1}^{N}\left[\varepsilon\sigma_2 a+\frac{\Delta t}{2}b-\delta\max(\varepsilon,\Delta t)-\frac{1}{2}C\varepsilon^2\right]u_{\bar{t}}^2$$

$$+\frac{1}{2}\varepsilon^2\|\sqrt{\sigma_3}u_{\bar{x}}\|_s^2+\frac{\Delta t}{2}\varepsilon^2\cdot\Delta x\Delta t\sum_{j=2}^{s}\sum_{i=1}^{N}\sigma_3 u_{\bar{x}\bar{t}}^2+\frac{1}{2}\|\sqrt{b}u\|_s^2$$

$$\leqslant\frac{\Delta x\Delta t}{2\delta\max(\varepsilon,\Delta t)}\sum_{j=2}^{s}\sum_{i=1}^{N}f^2+\frac{1}{2}C\varepsilon^2\Delta x\Delta t\sum_{j=2}^{s}\sum_{i=1}^{N}u_{\bar{x}}^2(x,t-\Delta t)$$

$$+\frac{1}{2}C\Delta x\Delta t\sum_{j=2}^{s}\sum_{i=1}^{N}u^2(x,t-\Delta t)$$

$$+\frac{1}{2}\varepsilon^2\|\sqrt{\sigma_1}u_{\bar{t}}\|_1^2+\frac{1}{2}C\varepsilon^2\|u_x\|_1^2+\frac{1}{2}C\|u\|_1^2$$

取δ适当小，当ε充分小时，可有

$$\varepsilon\sigma_2 a+\Delta t b/2-\delta\max(\varepsilon,\Delta t)-C\varepsilon^2/2>0$$

从而存在常数m，$M>0$，使得

$$m[\varepsilon^2\|\sqrt{\sigma_1}u_{\bar{t}}\|_i^2+\varepsilon^2\|u_{\bar{x}}\|_i^2+\|u\|_i^2]$$

$$\leqslant M\left\{\frac{\Delta x\Delta t}{\max(\varepsilon,\Delta t)}\sum_{j=2}^{s}\sum_{i=1}^{N}f^2+\varepsilon^2\|u_{\bar{t}}\|_1^2+\varepsilon^2\|u_x\|_1^2+\|u\|_1^2\right.$$

$$\left.+\Delta t\sum_{j=2}^{s-1}[\varepsilon^2\|u_x\|_j^2+\|u\|_j^2]\right\}$$

故由离散的Gronwall不等式(见[3])，得(5.1)成立。

(ii) $\Delta x/\varepsilon\geqslant\max(1,c_0)$，这时(5.8)不成立，在情况(i)的过程中$u_{\bar{x}}^2$项难以处理，必须对上面的过程作修改。以$\alpha u_{\bar{t}}+\beta u_{\bar{x}}$乘差分方程(4.2)的两端，其中$\alpha,\beta$为正常数，利用

$$buu_x=\frac{1}{2}(bu^2)_x+\frac{\Delta x}{2}bu_x^2-\frac{1}{2}b_xu^2(x-\Delta x,t)$$

$$\sigma_2 au_{\bar{t}}u_x\leqslant\frac{1}{2\delta}\sigma_2 au_{\bar{t}}^2+\frac{1}{2}\delta\sigma_2 au_x^2$$

$$\sigma_1u_{\bar{t}\bar{t}}u_x\leqslant\frac{\Delta t}{2}\sigma_1u_{\bar{t}\bar{t}}^2+\frac{1}{2\Delta t}\sigma_1u_x^2$$

$$\sigma_3u_{xx}u_x=\frac{1}{2}(\sigma_3u_x^2)_x-\frac{\Delta x}{2}\sigma_3u_{x\bar{x}}^2-\frac{1}{2}(\sigma_3)_xu_x^2(x+\Delta x,t)$$

结合(5.3)～(5.5)，差分方程(4.2)成为

$$\frac{1}{2}\alpha\varepsilon^2(\sigma_1u_{\bar{t}}^2)_{\bar{t}}+\frac{1}{2}\alpha\varepsilon^2\Delta t\sigma_1u_{\bar{t}\bar{t}}^2+\alpha\varepsilon\sigma_2 au_{\bar{t}}^2-\alpha\varepsilon^2(\sigma_3u_{\bar{t}}u_x)_x$$

$$+\frac{1}{2}\alpha\varepsilon^2(\sigma_3u_x^2)_{\bar{t}}+\frac{1}{2}\alpha\varepsilon^2\Delta t\sigma_3u_{x\bar{t}}^2+\frac{1}{2}\alpha(bu^2)_{\bar{t}}+\frac{1}{2}\alpha\Delta tbu_{\bar{t}}^2$$

$$+\frac{1}{2}\beta(bu^2)_x+\frac{1}{2}\beta b\Delta xu_x^2+\frac{1}{2}\beta\varepsilon^2\sigma_3\Delta xu_{x\bar{x}}^2-\frac{1}{2}\beta\varepsilon^2(\sigma_3u_x^2)_x$$

$$\leqslant\frac{\alpha f^2}{2\delta\max(\varepsilon,\Delta t)}+\frac{1}{2}\alpha\delta\max(\varepsilon,\Delta t)u_{\bar{t}}^2+\frac{\beta}{2\delta\Delta x}f^2$$

$$+\frac{1}{2}\beta\delta\Delta xu_x^2+C\alpha\varepsilon^2u_{\bar{t}}^2(x-\Delta x,t)+C\alpha\varepsilon^2u_x^2$$

$$+C\alpha\varepsilon^2u_x^2(x,t-\Delta t)+C\alpha u^2(x,t-\Delta t)+C\beta u^2(x-\Delta x,t)$$

$$+\frac{\beta}{2\delta}\varepsilon\sigma_2 au_{\bar{t}}^2+\frac{1}{2}\beta\delta\varepsilon\sigma_2 au_x^2+\frac{\beta}{2\delta}\varepsilon^2\Delta t\sigma_1u_{\bar{t}\bar{t}}^2$$

$$+\delta\frac{\varepsilon^2}{2\Delta t}\sigma_1u_{\bar{x}}^2+C\beta\varepsilon^2u_{\bar{x}}^2(x+\Delta x,t)$$

两边按 $\Delta x\Delta t\sum_{j=2}^{s}\sum_{i=1}^{N-1}$求和，得

$$
\begin{aligned}
&\frac{1}{2}\alpha\varepsilon^2\|\sqrt{\sigma_1}u_{\bar{t}}\|_s^{*2}+\frac{\Delta t}{2}\varepsilon^2(\alpha-\beta/\delta)\Delta x\Delta t\sum_{j=2}^{s}\sum_{i=1}^{N-1}\sigma_1 u_{\bar{t}t}^2\\
&+\Delta x\Delta t\sum_{j=2}^{s}\sum_{i=1}^{N-1}I_1u_{\bar{t}}^2+\frac{1}{2}\alpha\varepsilon^2\|\sqrt{\sigma_3}u_{x}\|_s^{*2}\\
&+\frac{\Delta t}{2}\alpha\varepsilon^2\Delta x\Delta t\sum_{j=2}^{s}\sum_{i=1}^{N-1}\sigma_3u_{\bar{x}\bar{t}}^2+\frac{1}{2}\alpha\|\sqrt{b}\,u\|_s^{*2}\\
&+\Delta x\Delta t\sum_{j=2}^{s}\sum_{i=1}^{N-1}I_2u_{\bar{x}}^2+\frac{1}{2}\beta\Delta t\sum_{j=2}^{s}bu^2((N-1)\Delta x,t)\\
&+\frac{\Delta x}{2}\varepsilon^2\beta\Delta x\Delta t\sum_{j=2}^{s}\sum_{i=1}^{N-1}\sigma_3u_{x\bar{x}}^2+I_3+\frac{1}{2}\beta\varepsilon^2\Delta t\sum_{j=2}^{s}\sigma_3u_{\bar{x}}^2(\Delta x,t)+I_4\\
&\leqslant\left(\frac{\alpha}{2\delta\max(\varepsilon,\Delta t)}+\frac{\beta}{2\delta\Delta x}\right)\Delta x\Delta t\sum_{j=2}^{s}\sum_{i=1}^{N-1}f^2\\
&+C(\alpha+\beta)\Delta t\sum_{j=2}^{s-1}\|u\|_j^{*2}+C\alpha(\varepsilon^2\|\sqrt{\sigma_1}u_{\bar{t}}\|_1^{*2}+\varepsilon^2\|\sqrt{\sigma_3}u_{\bar{x}}\|_1^{*2}\\
&+\|\sqrt{b}\,u\|_1^{*2})+I_5
\end{aligned}\tag{5.9}
$$

其中
$$I_1=\alpha\varepsilon\sigma_2a+\frac{\Delta t}{2}\alpha b-\frac{1}{2}\alpha\delta\max(\varepsilon,\Delta t)-\alpha C\varepsilon^2-\frac{\beta}{2\delta}\varepsilon\sigma_2a$$

$$I_2=\frac{1}{2}\beta b\Delta x-\beta\delta\Delta x-2C\alpha\varepsilon^2-\frac{1}{2}\beta\delta\varepsilon\sigma_2a-\beta\delta\frac{\varepsilon^2}{\Delta x}\sigma_1-C\beta\varepsilon^2$$

$$I_3=-\frac{1}{2}\beta\varepsilon^2\Delta t\sum_{j=2}^{s}\sigma_3u_{\bar{x}}^2(N\Delta x,t)$$

$$I_4=-\alpha\varepsilon^2\Delta t\sum_{j=2}^{s}(\sigma_3u_{\bar{t}}u_x)((N-1)\Delta x,t)$$

$$I_5=C\beta\varepsilon^2\Delta x\Delta t\sum_{j=2}^{s}u_{\bar{x}}^2(N\Delta x,t)$$

我们先对I_3，I_4，I_5作一些处理．不难有

$$I_3=-\frac{1}{2}\beta\frac{\varepsilon^2}{(\Delta x)^2}\Delta t\sum_{j=2}^{s}\sigma_3 u^2((N-1)\Delta x,t)$$

$$\geqslant-\frac{1}{2}\beta\Delta t\sum_{j=2}^{s}\sigma_3 u^2((N-1)\Delta x,t),\ (\text{因为}\Delta x/\varepsilon\geqslant 1)$$

$$I_4=-\alpha\frac{\varepsilon^2}{\Delta x}\Delta t\sum_{j=2}^{s}(\sigma_3 u_{\bar{x}} u)((N-1)\Delta x,t)$$

$$\geqslant-\frac{1}{2}\alpha\varepsilon^2\Delta t\sum_{j=2}^{s}\sigma_3 u_{\bar{x}}^2((N-1)\Delta x,t)$$

$$-\frac{1}{2}\alpha\frac{\varepsilon^2}{(\Delta x)^2}\Delta t\sum_{j=2}^{s}\sigma_3 u^2((N-1)\Delta x,t)$$

$$\geqslant-\frac{1}{2}\alpha\frac{\varepsilon^2}{(\Delta x)^2}\Delta t\sum_{j=2}^{s}\Delta x\sum_{i=1}^{N-1}\sigma_3 u_{\bar{x}}^2(x,t)$$

$$-\frac{1}{2}\alpha\frac{\varepsilon^2}{(\Delta x)^2}\Delta t\sum_{j=2}^{s}\sigma_3 u^2((N-1)\Delta x,t)$$

$$\geqslant-\frac{1}{2}\alpha\varepsilon\Delta x\Delta t\sum_{j=2}^{s}\sum_{i=1}^{N-1}\sigma_3 u_{\bar{x}}^2-\frac{1}{2}\alpha\Delta t\sum_{j=2}^{s}\sigma_3 u^2((N-1)\Delta x,t)$$

$$I_6=C\beta\varepsilon^2\Delta x\Delta t\sum_{j=2}^{s}\frac{1}{\Delta x^2}u^2((N-1)\Delta x,t)\leqslant C\beta\varepsilon\Delta t\sum_{j=2}^{s}u^2((N-1)\Delta x,t)$$

从而略去一些无关紧要的项后，不等式(5.9)成为

$$\frac{1}{2}\alpha\varepsilon^2\|\sqrt{\sigma_1}\,u_{\bar{x}}\|_s^{*2}+\frac{1}{2}\varepsilon^2\Delta t(\alpha-\beta/\delta)\Delta x\Delta t\sum_{j=2}^{s}\sum_{i=1}^{N-1}\sigma_1 u_{\bar{x}\bar{t}}^2$$

$$+\Delta x\Delta t\sum_{j=2}^{s}\sum_{i=1}^{N-1}\left(I_1-\frac{1}{2}\alpha\varepsilon\sigma_3\right)u_{\bar{x}}^2+\frac{1}{2}\alpha\varepsilon^2\|\sqrt{\sigma_3}u_{\bar{x}}\|_s^{*2}$$

$$+\frac{1}{2}\alpha\|\sqrt{b}\,u\|_s^{*2}+\Delta x\Delta t\sum_{j=2}^{s}\sum_{i=1}^{N-1}I_2 u_{\bar{x}}^2+\Delta t\sum_{j=2}^{s}I_6 u^2((N-1)\Delta x,t)$$

$$\leqslant M\Big[\Big(\frac{1}{\max(\varepsilon,\Delta t)}+\frac{1}{\Delta x}\Big)\Delta x\Delta t\sum_{j=2}^{s}\sum_{i=1}^{N-1}f^2+\varepsilon^2\|u_{\bar{x}}\|_1^{*2}$$

$$+\varepsilon^2\|u_{\bar{x}}\|_1^{*2}+\|u\|_1^{*2}+\Delta t\sum_{j=2}^{s-1}\|u\|_j^{*2}\Big]$$

取δ适当小，$\alpha\geqslant 4\beta/\delta$，选取$c_0$使当$\Delta x/\varepsilon\geqslant c_0$时 $\sigma_3<(1/2)\min(1,\beta/\alpha)\min\limits_{x,t}b(x,t)$，则在$\varepsilon$充分小时有

$$\alpha-\beta/\delta>0,\ I_1-\frac{1}{2}\alpha\varepsilon\sigma_3>0,\ I_2>0,\ I_6=\frac{1}{2}\beta b-\frac{1}{2}\beta\sigma_3-\frac{1}{2}\alpha\sigma_3-C\beta\varepsilon>0$$

从而存在常数m，$M>0$，使

$$m(\varepsilon^2\|\sqrt{\sigma_1}u_{\bar{t}}\|_i^{*2}+\varepsilon^2\|\sqrt{\sigma_3}u_{\bar{x}}\|_i^{*2}+\|u\|_i^{*2})$$

$$\leqslant M\Big[\frac{\Delta x\Delta t}{\max(\varepsilon,\Delta x)}\sum_{j=2}^{i}\sum_{i=1}^{N-1}f^2+\varepsilon^2\|u_{\bar{t}}\|_1^{*2}+\varepsilon^2\|u_{\bar{x}}\|_1^{*2}+\|u\|_1^{*2}+\Delta t\sum_{j=2}^{i-1}\|u\|_j^{*2}\Big]$$

仍由离散 Gronwall 不等式(见[3])得(5.2)成立．从而定理证毕．

六、差分解的误差估计

首先对拟合因子σ_1，σ_2，σ_3，不难有估计：

$$|\sigma_1-1|\leqslant C\Delta t/\varepsilon \tag{6.1}$$

$$|\sigma_2-1|\leqslant C\Delta t/\varepsilon \tag{6.2}$$

$$|\sigma_3-1|\leqslant C(\Delta x)^2/\varepsilon^2 \tag{6.3}$$

利用Taylor展式，不等式(6.1)～(6.3)以及导数估计(3.10)，可得

$$L_i^d(u(x,t)-u^d(x,t))=O\left(\frac{\Delta t}{\varepsilon}+\frac{(\Delta t)^2}{\varepsilon^2}+\frac{(\Delta x)^2}{\varepsilon^2}\right)$$

并且 $u(x,0)-u^d(x,0)=0$，$u(x,\Delta t)-u^d(x,\Delta t)=O((\Delta t)^2/\varepsilon^2)$

$$u(0,t)-u^d(0,t)=0,\ u(l,t)-u^d(l,t)=0$$

利用(4.3)及定理5.1，可得古典估计：

$$\|u(x,t)-u^d(x,t)\|_i^*\leqslant C\sqrt{\frac{1}{\max(\varepsilon,\Delta x)}}\left(\frac{\Delta x}{\varepsilon}+\frac{(\Delta x)^2}{\varepsilon^2}\right) \tag{6.4}$$

另一方面，我们还有

$$L_i^d(U_0(x,t)-u^d(x,t))=L_i^d(u_0(x,t)+\Pi_0(x,\tau)+Q_0(\xi,t)+\tilde{Q}_0(\tilde{\xi},t)-u^d(x,t))=L_i^d\Pi_0+L_i^dQ_0+L_i^d\tilde{Q}_0+O(\varepsilon) \tag{6.5}$$

由(3.7)可有

$$L_i^d\Pi_0(x,\tau)=\varepsilon^2\sigma_1\Pi_{0\bar{t}\bar{t}}+\varepsilon\sigma_2a(x,0)\Pi_{0\bar{t}}+b(x,0)\Pi_0+O(\varepsilon)$$

不难验证 $\varepsilon^2\sigma_1\Pi_{0\bar{t}\bar{t}}+\varepsilon\sigma_2a(x,0)\Pi_{0\bar{t}}+b(x,0)\Pi_0=0$，所以

$$L_i^d\Pi_0(x,\tau)=O(\varepsilon) \tag{6.6}$$

再由(3.8)，可有

$$L_i^dQ_0=-\varepsilon^2\sigma_3Q_{0\bar{x}\bar{x}}+bQ_0+O(\varepsilon)$$

类似[4]，不难得$-\varepsilon^2\sigma_3Q_{0\bar{x}\bar{x}}+bQ_0=O(\Delta x)$，所以

$$L_i^dQ_0=O(\varepsilon+\Delta x) \tag{6.7}$$

同理可得

$$L_i^d\tilde{Q}_0=O(\varepsilon+\Delta x) \tag{6.8}$$

从(6.5)～(6.8)得

$$L_i^d(U_0(x,t)-u^d(x,t))=O(\varepsilon+\Delta x)$$

又离散的初始条件和边界条件为(n为任意正整数)

$$U_0(x,0)-u^d(x,0)=O(\varepsilon^n),\ U_0(x,\Delta t)-u^d(x,\Delta t)=O(\varepsilon^n)+\Delta tO(1)$$

$$U_0(0,t)-u^d(0,t)=O(\varepsilon^n),\ U_0(l,t)-u^d(l,t)=O(\varepsilon^n)$$

类似[2]，先作变换使边界条件为齐次，然后用定理5.1及条件(4.3)得

$$\|U_0(x,t)-u^d(x,t)\|_s^*\leqslant C(\sqrt{\max(\varepsilon,\Delta x)}+\varepsilon^{n+1}/\Delta x)$$

再由余项估计(3.6)得

$$\|u(x,t)-U_0(x,t)\|_s^*\leqslant C\varepsilon$$

从而有下面的非古典估计

$$\|u(x,t)-u^d(x,t)\|_s^*\leqslant C(\sqrt{\max(\varepsilon,\Delta x)}+\varepsilon^{n+1}/\Delta x) \tag{6.9}$$

结合(6.4)和(6.9)可得下面的一致收敛估计。

定理6.1 设条件(4.1)和(4.3)成立。那么差分问题(4.2)，(4.4)，(4.5)的解 $u^d(x,t)$ 在离散的能量范数意义下一致收敛于问题(1.1)，(1.2)，(1.3)的解 $u(x,t)$，即有下面的误差估计

$$\|u(x,t)-u^d(x,t)\|_s^*\leqslant C(\Delta x)^{1/4}\qquad (s=0,1,\cdots,J) \tag{6.10}$$

其中C为与ε，Δx无关的正常数，范数$\|\cdot\|_s^*$定义如定理5.1。

证 当$\varepsilon^2\leqslant\Delta x$时，利用估计式(6.9)，当$\varepsilon^2\geqslant\Delta x$时，利用估计式(6.4)立刻得(6.10)。

注 条件(4.1)是由于我们在t方向采用完全指数型拟合所致。如果我们在t方向采用其它的差分近似，则条件(4.1)有可能除去。

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Uniform Difference Scheme for a Singularly Perturbed Linear 2nd Order Hyperbolic Problem with Zeroth Order Reduced Equation

Su Yu-cheng Lin Ping

(*Nanjing University, Nanjing*)

Abstract

In this paper a singularly perturbed linear second order hyperbolic problem with zeroth order reduced equation is discussed. Firstly, an energy inequality of the solution and an estimate of the remainder term of the asymptotic solution are given. Then an exponentially fitted difference scheme is developed in an equidistant mesh. Finally, uniform convergence in small parameter is proved in the sense of discrete energy norm.