

# 均布载荷下正交异性变厚度圆薄板大挠度问题\*

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(北京大学力学系, 1988年9月29日收到)

## 摘 要

本文导出了正交各向异性变厚度圆薄板大挠度问题的基本方程, 用修正迭代法求解了正交各向异性变厚度圆薄板在均布载荷下的大挠度问题。作为特例, 令 $\varepsilon=0$ , 则由本文结果得到的表达式与 J. Nowinski 用摄动法得到的正交各向异性等厚度圆薄板大挠度问题的解完全一致。

## 一、引 言

薄板大挠度问题, 在理论上和工程实践中都具有重要意义。我国学者对于各向同性圆薄板进行了一系列成功的研究工作。1947年, 钱伟长<sup>[1]</sup>用摄动法解决了均布载荷下圆薄板大挠度问题。1954年, 钱伟长、叶开沅<sup>[2]</sup>, 胡海昌<sup>[3]</sup>研究了在均布载荷和集中载荷作用下等厚度圆薄板的大挠度问题, 给出了在各种边条件下的解。1955年, 叶开沅<sup>[4]</sup>用小参数法给出了按指数规律变化的变厚度圆薄板的大挠度问题的解。1985年, 叶开沅<sup>[5]</sup>用修正迭代法给出了按幂级数展开的变厚度圆薄板在均布载荷下大挠度问题的解。

1960年, J. Nowinski<sup>[6]</sup>用摄动法研究了轴对称正交各向异性等厚度圆薄板在均布载荷下的大挠度问题。1967年, B. Banerjee<sup>[7]</sup>研究了在中心集中载荷作用下等厚度正交异性圆薄板大挠度问题。

本文用修正迭代法<sup>[8]</sup>解决了正交各向异性变厚度圆薄板在均布载荷作用下的大挠度问题, 并且将本文的结果与 J. Nowinski<sup>[6]</sup>及叶开沅<sup>[5]</sup>的结果进行了比较。作为特例, 令 $\varepsilon=0$ , 由本文结果可以得到正交各向异性等厚度圆薄板大挠度问题的解, 这个解和 J. Nowinski 的解完全一致。令 $\lambda=1$ ,  $\nu_{\theta r}=\nu$ , 由本文结果可以得到各向同性变厚度圆薄板大挠度问题的解, 它与叶开沅<sup>[5]</sup>的解完全一致。

## 二、正交各向异性变厚度圆薄板大挠度理论的基本方程

我们分别导出正交各向异性变厚度圆薄板大挠度理论的平衡方程和协调方程。

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### 1. 平衡方程

从变厚度圆薄板上截取一个微元, 作用于微元上的内力、内力矩如图1(a)、(b)所示。 $T_r, T_\theta$ 为薄膜力,  $M_r, M_\theta$ 为弯矩,  $Q_r$ 为横向剪力。由于轴对称性,  $T_{r\theta}=0, M_{r\theta}=0$ 。考虑径向力的平衡, 并略去横向载荷 $q$ 和横向剪力 $Q_r$ 作用方向的改变, 可以导出

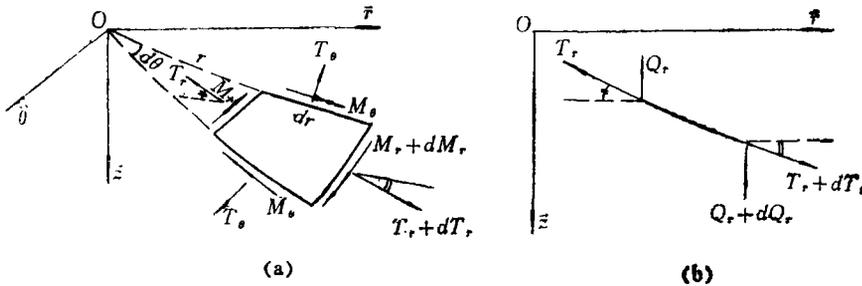


图 1

$$T_\theta = \frac{d}{dr} (rT_r) = r \frac{dT_r}{dr} + T_r \quad (2.1)$$

由 $z$ 方向力的平衡, 得出

$$rT_r \frac{dw}{dr} + rQ_r + \frac{qr^2}{2} = 0 \quad (2.2)$$

其中 $w$ 为板的挠度,  $q$ 为横向均布载荷。考虑环向力矩的平衡, 可得

$$rQ_r = \frac{d}{dr} (rM_r) - M_\theta \quad (2.3)$$

对于轴对称正交各向异性变厚度圆薄板, 有

$$M_r = -D_r \left( r \frac{d^2w}{dr^2} + \nu_{\theta r} \frac{dw}{dr} \right), \quad M_\theta = -D_\theta \left( \nu_{r\theta} \frac{d^2w}{dr^2} + \frac{1}{r} \frac{dw}{dr} \right) \quad (2.4)$$

其中 $\nu_{r\theta}, \nu_{\theta r}$ 为Poisson比,  $D_r, D_\theta$ 为弯曲刚度:

$$D_r = \frac{E_r h^3(r)}{12(1-\nu_{r\theta}\nu_{\theta r})}, \quad D_\theta = \frac{E_\theta h^3(r)}{12(1-\nu_{r\theta}\nu_{\theta r})}$$

这里 $h(r)$ 为板的厚度,  $E_r, E_\theta$ 分别为沿径向和沿环向的弹性模量。将(2.4)代入(2.3), 再代入(2.2), 导出

$$\begin{aligned} D_r \left( r \frac{d^3w}{dr^3} + \frac{d^2w}{dr^2} + \nu_{\theta r} \frac{d^2w}{dr^2} \right) - D_\theta \left( \nu_{r\theta} \frac{d^2w}{dr^2} + \frac{1}{r} \frac{dw}{dr} \right) \\ + \frac{dD_r}{dr} \left( r \frac{d^2w}{dr^2} + \nu_{\theta r} \frac{dw}{dr} \right) = rT_r \frac{dw}{dr} + \frac{qr^2}{2} \end{aligned} \quad (2.5)$$

### 2. 协调方程

在极坐标系中, 轴对称大挠度问题的 Green 应变为

$$\varepsilon_r^0 = \frac{du_r}{dr} + \frac{1}{2} \left( \frac{dw}{dr} \right)^2, \quad \varepsilon_\theta^0 = \frac{1}{r} \left( u_r + \frac{\partial u_\theta}{\partial \theta} \right) + \frac{1}{2r^2} \left( \frac{\partial w}{\partial \theta} \right)^2 = \frac{u_r}{r}, \quad \gamma_{r\theta}^0 = 0$$

其中  $\varepsilon_r^0$ ,  $\varepsilon_\theta^0$ ,  $\gamma_{r\theta}^0$  为中面的应变分量,  $u_r$ ,  $u_\theta$  分别为中面在径向和环向的位移分量。由此不难得到协调方程

$$\varepsilon_r^0 = \frac{d}{dr} (\varepsilon_\theta^0 r) + \frac{1}{2} \left( \frac{dw}{dr} \right)^2 \quad (2.6)$$

对于正交各向异性板, 有

$$T_r = \frac{E_r h}{1 - \nu_{r\theta} \nu_{\theta r}} \varepsilon_r^0 + \frac{\nu_{r\theta} E_\theta h}{1 - \nu_{r\theta} \nu_{\theta r}} \varepsilon_\theta^0, \quad T_\theta = \frac{\nu_{\theta r} E_r h}{1 - \nu_{r\theta} \nu_{\theta r}} \varepsilon_r^0 + \frac{E_\theta h}{1 - \nu_{r\theta} \nu_{\theta r}} \varepsilon_\theta^0$$

由此解出  $\varepsilon_r^0$ ,  $\varepsilon_\theta^0$  代入(2.6), 并利用(2.1)式及关系式  $\nu_{r\theta}/E_r = \nu_{\theta r}/E_\theta$ , 可得

$$\begin{aligned} r \frac{d^2}{dr^2} (rT_r) + \frac{d}{dr} (rT_r) - \lambda^2 T_r \\ = \frac{r}{h} \frac{dh}{dr} \frac{d}{dr} (rT_r) - \frac{\nu_{\theta r}}{h} \frac{dh}{dr} rT_r - \frac{1}{2} E_\theta h \left( \frac{dw}{dr} \right)^2 \end{aligned} \quad (2.7)$$

其中  $\lambda^2 = E_\theta/E_r = \nu_{\theta r}/\nu_{r\theta}$ ;  $E_r$ ,  $E_\theta$  分别为沿径向和沿环向的弹性模量。

方程(2.5)、(2.7)为正交异性变厚度圆薄板大挠度问题的基本方程组。

由于篇幅所限, 本文仅考虑具有固定夹紧边的变厚度圆薄板大挠度问题。边界条件为

$$\left. \begin{aligned} r=a \text{ 处 } \quad w = \frac{dw}{dr} = 0 \\ u_r = 0, \text{ 即 } \frac{d}{dr} (rT_r) - \nu_{\theta r} T_r = 0 \\ r=0 \text{ 处 } \quad \frac{dw}{dr}, T_r \text{ 取有限值} \end{aligned} \right\} \quad (2.8)$$

板厚度的变化规律可以表达为

$$h(r) = h_0 \left( 1 + \varepsilon_1 \frac{r}{a} + \varepsilon_2 \frac{r^2}{a^2} + \varepsilon_3 \frac{r^3}{a^3} + \dots \right)$$

其中  $h_0$  为板中心处的厚度,  $\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots$  为变厚度参数, 且  $|\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \dots| < 1$ 。为了简便, 在级数展开式中仅取两项, 即仅考虑厚度为线性变化的正交各向异性圆薄板:

$$h(r) = h_0 (1 + \varepsilon r/a) \quad |\varepsilon| < 1$$

不难看到, 本文的解法很容易推广到厚度为非线性变化的情况。

### 三、解 法

引进无量纲参数:

$$\begin{aligned} \rho = \frac{r}{a}, \quad x = \frac{w}{h_0}, \quad Q = \frac{6(1 - \nu_{r\theta} \nu_{\theta r}) a^4 q}{E_r h_0^4} \\ f = \frac{12(1 - \nu_{r\theta} \nu_{\theta r}) \rho a^2}{E_r h_0^3} T_r, \quad \tau = \frac{12(1 - \nu_{r\theta} \nu_{\theta r}) \rho a^2}{E_r h_0^3} T_\theta \end{aligned}$$

将它们代入基本方程(2.5)、(2.7), 并记  $\kappa = dx/d\rho$ , 则可以导出均布载荷下正交各向异性变厚度圆薄板大挠度问题的无量纲方程组:

$$(1+\varepsilon\rho)^3 \left[ \rho \frac{d^2\kappa}{d\rho^2} + \frac{d\kappa}{d\rho} - \lambda^2 \frac{\kappa}{\rho} \right] = t\kappa + \rho^2 Q - 3\varepsilon(1+\varepsilon\rho)^2 \left( \rho \frac{d\kappa}{d\rho} + \nu_{\theta r} \kappa \right) \quad (3.1)$$

$$\rho \frac{d^2 t}{d\rho^2} + \frac{dt}{d\rho} - \lambda^2 \frac{t}{\rho} = \frac{\varepsilon}{1+\varepsilon\rho} \left( \rho \frac{dt}{d\rho} - \nu_{\theta r} t \right) - 6(\lambda^2 - \nu_{\theta r}^2)(1+\varepsilon\rho)\kappa^2 \quad (3.2)$$

边界条件(2.8)变为

$$\left. \begin{array}{l} \rho=1 \text{ 处} \quad x=\kappa=0, \quad \frac{dt}{d\rho} - \nu_{\theta r} \frac{t}{\rho} = 0 \\ \rho=0 \text{ 处} \quad \kappa, t \text{ 取有限值} \end{array} \right\} \quad (3.3)$$

我们用修正迭代法, 在边界条件(3.3)下, 求解方程组(3.1)和(3.2). 令算子

$$L(\quad) = \rho \frac{d^2}{d\rho^2} + \frac{d}{d\rho} - \frac{\lambda^2}{\rho}$$

则修正迭代法的求解过程为

$$L(\kappa_1) = Q\rho^2$$

$$L(t_1) = -6(\lambda^2 - \nu_{\theta r}^2)\kappa_1^2(1+\varepsilon\rho)$$

$$L(\kappa_2) = Q\rho^2 + t_1\kappa_1 - \varepsilon\rho(3+3\varepsilon\rho+\varepsilon^2\rho^2)L(\kappa_1)$$

$$-3\varepsilon(1+\varepsilon\rho)^2 \left( \rho \frac{d\kappa_1}{d\rho} + \nu_{\theta r} \kappa_1 \right)$$

$$L(t_2) = \frac{\varepsilon}{1+\varepsilon\rho} \left( \rho \frac{dt_1}{d\rho} - \nu_{\theta r} t_1 \right) - 6(\lambda^2 - \nu_{\theta r}^2)(1+\varepsilon\rho)\kappa_1^2$$

$$L(\kappa_3) = Q\rho^2 + t_2\kappa_2 - \varepsilon\rho(3+3\varepsilon\rho+\varepsilon^2\rho^2)L(\kappa_2)$$

$$-3\varepsilon(1+\varepsilon\rho)^2 \left( \rho \frac{d\kappa_2}{d\rho} + \nu_{\theta r} \kappa_2 \right)$$

$$L(t_3) = \frac{\varepsilon}{1+\varepsilon\rho} \left( \rho \frac{dt_2}{d\rho} - \nu_{\theta r} t_2 \right) - 6(\lambda^2 - \nu_{\theta r}^2)(1+\varepsilon\rho)\kappa_2^2$$

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### 1. 一阶近似

$$\left\{ \begin{array}{l} L(\kappa_1) = Q\rho^2 \end{array} \right. \quad (3.4)$$

$$\left\{ \begin{array}{l} L(t_1) = -6(\lambda^2 - \nu_{\theta r}^2)\kappa_1^2(1+\varepsilon\rho) \end{array} \right. \quad (3.5)$$

相应的边界条件是

$$\left. \begin{array}{l} \rho=1 \text{ 时} \quad x_1=\kappa_1=0, \quad \frac{dt_1}{d\rho} - \nu_{\theta r} \frac{t_1}{\rho} = 0 \\ \rho=0 \text{ 时} \quad \kappa_1, t_1 \text{ 取有限值} \end{array} \right\} \quad (3.6)$$

在边界条件(3.6)下解方程(3.4), 得到

$$\kappa_1 = \frac{dx_1}{d\rho} = \frac{Q}{9-\lambda^2} (\rho^3 - \rho^\lambda) \quad (3.7)$$

$$x_1 = \frac{Q}{4(9-\lambda^2)(\lambda+1)} [(\lambda+1)\rho^4 - 4\rho^{\lambda+1} + 3 - \lambda] \quad (3.8)$$

注意, 这里限定  $\lambda > 0$ ,  $\lambda^2 \neq 9$ . 对于  $\lambda^2 = 9$  的情况, 将在下一节讨论.

令  $\rho=0$  时  $x_1 = x_0$ , 则

$$Q = 4(\lambda+1)(\lambda+3)x_0$$

于是

$$\kappa_1 = \frac{4(\lambda+1)}{3-\lambda} x_0 (\rho^3 - \rho^\lambda) \quad (3.9)$$

$$x_1 = \frac{x_0}{3-\lambda} [(\lambda+1)\rho^4 - 4\rho^{\lambda+1} + 3 - \lambda] \quad (3.10)$$

将(3.9)代入方程(3.5)的右端, 在边条件(3.6)下解方程(3.5), 得到

$$t_1 = \gamma \rho^\lambda - \beta \left\{ \frac{\rho^7}{49-\lambda^2} - \frac{\rho^{\lambda+4}}{4(\lambda+2)} + \frac{\rho^{2\lambda+1}}{(3\lambda+1)(\lambda+1)} \right. \\ \left. + e \left[ \frac{\rho^3}{64-\lambda^2} - \frac{2\rho^{\lambda+5}}{5(2\lambda+5)} + \frac{\rho^{2(\lambda+1)}}{(3\lambda+2)(\lambda+2)} \right] \right\} \quad (3.11)$$

其中

$$\beta = 96 \frac{(\lambda^2 - \nu_{\theta r}^2)(\lambda+1)^2}{(3-\lambda)^2} x_0^2 \quad (3.12)$$

$$\gamma = \frac{\beta}{\lambda - \nu_{\theta r}} (A_0 + eA_1) \quad (3.13)$$

$$A_0 = \frac{7 - \nu_{\theta r}}{49 - \lambda^2} - \frac{\lambda + 4 - \nu_{\theta r}}{4(\lambda + 2)} + \frac{2\lambda + 1 - \nu_{\theta r}}{(3\lambda + 1)(\lambda + 1)} \quad (3.14)$$

$$A_1 = \frac{8 - \nu_{\theta r}}{64 - \lambda^2} - \frac{2(\lambda + 5 - \nu_{\theta r})}{5(2\lambda + 5)} + \frac{2\lambda + 2 - \nu_{\theta r}}{(3\lambda + 2)(\lambda + 2)} \quad (3.15)$$

## 2. 二阶近似

$$L(\kappa_2) = Q\rho^2 + t_1\kappa_1 - \varepsilon\rho(3 + 3\varepsilon\rho + \varepsilon^2\rho^2)L(\kappa_1) \\ - 3\varepsilon(1 + \varepsilon\rho)^2 \left( \rho \frac{d\kappa_1}{d\rho} + \nu_{\theta r}\kappa_1 \right) \quad (3.16)$$

$$L(t_2) = \frac{\varepsilon}{1 + \varepsilon\rho} \left( \rho \frac{dt_1}{d\rho} - \nu_{\theta r}t_1 \right) - 6(\lambda^2 - \nu_{\theta r}^2)(1 + \varepsilon\rho)\kappa_1^2 \quad (3.17)$$

相应的边界条件是

$$\left. \begin{aligned} \rho=1 \text{ 时} \quad & x_2 = \kappa_2 = 0, \quad \frac{dt_2}{d\rho} - \nu_{\theta r} \frac{t_2}{\rho} = 0 \\ \rho=0 \text{ 时} \quad & \kappa_2, t_2 \text{ 取有限值} \end{aligned} \right\} \quad (3.18)$$

在边界条件(3.18)下解方程(3.16), 可以得到

$$\begin{aligned}
 x_2 = & \frac{C}{\lambda+1}(\rho^{\lambda+1}-1) - A + \frac{Q}{4(9-\lambda^2)}\rho^4 \\
 & + 384 \frac{(\lambda^2 - \nu_{\theta r}^2)(\lambda+1)^3}{(3-\lambda)^3} x_3^3 \left\{ \frac{A_0 + \varepsilon A_1}{\lambda - \nu_{\theta r}} \left[ \frac{\rho^{\lambda+5}}{8(\lambda+2)(\lambda+5)} \right. \right. \\
 & \left. \left. - \frac{\rho^{2\lambda+2}}{2(\lambda+1)^2(3\lambda+1)} \right] + C_3 \rho^{12} + C_4 \rho^{\lambda+9} + C_5 \rho^{2\lambda+6} + C_6 \rho^{3\lambda+3} \right. \\
 & \left. + \varepsilon (C_7 \rho^{13} + C_8 \rho^{\lambda+10} + C_9 \rho^{2\lambda+7} + C_{10} \rho^{3\lambda+4}) \right\} - 12(\lambda+1)(\lambda+3) x_0 \varepsilon \\
 & \cdot \left( \frac{\rho^5}{5(16-\lambda^2)} + \varepsilon \frac{\rho^6}{6(25-\lambda^2)} + \varepsilon^2 \frac{\rho^7}{21(36-\lambda^2)} \right) - 12(\lambda+1) \\
 & \cdot \frac{3+\nu_{\theta r}}{3-\lambda} x_0 \varepsilon \left( \frac{\rho^5}{5(16-\lambda^2)} + \varepsilon \frac{\rho^6}{3(25-\lambda^2)} + \varepsilon^2 \frac{\rho^7}{7(36-\lambda^2)} \right) \\
 & + 12(\lambda+1) \frac{\lambda+\nu_{\theta r}}{3-\lambda} x_0 \varepsilon \left( \frac{\rho^{\lambda+2}}{(\lambda+2)(2\lambda+1)} + \varepsilon \frac{\rho^{\lambda+3}}{2(\lambda+1)(\lambda+3)} \right. \\
 & \left. + \varepsilon^2 \frac{\rho^{\lambda+4}}{3(\lambda+4)(2\lambda+3)} \right) \tag{3.19}
 \end{aligned}$$

其中

$$\begin{aligned}
 C_3 = & -\frac{1}{12(49-\lambda^2)(121-\lambda^2)} \\
 C_4 = & \frac{57+4\lambda-\lambda^2}{64(49-\lambda^2)(\lambda+2)(\lambda+4)(\lambda+9)} \\
 C_5 = & -\frac{3\lambda^2+8\lambda+9}{8(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+5)(3\lambda+1)(3\lambda+5)} \\
 C_6 = & \frac{1}{12(\lambda+1)^2(3\lambda+1)(2\lambda+1)} \\
 C_7 = & -\frac{1}{13(64-\lambda^2)(144-\lambda^2)} \\
 C_8 = & \frac{-2\lambda^2+10\lambda+153}{45(64-\lambda^2)(\lambda+10)(2\lambda+5)(2\lambda+9)} \\
 C_9 = & -\frac{6\lambda^2+26\lambda+33}{15(\lambda+2)^2(\lambda+6)(2\lambda+5)(2\lambda+7)(3\lambda+2)} \\
 C_{10} = & \frac{1}{(\lambda+2)(2\lambda+3)(3\lambda+2)(3\lambda+4)(4\lambda+3)}
 \end{aligned}$$

$$\begin{aligned}
C = & -\frac{Q}{(9-\lambda^2)} - 384x_0^2 \frac{(\lambda^2 - \nu_{\theta r}^2)(\lambda+1)^3}{(3-\lambda)^3} \left\{ \frac{A_0 + \varepsilon A_1}{\lambda - \nu_{\theta r}} \left[ \frac{1}{8(\lambda+2)} \right. \right. \\
& \left. \left. - \frac{1}{(\lambda+1)(3\lambda+1)} \right] + 12C_3 + (\lambda+9)C_4 + (\lambda+3)C_5 + 3(\lambda+1)C_6 \right. \\
& \left. + \varepsilon [13C_7 + (\lambda+10)C_8 + (2\lambda+7)C_9 + (3\lambda+4)C_{10}] \right\} + 12(\lambda+1) \\
& \cdot (\lambda+3)x_0\varepsilon \left( \frac{1}{16-\lambda^2} + \frac{\varepsilon}{25-\lambda^2} + \frac{\varepsilon^2}{3(36-\lambda^2)} \right) + 12\frac{\lambda+1}{3-\lambda}(3+\nu_{\theta r})x_0\varepsilon \\
& \cdot \left( \frac{1}{16-\lambda^2} + \frac{2\varepsilon}{25-\lambda^2} + \frac{\varepsilon^2}{36-\lambda^2} \right) - 12\frac{\lambda+1}{3-\lambda}(\lambda+\nu_{\theta r})x_0\varepsilon \left( \frac{1}{2\lambda+1} \right. \\
& \left. + \frac{\varepsilon}{2(\lambda+1)} + \frac{\varepsilon^2}{3(2\lambda+3)} \right) \tag{3.20}
\end{aligned}$$

$$\begin{aligned}
A = & \frac{Q}{4(9-\lambda^2)} + 384x_0^2 \frac{(\lambda^2 - \nu_{\theta r}^2)(\lambda+1)^3}{(3-\lambda)^3} \left\{ \frac{A_0 + \varepsilon A_1}{\lambda - \nu_{\theta r}} \left[ \frac{1}{8(\lambda+2)(\lambda+5)} \right. \right. \\
& \left. \left. - \frac{1}{2(\lambda+1)^2(3\lambda+1)} \right] + C_3 + C_4 + C_5 + C_6 + \varepsilon(C_7 + C_8 + C_9 + C_{10}) \right\} \\
& - 12(\lambda+1)(\lambda+3)x_0\varepsilon \left( \frac{1}{5(16-\lambda^2)} + \frac{\varepsilon}{6(25-\lambda^2)} + \frac{\varepsilon^2}{21(36-\lambda^2)} \right) \\
& - 12\frac{\lambda+1}{3-\lambda}(3+\nu_{\theta r})x_0\varepsilon \left( \frac{1}{5(16-\lambda^2)} + \frac{\varepsilon}{3(25-\lambda^2)} + \frac{\varepsilon^2}{7(36-\lambda^2)} \right) \\
& + 12\frac{\lambda+1}{3-\lambda}(\lambda+\nu_{\theta r})x_0\varepsilon \left( \frac{1}{(\lambda+2)(2\lambda+1)} + \frac{\varepsilon}{2(\lambda+1)(\lambda+3)} \right. \\
& \left. + \frac{\varepsilon^2}{3(\lambda+4)(2\lambda+3)} \right) \tag{3.21}
\end{aligned}$$

式(3.19)近似表达了均布载荷作用下正交各向异性变厚度圆薄板大挠度问题的无量纲挠度。通过与 J. Nowinski 的结果比较, 可知二阶近似解(3.19), 已达到较好的精度。

令  $\rho=0$  时  $x_2=x_0$ , 则由式(3.19)可以导出无量纲中心挠度  $x_0$  与无量纲载荷  $Q$  的关系:

$$\begin{aligned}
Q = & 4(\lambda+1)(\lambda+3)x_0 \left\{ 1 + 12 \frac{\varepsilon}{3-\lambda} \left[ \frac{12+\nu_{\theta r}-\lambda^2}{5(\lambda+4)} - \frac{\lambda+\nu_{\theta r}}{(\lambda+2)(2\lambda+1)} \right. \right. \\
& \left. \left. + \frac{15+2\nu_{\theta r}-\lambda^2}{6(\lambda+5)} \varepsilon - \frac{\lambda+\nu_{\theta r}}{(\lambda+1)(\lambda+3)} \varepsilon + \frac{18+3\nu_{\theta r}-\lambda^2}{21(\lambda+6)} \varepsilon^2 \right. \right. \\
& \left. \left. - \frac{\lambda+\nu_{\theta r}}{(\lambda+4)(2\lambda+3)} \varepsilon^2 \right] \right\} - 1536 \frac{(\lambda^2 - \nu_{\theta r}^2)(\lambda+1)^3(\lambda+3)}{(3-\lambda)^3} x_0^2 \left\{ \frac{A_0 + \varepsilon A_1}{\lambda - \nu_{\theta r}} C_0 \right. \\
& \left. + (11-\lambda)C_3 + 8C_4 + (\lambda+5)C_5 + 2(\lambda+1)C_6 + \varepsilon [(12-\lambda)C_7 \right.
\end{aligned}$$

$$+9C_8+(\lambda+6)C_9+(2\lambda+3)C_{10} \Big\} \quad (3.22)$$

#### 四、结论与讨论

##### 1. 与 J. Nowinski 等厚度解的比较

为了与 J. Nowinski 的正交异性等厚度圆薄板的解相比较, 令  $\varepsilon=0$ , 将(3.19)式重新整理, 可以得到

$$x_2(\rho) = x_0\delta_1 + x_0^3\delta_3 \quad (4.1)$$

其中

$$\begin{aligned} \delta_1 &= \frac{1}{3-\lambda} [(\lambda+1)\rho^4 - 4\rho^{\lambda+1} + 3 - \lambda] \\ \delta_3 &= 384 \frac{(\lambda^2 - \nu_{\theta r}^2)(\lambda+1)^3}{(3-\lambda)^4} \left\{ B_1\rho^{\lambda+1} - (\lambda-3)\rho^4 \left[ C_1 \left( \rho^{\lambda+1} + \frac{4}{\lambda-3} \right) \right. \right. \\ &\quad \left. \left. + C_2 \left( \rho^{2\lambda-2} + \frac{\lambda+1}{\lambda-3} \right) + C_3 \left( \rho^8 - \frac{\lambda-11}{\lambda-3} \right) + C_4 \left( \rho^{\lambda+5} + \frac{8}{\lambda-3} \right) \right. \right. \\ &\quad \left. \left. + C_5 \left( \rho^{2\lambda+2} + \frac{\lambda+5}{\lambda-3} \right) + C_6 \left( \rho^{3\lambda-1} + \frac{2\lambda+2}{\lambda-3} \right) \right] \right\} \end{aligned}$$

$$C_1 = \frac{A_2}{8(\lambda+2)(\lambda+5)}, \quad C_2 = -\frac{A_2}{2(\lambda+1)^2(3\lambda+1)}$$

$$A_2 = \frac{A_0}{\lambda - \nu_{\theta r}} = \frac{1}{\lambda - \nu_{\theta r}} \left[ \frac{7 - \nu_{\theta r}}{64 - \lambda^2} - \frac{\lambda + 4 - \nu_{\theta r}}{4(\lambda+2)} + \frac{2\lambda + 1 - \nu_{\theta r}}{(3\lambda+1)(\lambda+1)} \right]$$

$$B_1 = (\lambda+1)C_1 + 2(\lambda-1)C_2 + 8C_3 + (\lambda+5)C_4 + 2(\lambda+1)C_5 + (3\lambda-1)C_6$$

这里得到的结果与 J. Nowinski 用摄动法求得的无量纲挠度表达式完全一致<sup>〔8〕,〔9〕</sup>。(顺便指出,〔8〕中将  $C_3(\rho^8 - (\lambda-11)/(\lambda-3))$  误写为  $C_3(\rho^8 - (\lambda-1)/(\lambda-3))$ 。)

令  $\varepsilon=0$ , 由(3.22)可以得到

$$Q = 4(\lambda+1)(\lambda+3)x_0 + 4B_2(9-\lambda^2)x_0^3 = \omega_1 x_0 + \omega_3 x_0^3 \quad (4.2)$$

式中

$$\omega_1 = 4(\lambda+1)(\lambda+3), \quad \omega_3 = 4B_2(9-\lambda^2)$$

$$\begin{aligned} B_2 &= 384 \frac{(\lambda^2 - \nu_{\theta r}^2)(\lambda+1)^3}{(3-\lambda)^4} [A_2 C_0 + (11-\lambda)C_3 + 8C_4 \\ &\quad + (\lambda+5)C_5 + 2(\lambda+1)C_6] \end{aligned}$$

$$C_0 = \frac{2\lambda^2 - 3\lambda - 9}{2(\lambda+1)(\lambda+2)(\lambda+5)(3\lambda+1)}$$

只要注意到  $A_2 C_0 = 4C_1 + C_2(\lambda+1)$ , 可知(4.2)式与 J. Nowinski 得到的中心挠度与载荷的关系式完全一致<sup>〔8〕</sup>。

## 2. 与各向同性变厚度解的比较

为了得到各向同性变厚度圆薄板大挠度问题的解, 只须令 $\lambda=1$ ,  $\nu_{\theta r}=\nu$ . 此时, 由(3.8)式得到

$$x_1 = Q(\rho^4 - 2\rho^2 + 1)/32$$

由(3.11)式得到

$$t_1 = -2(1-\nu^2)x_0^2 \left[ \rho^7 - 4\rho^5 + 6\rho^3 - \frac{5-3\nu}{1-\nu} \rho \right] \\ - \frac{32}{105} (1-\nu^2)x_0^2 e \left[ 5\rho^8 - 18\rho^6 + 21\rho^4 + \frac{8(2-\nu)}{1-\nu} \rho \right]$$

由(3.19)式得到

$$x_2 = \frac{Q}{32}(\rho^4 - 2\rho^2 + 1) - 16x_0 e \left[ \frac{1}{25}(2\rho^5 - 5\rho^3 + 3) + \frac{e}{24}(\rho^8 - 3\rho^2 + 2) \right. \\ \left. + \frac{e^2}{245}(2\rho^7 - 7\rho^2 + 5) \right] - 6x_0 e(3+\nu) \left[ \frac{1}{75}(2\rho^5 - 5\rho^3 + 3) \right. \\ \left. + \frac{e}{36}(\rho^8 - 3\rho^2 + 2) + \frac{e}{245}(2\rho^7 - 7\rho^2 + 5) \right] + 6x_0 e(1+\nu) \\ \cdot \left[ \frac{1}{9}(2\rho^3 - 3\rho^2 + 1) + \frac{e}{8}(\rho^4 - 2\rho^2 + 1) + \frac{e^2}{75}(2\rho^5 - 5\rho^3 + 3) \right] \\ - x_0^3 \left[ (1-\nu^2) \left( \frac{\rho^{12}}{180} - \frac{\rho^{10}}{20} + \frac{5}{24}\rho^8 + \frac{\rho^6}{3} + \frac{69}{180}\rho^2 - \frac{77}{360} \right) \right. \\ \left. + \frac{(1+\nu)(3\nu-5)}{36}(2\rho^8 - 9\rho^4) - \frac{1+\nu}{18}(12\rho^2 - 5) \right] - \frac{384}{315}x_0^3 e \\ \cdot (1-\nu^2) \left[ \frac{5}{11859}\rho^{13} - \frac{23}{1089}\rho^{11} + \frac{13}{189}\rho^9 - \frac{3}{35}\rho^7 + \frac{4016}{45045}\rho^5 \right. \\ \left. - \frac{415808}{3864861} \right] + \frac{64}{315}x_0^3 e(1+\nu) \left[ \frac{2-\nu}{3}\rho^8 - \frac{3(2-\nu)}{2}\rho^4 + 4\rho^2 - 4 \right]$$

这些结果与叶开沅在文[5]中得到的结果完全一致。

## 3. $\lambda^2=9$ 情况的讨论

在前述讨论中, 从物理意义上考虑, 限定 $\lambda > 0$ . 在解方程(3.4)和(3.5)时, 我们限定 $\lambda^2 \neq 9$ . 这是因为,  $\lambda=3$ 是方程(3.4)、(3.5)所对应的齐次常微分方程(Euler方程)的特征根.  $\lambda=3$ 时, 考虑到边界条件(3.6), 可以由方程(3.4)得到

$$\kappa_1 = \frac{Q}{6} \rho^3 \ln \rho, \quad x_1 = \frac{Q}{24} \left( \rho^4 \ln \rho - \frac{1}{4} \rho^4 + \frac{1}{4} \right)$$

由这些表达式出发, 采用类似于上述解法的步骤, 可以给出 $\lambda^2=9$ 时正交各向异性变厚度圆薄板大挠度问题的解。

本文用修正迭代法, 对于在均布载荷下, 在固定夹紧边界条件下的正交各向异性变厚度圆薄板, 给出了大挠度问题的二阶近似解。通过和正交异性等厚度圆薄板大挠度问题及各向同性变厚度圆薄板大挠度问题的已知解的比较, 证实了本文结果的可靠性。

对于工程中用到的正交各向异性变厚度圆薄板, 只须将相应的 $\lambda$ ,  $\nu_{\theta\theta}$ ,  $\epsilon$ 值代入(3.19)式及(3.22)式, 便可得到所需要的挠度表达式、中心挠度与载荷的关系式。

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## Large Deflection Problem of Thin Orthotropic Circular Plate with Variable Thickness under Uniform Pressure

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### Abstract

Basic equations for large deflection theory of thin orthotropic circular plate with variable thickness are derived in this paper. The modified iteration method is adopted to solve the large deflection problem of thin orthotropic circular plate with variable thickness under uniform pressure. If  $\epsilon=0$ , then the solution derived from the result in this paper coincides completely with the result given by J. Nowinski (using perturbation method) for solving large deflection problem of thin orthotropic circular plate with constant thickness under uniform pressure.