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弹性矩形薄板受迫振动的功的互等定理法(Ⅰ)——四边固定的矩形板和三边固定的矩形板

付宝连　　李　农

(燕山大学)　(洛阳工学院)

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摘　要

本文将功的互等定理法(MRT)推广于求解在简谐干扰力作用下矩形板的稳态响应．给出了各种边界条件矩形板的一系列封闭解并提供了一些有实用价值的图表．

功的互等定理法(MRT)是求解在各种简谐干扰力作用下的矩形板稳态响应的一个简便、通用的方法．

本文包括三部分：(Ⅰ)四边固定的矩形板和三边固定的矩形板；(Ⅱ)二邻边固定的矩形板；(Ⅲ)悬臂矩形板．

我们准备分三次陆续发表它们．

一、引　言

矩形板的受迫振动具有重要的实际意义．

世界上很多国家的学者和专家曾研究过这些问题并且对它们做了大量的工作．

Stanisic[1]利用傅里叶变换研究了沿其对角线受简谐分布弯矩作用的四边简支矩形板的受迫振动并给出了封闭解．

Gorman[2]采用Levy法研究了同一问题．

E．H．Dill和K．S．Pister[3]应用迭加原理研究了矩形板和连续矩形板的受迫振动．但是他们的方法不能处理板的角点位移不为零的情况．

E．A．Susemih和P．A．A.Laura[4]利用Galerkin方法研究了具有弹性边界的矩形板的受迫振动．然而，对较为复杂的边界条件，这一方法假设容许位移有时是困难的并且经常是不可能的．

B．K．Donaldson[5]提出了扩域法．它能解决某些复杂问题．但是我们认为，该法不是十分简便的．

B Новацкий[6]的工作有重大的价值．

我国科学家的工作有[7]和[8]，等等．

在论文[9]中，我们首次应用功的互等定理求解矩形板的弯曲问题。后来在论文[10～16]中，我们应用功的互等定理解决了一些其它问题并开始称这一方法为功的互等定理法(MRT)[14]。

在本文中，我们将推广功的互等定理法于求解矩形板的受迫振动问题。

首先，本文给出了基本系统的动力基本解。然后，在基本系统与实际系统之间应用功的互等定理，得到了实际系统的振幅挠曲面方程，最后，将三角级数转换成双曲函数并满足边界条件，我们得到最后结果。

计算表明，功的互等定理法对于求解在简谐干扰力作用下的矩形板稳态响应是一个简便、通用的方法。

二、基　本　原　理

如所周知，弹性薄板振动的控制方程为

$$\frac{\partial^4 W}{\partial x^4}+2\frac{\partial^4 W}{\partial x^2\partial y^2}+\frac{\partial^4 W}{\partial y^4}+\frac{\rho}{D}\frac{\partial^2 W}{\partial t^2}=\frac{F(x,y,t)}{D} \tag{2.1}$$

如果板是在简谐干扰力作用下，并且可忽略阻尼，那么我们可以假设

$$F(x,y,t)=q(x,y)\sin\omega t,\quad W(x,y,t)=w(x,y)\sin\omega t \tag{2.2}$$

这里$q(x,y)$是简谐干扰力的振幅，$w(x,y)$是振幅挠曲面方程。

将(2.2)代入(2.1)，我们有

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}-\lambda^2 w=\frac{q(x,y)}{D} \tag{2.3}$$

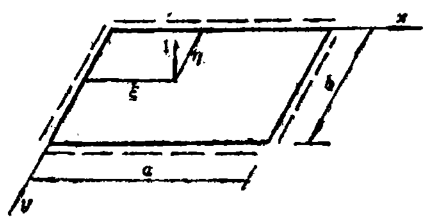

图　1

这里$\lambda^2=\rho\omega^2/D$，ρ是板单位面积的质量。

我们取一四边简支的矩形板作为基本系统。简谐干扰力是一简谐集中载荷，它作用在板的(ξ,η)点，如图1所示。在这种情况下，(2.3)成为

$$\frac{\partial^4 w_1}{\partial x^4}+2\frac{\partial^4 w_1}{\partial x^2\partial y^2}+\frac{\partial^4 w_1}{\partial y^4}-\lambda^2 w_1=\frac{\delta(x-\xi,y-\eta)}{D} \tag{2.4}$$

假设

$$w_1(x,y;\ \xi,\eta)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{2.5}$$

且将(2.5)代入(2.4)，则得

$$w_1(x,y;\ \xi,\eta)=\frac{4}{abD}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{K_{mn}}\sin k_m\xi\sin k_n\eta\sin k_m x\sin k_n y \tag{2.6}$$

这里　$k_m=\frac{m\pi}{a}$，$k_n=\frac{n\pi}{b}$，$K_{mn}=(k_m^2+k_n^2)^2-\lambda^2$

我们称(2.6)为基本系统的动力基本解或简称为基本解。

三、简谐干扰力作用下矩形板的功的互等定理法

我们假设，实际系统为一具有广义支撑边的并且四个角点的位移不为零的矩形板。在基本系统与实际系统之间应用功的互等定理，得

$$
\begin{aligned}
w(\xi,\eta)=&\int_0^a\int_0^b q(x,y)w_1(x,y;\ \xi,\eta)dxdy+\sum_{i=1}^m P_i w_1(x_i,y_i;\ \xi,\eta)\\
&+\sum_{j=1}^n M_{xj}\frac{\partial}{\partial x}w_1(x_j,y_j;\ \xi,\eta)+\sum_{k=1}^p M_{yk}\frac{\partial}{\partial y}w_1(x_k,y_k;\ \xi,\eta)\\
&-\left[\int_0^b M_x(y)\frac{\partial}{\partial x}w_1(x,y;\ \xi,\eta)dy\right]_0^a-\left[\int_0^a M_y(x)\frac{\partial}{\partial y}w_1(x,y;\ \xi,\eta)dx\right]_0^b\\
&-\left[\int_0^b\left(\frac{\partial M_{x1}}{\partial x}-2\frac{\partial M_{xy1}}{\partial y}\right)_{(x,y;\xi,\eta)}w(y)dy\right]_0^a\\
&-\left[\int_0^a\left(\frac{\partial M_{y1}}{\partial y}-2\frac{\partial M_{xy1}}{\partial x}\right)_{(x,y;\xi,\eta)}w(x)dx\right]_0^b\\
&+[2M_{xy1}(x,y;\xi,\eta)w(x,y)]_{0;b}^{a;b}-[2M_{xy1}(x,y;\xi,\eta)w(x,y)]_{0;0}^{a;0}
\end{aligned}\tag{3.1}
$$

这里，$q(x,y)$为简谐分布干扰力幅值，P_i为作用在i点的简谐集中干扰力幅值，M_{xj}为作用在j点在x方向上的简谐集中弯矩的幅值，M_{yk}为作用在k点在y方向上的简谐集中弯矩的幅值。

式(3.1)是被求的振幅方程。使其满足边界条件，我们将得到实际系统的解。

四、四边固定的矩形板

4.1 在简谐均布干扰力作用下的矩形板

首先，我们应用功的互等定理求解在简谐均布干扰力作用下四边固定矩形板的稳态响应。

实际系统如图2所示。解除四个固定边的弯曲约束并代以分布弯矩，我们得到图3所示的矩形板。

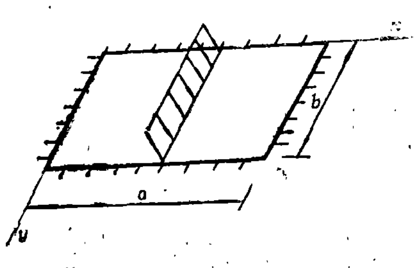

图 2

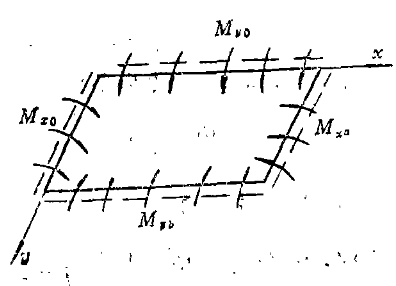

图 3

假设沿板的四个边的分布弯矩分别为

$$M_{y0}=M_{yb}=\sum_{m=1,3}^{\infty}A_m\sin\frac{m\pi x}{a},\quad M_{x0}=M_{xa}=\sum_{n=1,3}^{\infty}B_n\sin\frac{n\pi y}{b} \tag{4.1}$$

在图3所示实际系统与图1所示基本系统之间应用功的互等定理，我们得到

$$w(\xi,\eta)=\int_0^a\int_0^b qw_1dxdy+2\int_0^b M_{x0}\left(\frac{\partial w_1}{\partial x}\right)_{x=0}dy+2\int_0^a M_{y0}\left(\frac{\partial w_1}{dy}\right)_{y=0}dx \tag{4.2}$$

将(2.6)代入(4.2)并注意到附录Ⅰ

对于$\lambda<k_m^2$，$\lambda<k_n^2$，我们得到

$$w(\xi,\eta)=\frac{4q}{\pi D}\sum_{m=1,3}^{\infty}\frac{1}{m}\left\{\frac{1}{\alpha_m^2-\beta_m^2}\left[\frac{\text{ch}\alpha_m(b/2-\eta)}{\alpha_m^2\text{ch}\alpha_m(b/2)}-\frac{\text{ch}\beta_m(b/2-\eta)}{\beta_m^2\text{ch}\beta_m(b/2)}\right]+\frac{1}{\alpha_m^2\beta_m^2}\right\}\sin k_m\xi$$

$$\left(\text{或}\ \frac{4q}{\pi D}\sum_{n=1,3}^{\infty}\frac{1}{n}\left\{\frac{1}{\alpha_n^2-\beta_n^2}\left[\frac{\text{ch}\alpha_n(a/2-\xi)}{\alpha_n^2\text{ch}\alpha_n(a/2)}-\frac{\text{ch}\beta_n(a/2-\xi)}{\beta_n^2\text{ch}\beta_n(a/2)}\right]+\frac{1}{\alpha_n^2\beta_n^2}\right\}\sin k_n\eta\right)$$

$$+\frac{1}{D}\sum_{n=1,3}^{\infty}\frac{B_n}{\alpha_n^2-\beta_n^2}\left[-\frac{\text{ch}\alpha_n(a/2-\xi)}{\text{ch}\alpha_n(a/2)}+\frac{\text{ch}\beta_n(a/2-\xi)}{\text{ch}\beta_n(a/2)}\right]\sin k_n\eta$$

$$+\frac{1}{D}\sum_{m=1,3}^{\infty}\frac{A_m}{\alpha_m^2-\beta_m^2}\left[-\frac{\text{ch}\alpha_m(b/2-\eta)}{\text{ch}\alpha_m(b/2)}+\frac{\text{ch}\beta_m(b/2-\eta)}{\text{ch}\beta_m(b/2)}\right]\sin k_m\xi \tag{4.3}$$

这里，$\alpha_m=\sqrt{k_m^2+\lambda}$，$\beta_m=\sqrt{k_m^2-\lambda}$ 及 $\alpha_n=\sqrt{k_n^2+\lambda}$，$\beta_n=\sqrt{k_n^2-\lambda}$。对于 $\lambda>k_m^2$，$\lambda>k_n^2$，我们有

$$w(\xi,\eta)=\frac{4q}{\pi D}\sum_{m=1,3}^{\infty}\frac{1}{m}\left\{\frac{1}{\alpha_m^2+\beta_m'^2}\left[\frac{\text{ch}\alpha_m(b/2-\eta)}{\alpha_m^2\text{ch}\alpha_m(b/2)}+\frac{\cos\beta_m'(b/2+\eta)}{\beta_m'^2\cos\beta_m'(b/2)}\right]-\frac{1}{\alpha_m^2\beta_m'^2}\right\}\sin k_m\xi$$

$$\left(\text{或}\ \frac{4q}{\pi D}\sum_{n=1,3}^{\infty}\frac{1}{n}\left\{\frac{1}{\alpha_n^2+\beta_n'^2}\left[\frac{\text{ch}\alpha_n(a/2-\xi)}{\alpha_n^2\text{ch}\alpha_n(a/2)}+\frac{\cos\beta_n'(a/2-\xi)}{\beta_n'^2\cos\beta_n'(a/2)}\right]-\frac{1}{\alpha_n^2\beta_n'^2}\right\}\sin k_n\eta\right)$$

$$+\frac{1}{D}\sum_{n=1,3}^{\infty}\frac{B_n}{\alpha_n^2+\beta_n'^2}\left[-\frac{\text{ch}\alpha_n(a/2-\xi)}{\text{ch}\alpha_n(a/2)}+\frac{\cos\beta_n'(a/2-\xi)}{\cos\beta_n'(a/2)}\right]\sin k_n\eta$$

$$+\frac{1}{D}\sum_{m=1,3}^{\infty}\frac{A_m}{\alpha_m^2+\beta_m'^2}\left[-\frac{\text{ch}\alpha_m(b/2-\eta)}{\text{ch}\alpha_m(b/2)}+\frac{\cos\beta_m'(b/2-\eta)}{\cos\beta_m'(b/2)}\right]\sin k_m\xi \tag{4.4}$$

这里， $\beta_m'=\sqrt{\lambda-k_m^2}$，$\beta_n'=\sqrt{\lambda-k_n^2}$

让式(4.3)和(4.4)满足边界条件

$$\left[\frac{\partial w(\xi,\eta)}{\partial\xi}\right]_{\xi=0}=\left[\frac{\partial w(\xi,\eta)}{\partial\eta}\right]_{\eta=0}=0 \tag{4.5}$$

当$\lambda<k_m^2$，$\lambda<k_n^2$时，我们有

$$\frac{4q}{n\pi}\left(-\frac{\text{th}\alpha_n(a/2)}{\alpha_n}+\frac{\text{th}\beta_n(a/2)}{\beta_n}\right)+B_n\left(\alpha_n\text{th}\alpha_n\frac{a}{2}-\beta_n\text{th}\beta_n\frac{a}{2}\right)+\frac{8\lambda}{b}\sum_{m=1,3}^{\infty}\frac{A_mk_mk_n}{K_{mn}}=0$$

$$\frac{4q}{m\pi}\left(-\frac{\text{th}\alpha_m(b/2)}{\alpha_m}+\frac{\text{th}\beta_m(b/2)}{\beta_m}\right)+A_m\left(\alpha_m\text{th}\alpha_m\frac{b}{2}-\beta_m\text{th}\beta_m\frac{b}{2}\right)\frac{8\lambda}{a}\sum_{n=1,3}^{\infty}\frac{B_nk_mk_n}{K_{mn}}=0 \tag{4.6}$$

而当$\lambda>k_m^2$，$\lambda>k_n^2$时，我们有

$$\frac{4q}{n\pi}\left(-\frac{\text{th}\alpha_n(a/2)}{\alpha_n}+\frac{\text{tg}\beta_n'(a/2)}{\beta_n'}\right)+B_n\left(\alpha_n\text{th}\alpha_n\frac{a}{2}+\beta_n'\text{tg}\beta_n'\frac{a}{2}\right)+\frac{8\lambda}{b}\sum_{m=1,3}^{\infty}\frac{A_mk_mk_n}{K_{mn}}=0$$

$$\frac{4q}{m\pi}\left(-\frac{\text{th}\alpha_m(b/2)}{\alpha_m}+\frac{\text{tg}\beta_m'(b/2)}{\beta_m'}\right)+A_m\left(\alpha_m\text{th}\alpha_m\frac{b}{2}+\beta_m'\text{tg}\beta_m'\frac{b}{2}\right)+\frac{8\lambda}{a}\sum_{n=1,3}^{\infty}\frac{B_nk_mk_n}{K_{mn}}=0 \tag{4.7}$$

在上面二组无穷联立方程中，我们从每个A_m和B_n中各截取10个系数，组成20×20阶的线性方程组。在M6800微机上计算，我们得到图4表1。

表1　**固定边弯矩**(单位qa^2)**和最大振幅**(单位qa^4/D)　$a/b=2$

ω/ω_{11}	$x/a(y/b)$ / M	0.1	0.2	0.3	0.4	0.5	w_{max}
0.0	M_{x0}	−0.001998	−0.006459	−0.01058	−0.01331	−0.01426	0.000158
	M_{y0}	−0.006506	−0.01425	−0.01849	−0.02028	−0.02074	
0.3	M_{x0}	−0.002041	−0.006753	−0.01117	−0.01414	−0.01517	0.000175
	M_{y0}	−0.006804	−0.01524	−0.02004	−0.02217	−0.02275	
0.5	M_{x0}	−0.002131	−0.00742	−0.01254	−0.01604	−0.01726	0.000216
	M_{y0}	−0.007494	−0.01753	−0.02371	−0.02668	−0.02753	
0.8	M_{x0}	−0.002667	−0.01131	−0.02054	−0.02719	−0.02958	0.000472
	M_{y0}	−0.01436	−0.03119	−0.04617	−0.05486	−0.05768	

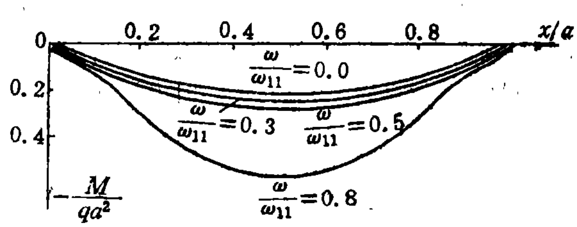

图4a　**固定边$y=0$弯矩分布曲线**$(a/b=2)$

为了证明结果的准确性，让我们比较我们的结果与Galerkin方法求得的结果，列于表2，3。

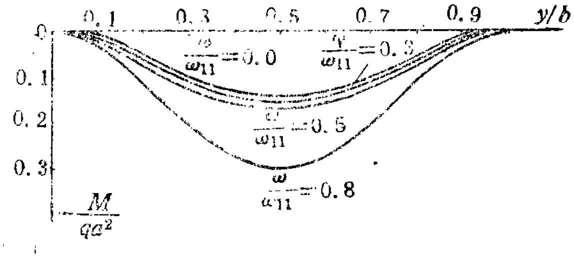

图4b　固定边$x=0$弯矩分布曲线($a/b=2$)

表2　　w_{max}**数值比较**　　单位qa^4/D

a/b 文献 ω/ω_{11}	1 Galerkin	1 本　文	2 Galerkin	2 本　文
0.3	0.00139	0.001397	0.00017	0.000175
0.5	0.00170	0.00171	0.00021	0.000216
0.8	0.00362	0.00361	0.00046	0.000472

表3　　**固定边弯矩数值比较**　　单位qa^2

a/b 文献 ω/ω_{11}	1 Galerkin	1 本　文	2 Galerkin	2 本　文
0.3	-0.0555	-0.0558	-0.0134 -0.0223	-0.01516 -0.02217
0.5	-0.0658	-0.06635	-0.0154 -0.0270	-0.01726 -0.0275
0.8	-0.1289	-0.1318	-0.0269 -0.0570	-0.0296 -0.0577
备　注	当$a/b=2$，上格为M_{x0}，下格为M_{y0}			

4.2　在简谐集中干扰力作用下的矩形板

如图5所示矩形板，一简谐集中干扰力作用在该板的(x_0,y_0)点。解除固定边弯曲约束并以弯矩M_{x0}，M_{xa}，M_{y0}和M_{yb}代替它们，我们得到如图6所示的实际系统。

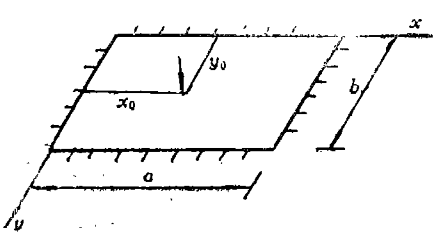

图　5

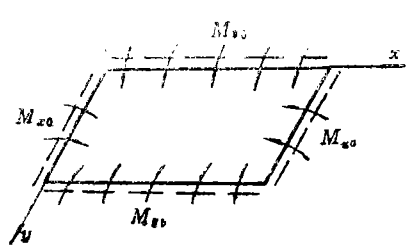

图　6

我们假设

$$\left.\begin{aligned} M_{y0}&=\sum_{m=1,2}^{\infty}A_{1m}\sin k_m x, \quad M_{yb}=\sum_{m=1,2}^{\infty}A_{2m}\sin k_m x \\ M_{x0}&=\sum_{n=1,2}^{\infty}B_{1n}\sin k_n y, \quad M_{xa}=\sum_{n=1,2}^{\infty}B_{2n}\sin k_n y \end{aligned}\right\} \tag{4.8}$$

在图1的基本系统与图6的实际系统之间应用功的互等定理，我们得到

$$w(\xi,\eta)=Pw_1(x_0,y_0;\xi,\eta)+\int_0^b M_{x0}\left(\frac{\partial w_1}{\partial x}\right)_{x=0}dy-\int_0^b M_{xa}\left(\frac{\partial w_1}{\partial x}\right)_{x=a}dy$$
$$+\int_0^a M_{y0}\left(\frac{\partial w_1}{\partial y}\right)_{y=0}dx-\int_0^a M_{yb}\left(\frac{\partial w_1}{\partial y}\right)_{y=b}dx \tag{4.9}$$

用与前节相似的方法，对于$\lambda<k_m^2$和$\lambda<k_n^2$，我们得

$$w(\xi,\eta)=w_a(\xi,\eta)+\frac{1}{D}\sum_{n=1,2}^{\infty}\frac{B_{1n}}{\alpha_n^2-\beta_n^2}\left[-\frac{\operatorname{sh}\alpha_n(a-\xi)}{\operatorname{sh}\alpha_n a}+\frac{\operatorname{sh}\beta_n(a-\xi)}{\operatorname{sh}\beta_n a}\right]\sin k_n\eta$$
$$-\frac{1}{D}\sum_{n=1,2}^{\infty}\frac{B_{2n}}{\alpha_n^2-\beta_n^2}\left[\frac{\operatorname{sh}\alpha_n\xi}{\operatorname{sh}\alpha_n a}-\frac{\operatorname{sh}\beta_n\xi}{\operatorname{sh}\beta_n a}\right]\sin k_n\eta$$
$$+\frac{1}{D}\sum_{m=1,2}^{\infty}\frac{A_{1m}}{\alpha_m^2-\beta_m^2}\left[-\frac{\operatorname{sh}\alpha_m(b-\eta)}{\operatorname{sh}\alpha_m b}+\frac{\operatorname{sh}\beta_m(b-\eta)}{\operatorname{sh}\beta_m b}\right]\sin k_m\xi$$
$$-\frac{1}{D}\sum_{m=1,2}^{\infty}\frac{A_{2m}}{\alpha_m^2-\beta_m^2}\left[\frac{\operatorname{sh}\alpha_m\eta}{\operatorname{sh}\alpha_m b}-\frac{\operatorname{sh}\beta_m\eta}{\operatorname{sh}\beta_m b}\right]\sin k_m\xi \tag{4.10}$$

这里对于$w_a(\xi,\eta)$，有四种表达式

$$w_a(\xi,\eta)=\frac{2P}{bD}\sum_{n=1,2}^{\infty}\frac{1}{\alpha_n^2-\beta_n^2}\left[-\frac{\operatorname{sh}\alpha_n(a-\xi)\operatorname{sh}\alpha_n x_0}{\alpha_n\operatorname{sh}\alpha_n a}\right.$$
$$\left.+\frac{\operatorname{sh}\beta_n(a-\xi)\operatorname{sh}\beta_n x_0}{\beta_n\operatorname{sh}\beta_n a}\right]\sin k_n y_0\sin k_n\eta \qquad (\xi\geqslant x_0) \tag{4.11a}$$

$$w_a(\xi,\eta)=\frac{2P}{bD}\sum_{n=1,2}^{\infty}\frac{1}{\alpha_n^2-\beta_n^2}\left[-\frac{\operatorname{sh}\alpha_n(a-x_0)\operatorname{sh}\alpha_n\xi}{\alpha_n\operatorname{sh}\alpha_n a}\right.$$
$$\left.+\frac{\operatorname{sh}\beta_n(a-x_0)\operatorname{sh}\beta_n\xi}{\beta_n\operatorname{sh}\beta_n a}\right]\sin k_n y_0\sin k_n\eta \qquad (\xi\leqslant x_0) \tag{4.11b}$$

$$w_a(\xi,\eta)=\frac{2P}{aD}\sum_{m=1,2}^{\infty}\frac{1}{\alpha_m^2-\beta_m^2}\left[-\frac{\operatorname{sh}\alpha_m(b-\eta)\operatorname{sh}\alpha_m y_0}{\alpha_m\operatorname{sh}\alpha_m b}\right.$$
$$\left.+\frac{\operatorname{sh}\beta_m(b-\eta)\operatorname{sh}\beta_m y_0}{\beta_m\operatorname{sh}\beta_m b}\right]\sin k_m x_0\sin k_m\xi \qquad (\eta\geqslant y_0) \tag{4.11c}$$

$$w_a(\xi,\eta)=\frac{2P}{aD}\sum_{m=1,2}^{\infty}\frac{1}{\alpha_m^2-\beta_m^2}\Big[-\frac{\text{sh}\alpha_m(b-y_0)\text{sh}\alpha_m\eta}{\alpha_m\text{sh}\alpha_m b}$$

$$+\frac{\text{sh}\beta_m(b-y_0)\text{sh}\beta_m\eta}{\beta_m\text{sh}\beta_m b}\Big]\sin k_m x_0\sin k_m\xi \qquad (\eta\leqslant y_0) \tag{4.11d}$$

对于$\lambda>k_m^2$和$\lambda>k_n^2$，我们只需在(4.10)和(4.11)中分别以$i\beta_m'$和$i\beta_n'$代替β_m和β_n，易于得到相应的表达式，故从略。

对于$\lambda<k_m^2$和$\lambda<k_n^2$，执行边界条件$(\partial w/\partial\xi)_{\xi=0}=(\partial w/\partial\xi)_{\xi=a}=(\partial w/\partial\eta)_{\eta=0}=(\partial w/\partial\eta)_{\eta=b}=0$，我们分别得到

$$\frac{2P}{b}\Big[-\frac{\text{sh}\alpha_n(a-x_0)}{\text{sh}\alpha_n a}+\frac{\text{sh}\beta_n(a-x_0)}{\text{sh}\beta_n a}\Big]\sin k_n y_0+B_{1n}(\alpha_n\text{cth}\alpha_n a-\beta_n\text{cth}\beta_n a)$$

$$-B_{2n}\left(\frac{\alpha_n}{\text{sh}\alpha_n a}-\frac{\beta_n}{\text{sh}\beta_n a}\right)+\frac{4\lambda}{b}\sum_{m=1,2}^{\infty}\frac{A_{1m}k_m k_n}{K_{mn}}-\frac{4\lambda}{b}\sum_{m=1,2}^{\infty}\frac{A_{2m}k_m k_n}{K_{mn}}\cos n\pi=0 \tag{4.12}$$

$$\frac{2P}{b}\left(\frac{\text{sh}\alpha_n x_0}{\text{sh}\alpha_n a}-\frac{\text{sh}\beta_n x_0}{\text{sh}\beta_n a}\right)\sin k_n y_0+B_{1n}\left(\frac{\alpha_n}{\text{sh}\alpha_n a}-\frac{\beta_n}{\text{sh}\beta_n a}\right)$$

$$-B_{2n}(\alpha_n\text{cth}\alpha_n a-\beta_n\text{ cth }\beta_n a)+\frac{4\lambda}{b}\sum_{m=1,2}^{\infty}\frac{A_{1m}k_m k_n}{K_{mn}}\cos m\pi$$

$$-\frac{4\lambda}{b}\sum_{m=1,2}^{\infty}\frac{A_{2m}k_m k_n}{K_{mn}}\cos m\pi\cos n\pi=0 \tag{4.13}$$

$$\frac{2P}{a}\Big[-\frac{\text{sh}\alpha_m(b-y_0)}{\text{sh}\alpha_m b}+\frac{\text{sh}\beta_m(b-y_0)}{\text{sh}\beta_m b}\Big]\sin k_m x_0+\frac{4\lambda}{a}\sum_{n=1,2}^{\infty}\frac{B_{1n}k_m k_n}{K_{mn}}$$

$$-\frac{4\lambda}{a}\sum_{n=1,2}^{\infty}\frac{B_{2n}k_m k_n}{K_{mn}}\cos m\pi+A_{1m}(\alpha_m\text{cth}\alpha_m b-\beta_m\text{cth}\beta_m b)-A_{2m}\left(\frac{\alpha_m}{\text{sh}\alpha_m b}-\frac{\beta_m}{\text{sh}\beta_m b}\right)=0 \tag{4.14}$$

$$\frac{2P}{a}\left(\frac{\text{sh}\alpha_m y_0}{\text{sh}\alpha_m b}-\frac{\text{sh}\beta_m y_0}{\text{sh}\beta_m b}\right)\sin k_m x_0+\frac{4\lambda}{a}\sum_{n=1,2}^{\infty}\frac{B_{1n}k_m k_n}{K_{mn}}\cos n\pi-\frac{4\lambda}{a}\sum_{n=1,2}^{\infty}\frac{B_{2n}k_m k_n}{K_{mn}}$$

$$\cdot\cos m\pi\cos n\pi+A_{1m}\left(\frac{\alpha_m}{\text{sh}\alpha_m b}-\frac{\beta_m}{\text{sh}\alpha_m b}\right)-A_{2m}(\alpha_m\text{cth}\,\alpha_m b-B_m\text{cth}\beta_m b)=0 \tag{4.15}$$

而对于$\lambda>k_m^2$和$\lambda>k_n^2$，相应的边界条件分别成为

$$\frac{2P}{b}\Big[-\frac{\text{sh}\alpha_n(a-x_0)}{\text{sh}\alpha_n a}+\frac{\sin\beta_n'(a-x_0)}{\sin\beta_n' a}\Big]\sin k_n y_0+B_{1n}(\alpha_n\text{cth}\alpha_n a-\beta_n'\text{ctg}\beta_n' a)$$

$$-B_{2n}\left(\frac{\alpha_n}{\text{sh}\alpha_n a}-\frac{\beta_n'}{\sin\beta_n' a}\right)+\frac{4\lambda}{b}\sum_{m=1,2}^{\infty}\frac{A_{1m}k_m k_n}{K_{mn}}-\frac{4\lambda}{b}\sum_{m=1,2}^{\infty}\frac{A_{2m}k_m k_n}{K_{mn}}\cos n\pi=0 \tag{4.16}$$

$$\frac{2P}{b}\left(\frac{\text{sh}\alpha_n x_0}{\text{sh}\alpha_n a}-\frac{\sin\beta_n' x_0}{\sin\beta_n' a}\right)\sin k_n y_0+B_{1n}\left(\frac{\alpha_n}{\text{sh}\alpha_n a}-\frac{\beta_n'}{\sin\beta_n' a}\right)-B_{2n}(\alpha_n\text{cth}\alpha_n a-\beta_n'\text{ctg}\beta_n' a)$$

$$+\frac{4\lambda}{b}\sum_{m=1,2}^{\infty}\frac{A_{1m}k_mk_n}{K_{mn}}\cos m\pi-\frac{4\lambda}{b}\sum_{m=1,2}^{\infty}\frac{A_{2m}k_mk_n}{K_{mn}}\cos m\pi\cos n\pi=0 \tag{4.17}$$

$$\frac{2P}{a}\left[-\frac{\mathrm{sh}\alpha_m(b-y_0)}{\mathrm{sh}\alpha_m b}+\frac{\sin\beta'_m(b-y_0)}{\sin\beta'_m b}\right]\sin k_m x_0+\frac{4\lambda}{a}\sum_{n=1,2}^{\infty}\frac{B_{1n}k_mk_n}{K_{mn}}$$

$$-\frac{4\lambda}{a}\sum_{n=1,2}^{\infty}\frac{B_{2n}k_mk_n}{K_{mn}}\cos m\pi+A_{1m}(\alpha_m\mathrm{cth}\alpha_m b-\beta'_m\,\mathrm{ctg}\,\beta'_m b)$$

$$-A_{2m}\left(\frac{\alpha_m}{\mathrm{sh}\alpha_m b}-\frac{\beta'_m}{\sin\beta'_m b}\right)=0 \tag{4.18}$$

$$\frac{2P}{a}\left(\frac{\mathrm{sh}\alpha_m y_0}{\mathrm{sh}\alpha_m b}-\frac{\sin\beta'_m y_0}{\sin\beta'_m b}\right)\sin k_m x_0+\frac{4\lambda}{a}\sum_{n=1,2}^{\infty}\frac{B_{1n}k_mk_n}{K_{mn}}\cos n\pi$$

$$-\frac{4\lambda}{a}\sum_{n=1,2}^{\infty}\frac{B_{2n}k_mk_n}{K_{mn}}\cos m\pi\cos n\pi+A_{1m}\left(\frac{\alpha_m}{\mathrm{sh}\alpha_m b}-\frac{\beta'_m}{\sin\beta'_m b}\right)$$

$$-A_{2m}(\alpha_m\mathrm{cth}\alpha_m b-\beta'_m\mathrm{ctg}\beta'_m b)=0 \tag{4.19}$$

当集中载荷作用在板的中点时，用微机M6800计算，我们得到相应的图7表4。

表4 **固定边弯矩**(单位P)**和最大振幅**(单位 Pa^2/D) $a/b=2$

ω/ω_{11}	$x/a(y/b)$ / M	0.1	0.2	0.3	0.4	0.5	w_{max}
0.0	M_{x0}	−0.000209	−0.003265	−0.008805	−0.01354	−0.01537	0.001804
	M_{y0}	−0.00322	−0.02315	−0.0658	−0.1298	−0.16750	
0.3	M_{x0}	−0.000099	−0.00440	−0.01128	−0.01707	−0.01930	0.001935
	M_{y0}	−0.00438	−0.02803	−0.07535	−0.14362	−0.1831	
0.5	M_{x0}	−0.000136	−0.00726	−0.01753	−0.02597	−0.02919	0.002249
	M_{y0}	−0.007424	−0.0402	−0.09841	−0.1761	−0.2195	
0.8	M_{x0}	−0.002795	−0.02901	−0.06338	−0.09054	−0.1007	0.004079
	M_{y0}	−0.02932	−0.1233	−0.2465	−0.3746	−0.4374	

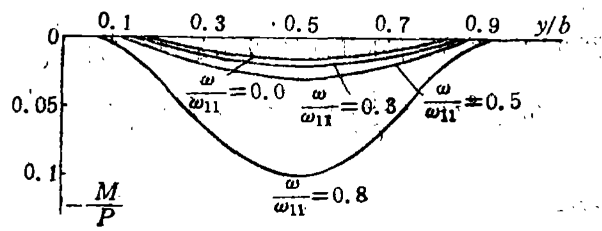

图7a 固定边$x=0$弯矩分布曲线$(a/b=2)$

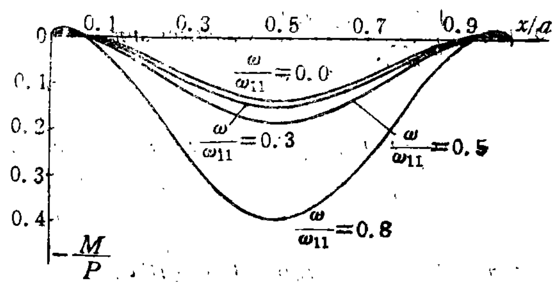

图7b　固定边$y=0$弯矩分布曲线($a/b=2$)

五、三边固定的矩形板

5.1　在简谐均布干扰力作用下三边固定一边自由的矩形板

现在让我们来考虑如图8所示的矩形板，解除固定边的弯曲约束，代以相应的分布弯矩，我们得到如图9所示的实际系统。

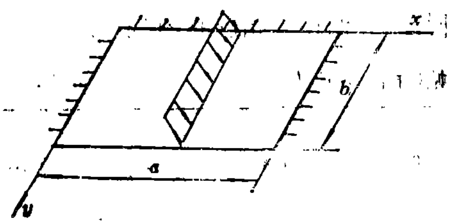

图　8

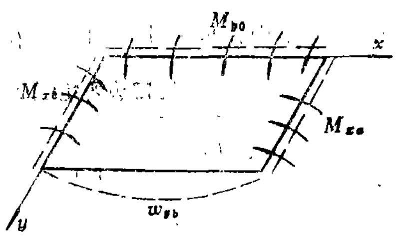

图　9

假设

$$
\left.
\begin{aligned}
M_{y0}(x) &= \sum_{m=1,3}^{\infty} A_m \sin k_m x \\
M_{x0}(y) = M_{xa}(y) &= \sum_{n=1,2}^{\infty} B_n \sin k_n y \\
W_{yb}(x) &= \sum_{m=1,3}^{\infty} C_m \sin k_m x
\end{aligned}
\right\}
\tag{5.1}
$$

在图1基本系统与图9实际系统之间应用功的互等定理，我们得到

$$
w(\xi,\eta)=\int_0^a\int_0^b q w_1\,dx dy+\int_0^a M_{y0}\left(\frac{\partial w_1}{\partial y}\right)_{y=0} dx+2\int_0^b M_{x0}\left(\frac{\partial w_1}{\partial x}\right)_{x=0} dy
$$

$$
-\int_0^a (V_{1y})_{y=b}\, w_{yb}\, dx \tag{5.2}
$$

对于$\lambda<k_m^2$和$\lambda<k_n^2$，利用与前节相同的方法，我们得到

$$w(\xi,\eta)=\frac{4q}{\pi D}\sum_{m=1,3}^{\infty}\frac{1}{m}\left\{\frac{1}{\alpha_m^2-\beta_m^2}\left[\frac{\operatorname{ch}\alpha_m(b/2-\eta)}{\alpha_m^2\operatorname{ch}\alpha_m(b/2)}-\frac{\operatorname{ch}\beta_m(b/2-\eta)}{\beta_m^2\operatorname{ch}\beta_m(b/2)}\right]\right.$$

$$\left.+\frac{1}{\alpha_m^2\beta_m^2}\right\}\sin k_m\xi$$

$$\left(\text{或 }\frac{4q}{\pi D}\sum_{n=1,3}^{\infty}\frac{1}{n}\left\{\frac{1}{\alpha_n^2-\beta_n^2}\left[\frac{\operatorname{ch}\alpha_n(a/2-\xi)}{\alpha_n^2\operatorname{ch}\alpha_n(a/2)}-\frac{\operatorname{ch}\beta_n(a/2-\xi)}{\beta_n^2\operatorname{ch}\beta_n(a/2)}\right]+\frac{1}{\alpha_n^2\beta_n^2}\right\}\sin k_n\eta\right)$$

$$+\frac{1}{D}\sum_{m=1,3}^{\infty}\frac{A_m}{\alpha_m^2-\beta_m^2}\left[-\frac{\operatorname{sh}\alpha_m(b-\eta)}{\operatorname{sh}\alpha_m b}+\frac{\operatorname{sh}\beta_m(b-\eta)}{\operatorname{sh}\beta_m b}\right]\sin k_m\xi$$

$$+\frac{1}{D}\sum_{n=1,2}^{\infty}\frac{B_n}{\alpha_n^2-\beta_n^2}\left[-\frac{\operatorname{ch}\alpha_n(a/2-\xi)}{\operatorname{ch}\alpha_n(a/2)}+\frac{\operatorname{ch}\beta_n(a/2-\xi)}{\operatorname{ch}\beta_n(a/2)}\right]\sin k_n\eta$$

$$+\sum_{m=1,3}^{\infty}\frac{C_m}{\alpha_m^2-\beta_m^2}\left\{\frac{[\alpha_m^2-k_m^2(2-\mu)]}{\operatorname{sh}\alpha_m b}\operatorname{sh}\alpha_m\eta\right.$$

$$\left.-\frac{[\beta_m^2-k_m^2(2-\mu)]}{\operatorname{sh}\beta_m b}\operatorname{sh}\beta_m\eta\right\}\sin k_m\xi \tag{5.3}$$

对于$\lambda>k_m^2$和$\lambda>k_n^2$，在(5.3)中令$\beta_m\to i\beta_m'$和$\beta_n\to i\beta_n'$，我们即可得相应的表达式，故从略。

对于$\lambda<k_m^2$和$\lambda<k_n^2$，让$w(\xi,\eta)$满足边界条件，我们分别得到

$$\frac{4q}{m\pi D}\left(-\frac{\operatorname{th}\alpha_m(b/2)}{\alpha_m}+\frac{\operatorname{th}\beta_m(b/2)}{\beta_m}\right)+\frac{A_m}{D}(\alpha_m\operatorname{cth}\alpha_m b-\beta_m\operatorname{cth}\beta_m b)$$

$$+\frac{8\lambda}{aD}\sum_{n=1,2}^{\infty}\frac{B_nk_mk_n}{K_{mn}}+C_m\left\{\frac{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]}{\operatorname{sh}\alpha_m b}-\frac{\beta_m[\beta_m^2-k_m^2(2-\mu)]}{\operatorname{sh}\beta_m b}\right\}=0 \tag{5.4}$$

$$\frac{2[1-(-1)^n]q}{n\pi D}\left(-\frac{\operatorname{th}\alpha_n(a/2)}{\alpha_n}+\frac{\operatorname{th}\beta_n(a/2)}{\beta_n}\right)+\frac{4\lambda}{bD}\sum_{m=1,3}^{\infty}\frac{A_mk_mk_n}{K_{mn}}$$

$$+B_n\left(\alpha_n\operatorname{th}\alpha_n\frac{a}{2}-\beta_n\operatorname{th}\beta_n\frac{a}{2}\right)+\frac{4\lambda}{b}\sum_{m=1,3}^{\infty}\frac{C_mk_mk_n[k_n^2+k_m^2(2-\mu)]}{K_{mn}}\cos n\pi=0 \tag{5.5}$$

$$\frac{4q}{m\pi D}\left\{\frac{[\alpha_m^2-k_m^2(2-\mu)]}{\alpha_m^2}\operatorname{th}\frac{\alpha_m b}{2}-\frac{[\beta_m^2-k_m^2(2-\mu)]}{\beta_m^2}\operatorname{th}\frac{\beta_m b}{2}\right\}$$

$$+\frac{A_m}{D}\left\{\frac{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]}{\operatorname{sh}\alpha_m b}-\frac{\beta_m[\beta_m^2-k_m^2(2-\mu)]}{\operatorname{sh}\beta_m b}\right\}$$

$$-\frac{8\lambda}{aD}\sum_{n=1,2}^{\infty}\frac{B_nk_mk_n[k_n^2+k_m^2(2-\mu)]}{K_{mn}}+C_m\{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]^2$$

$$\cdot \operatorname{cth} \alpha_m b-\beta_m[\beta_m^2-k_m^2(2-\mu)]^2 \operatorname{cth} \beta_m b\}=0 \tag{5.6}$$

而对于$\lambda>k_m^2$和$\lambda>k_n^2$，(5.4)～(5.6)分别变为

$$\frac{4q}{m\pi D}\left(-\frac{\operatorname{th} \alpha_m(b/2)}{\alpha_m}+\frac{\operatorname{tg} \beta_m'(b/2)}{\beta_m'}\right)+\frac{A_m}{D}(\alpha_m \operatorname{cth} \alpha_m b-\beta_m' \operatorname{ctg} \beta_m' b)$$

$$+\frac{8\lambda}{aD}\sum_{n=1,2}^{\infty}\frac{B_n k_m k_n}{K_{mn}}+C_m\left\{\frac{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]}{\operatorname{sh} \alpha_m b}+\frac{\beta_m'[\beta_m'^2+k_m^2(2-\mu)]}{\sin \beta_m' b}\right\}=0 \tag{5.7}$$

$$\frac{2[1-(-1)^n]q}{n\pi D}\left(-\frac{\operatorname{th} \alpha_n(a/2)}{\alpha_n}+\frac{\operatorname{tg} \beta_n'(a/2)}{\beta_n'}\right)+\frac{4\lambda}{bD}\sum_{m=1,3}^{\infty}\frac{A_m k_m k_n}{K_{mn}}$$

$$+B_n\left(\alpha_n \operatorname{th} \alpha_n \frac{a}{2}+\beta_n' \operatorname{tg}\beta_n' \frac{a}{2}\right)-\frac{4\lambda}{b}\sum_{m=1,3}^{\infty}\frac{C_m k_m k_n[k_n^2+k_m^2(2-\mu)]}{K_{mn}}\cos n\pi=0 \tag{5.8}$$

$$\frac{4q}{m\pi D}\left\{\frac{[\alpha_m^2-k_m^2(2-\mu)]}{\alpha_m^2}\operatorname{th}\frac{\alpha_m b}{2}+\frac{[\beta_m'^2+k_m^2(2-\mu)]}{\beta_m'^2}\operatorname{tg}\frac{\beta_m' b}{2}\right\}$$

$$+\frac{A_m}{D}\left\{\frac{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]}{\operatorname{sh} \alpha_m b}+\frac{\beta_m'[\beta_m'^2+k_m^2(2-\mu)]}{\sin \beta_m' b}\right\}$$

$$-\frac{8\lambda}{aD}\sum_{n=1,2}^{\infty}\frac{B_n k_m k_n[k_n^2+k_m^2(2-\mu)]}{K_{mn}}+C_m\{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]^2$$

$$\cdot \operatorname{cth} \alpha_m b-\beta_m'[\beta_m'^2+k_m^2(2-\mu)]^2 \operatorname{ctg} \beta_m' b\}=0 \tag{5.9}$$

(5.4)～(5.6)和(5.7)～(5.9)均为具有系数A_m，B_n和C_m的三组无穷联立方程。

我们必须指出，当矩形板具有自由边时，其解与μ有关。然而我们发现，μ对解的影响不是很大。这里取$\mu=1/6$并各取A_m，B_n和C_m10个系数。对于不同的关系ω/ω_{11}，我们给出固定边的弯矩幅值和自由边的挠度幅值如图10表5。

表 5　**固定边弯矩**（单位qa^2）**和自由边振幅**（单位qa^4/D）　$a/b=1$

ω/ω_{11}	$x/a(y/b)$ / $M(w)$	0.05	0.15	0.35	0.5	0.7	0.9	0.95
0.0	M_{x0}	−0.001481	−0.01689	−0.05122	−0.06388	−0.07334	−0.07346	−0.1178
	M_{y0}	−0.001435	−0.01642	−0.04814	−0.05648	−0.04191	−0.007784	−0.001435
	w_{yb}	0.000104	0.000725	0.002302	0.002777	0.001963	0.000363	0.000104
0.3	M_{x0}	−0.001452	−0.01754	−0.05458	−0.06877	−0.07998	−0.08088	−0.1303
	M_{y0}	−0.001391	−0.01702	−0.05102	−0.06008	−0.04427	−0.007943	−0.001391
	w_{yb}	0.000115	0.000806	0.002568	0.003101	0.002188	0.000402	0.000115
0.5	M_{x0}	−0.001382	−0.01904	−0.06244	−0.08034	−0.09589	−0.09888	−0.1608
	M_{y0}	−0.001284	−0.01838	−0.05768	−0.06847	−0.04973	−0.008299	−0.001284
	w_{yb}	0.000143	0.001003	0.003217	0.003892	0.002738	0.000499	0.000143
0.8	M_{x0}	−0.000951	−0.02814	−0.1118	−0.1543	−0.2014	−0.2216	−0.3719
	M_{y0}	−0.000516	−0.02639	−0.09853	−0.12010	−0.08308	−0.01025	−0.000516
	w_{yb}	0.000335	0.002363	0.007686	0.009332	0.006522	0.001173	0.000335

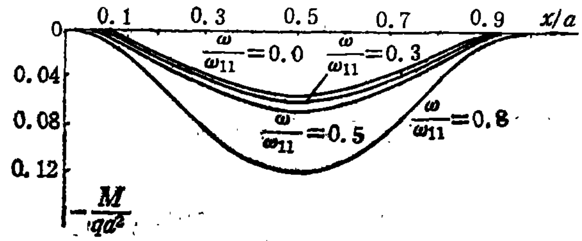

图10a　固定边$y=0$弯矩分布曲线($a/b=1$)

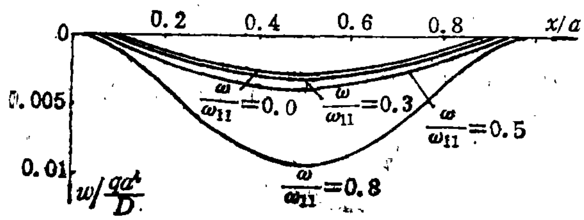

图10b　自由边$y=b$振幅曲线($a/b=1$)

5.2　在任意点上的简谐集中干扰力作用下具有三边固定一边自由的矩形板

我们考虑图11所示矩形板。解除三个固定边的弯曲约束代以M_{x0}，M_{xa}和M_{y0}，我们得到图12所示的实际系统。

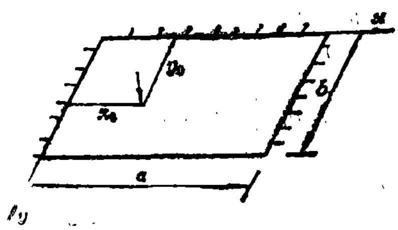

图　11

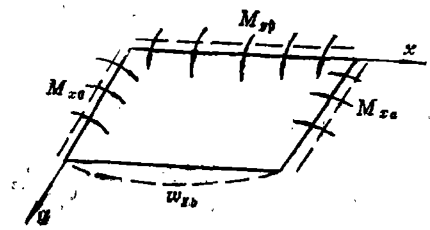

图　12

假设

$$\left.\begin{aligned}
M_{y0}(x)&=\sum_{m=1,2}^{\infty}A_m\sin k_m x,\quad M_{x0}(y)=\sum_{n=1,2}^{\infty}B_{1n}\sin k_n y\\
M_{xa}(y)&=\sum_{n=1,2}^{\infty}B_{2n}\sin k_n y,\quad w_{yb}(x)=\sum_{m=1,2}^{\infty}C_m\sin k_m x
\end{aligned}\right\}\tag{5.10}$$

在图1基本系统与图12实际系统之间应用功的互等定理，我们得到振幅挠曲面方程

$$\begin{aligned}
w(\xi,\eta)=&Pw_1(x_0,y_0;\ \xi,\eta)+\int_0^a M_{y0}\left(\frac{\partial w_1}{\partial y}\right)_{y=0}dx+\int_0^b M_{x0}\left(\frac{\partial w_1}{\partial x}\right)_{x=0}dy\\
&-\int_0^b M_{xa}\left(\frac{\partial w_1}{\partial x}\right)_{x=a}dy-\int_0^a (V_{1y})_{y=b}w_{yb}dx
\end{aligned}\tag{5.11}$$

让式(5.11)满足相应的边界条件，我们得到

$$\frac{2P}{aD}\left[-\frac{\operatorname{sh}\alpha_m(b-y_0)}{\operatorname{sh}\alpha_m b}+\frac{\operatorname{sh}\beta_m(b-y_0)}{\operatorname{sh}\beta_m b}\right]\sin k_m x_0+\frac{A_m}{D}(\alpha_m\operatorname{cth}\alpha_m b-\beta_m\operatorname{cth}\beta_m b)$$

$$+\frac{4\lambda}{aD}\sum_{n=1,2}^{\infty}\frac{B_{1n}k_m k_n}{K_{mn}}-\frac{4\lambda}{aD}\sum_{n=1,2}^{\infty}\frac{B_{2n}k_m k_n}{K_{mn}}\cos m\pi$$

$$+C_m\left\{\frac{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]}{\operatorname{sh}\alpha_m b}-\frac{\beta_m[\beta_m^2-k_m^2(2-\mu)]}{\operatorname{sh}\beta_m b}\right\}=0 \tag{5.12}$$

$$\frac{2P}{bD}\left[-\frac{\operatorname{sh}\alpha_n(a-x_0)}{\operatorname{sh}\alpha_n a}+\frac{\operatorname{sh}\beta_n(a-x_0)}{\operatorname{sh}\beta_n a}\right]\sin k_n y_0+\frac{4\lambda}{bD}\sum_{m=1,2}^{\infty}\frac{A_m k_m k_n}{K_{mn}}$$

$$+\frac{B_{1n}}{D}(\alpha_n\operatorname{cth}\alpha_n a-\beta_n\operatorname{cth}\beta_n a)-\frac{B_{2n}}{D}\left(\frac{\alpha_n}{\operatorname{sh}\alpha_n a}-\frac{\beta_n}{\operatorname{sh}\beta_n a}\right)$$

$$-\frac{4\lambda}{b}\sum_{m=1,2}^{\infty}\frac{C_m k_m k_n}{K_{mn}}[k_n^2+k_m^2(2-\mu)]\cos n\pi=0 \tag{5.13}$$

$$\frac{2P}{bD}\left[\frac{\operatorname{sh}\alpha_n x_0}{\operatorname{sh}\alpha_n a}-\frac{\operatorname{sh}\beta_n x_0}{\operatorname{sh}\beta_n a}\right]\sin k_n y_0+\frac{4\lambda}{bD}\sum_{m=1,2}^{\infty}\frac{A_m k_m k_n}{K_{mn}}\cos m\pi$$

$$+\frac{B_{1n}}{D}\left(\frac{\alpha_n}{\operatorname{sh}\alpha_n a}-\frac{\beta_n}{\operatorname{sh}\beta_n a}\right)-\frac{B_{2n}}{D}(\alpha_n\operatorname{cth}\alpha_n a-\beta_n\operatorname{cth}\beta_n a)$$

$$-\frac{4\lambda}{b}\sum_{m=1,2}^{\infty}\frac{C_m k_m k_n}{K_{mn}}[k_n^2+k_m^2(2-\mu)]\cos m\pi\cos n\pi=0 \tag{5.14}$$

$$\frac{2P}{aD}\left\{\frac{[\alpha_m^2-k_m^2(2-\mu)]}{\operatorname{sh}\alpha_m b}\operatorname{sh}\alpha_m y_0-\frac{[\beta_m^2-k_m^2(2-\mu)]}{\operatorname{sh}\beta_m b}\operatorname{sh}\beta_m y_0\right\}\sin k_m x_0$$

$$+\frac{A_m}{D}\left\{\frac{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]}{\operatorname{sh}\alpha_m b}-\frac{\beta_m[\beta_m^2-k_m^2(2-\mu)]}{\operatorname{sh}\beta_m b}\right\}$$

$$-\frac{4\lambda}{aD}\sum_{n=1,2}^{\infty}\frac{B_{1n}k_m k_n}{K_{mn}}[k_m^2+k_n^2(2-\mu)]\cos n\pi+\frac{4\lambda}{aD}\sum_{n=1,2}^{\infty}\frac{B_{2n}k_m k_n}{K_{mn}}$$

$$\cdot[k_m^2+k_n^2(2-\mu)]\cos m\pi\cos n\pi+C_m\{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]^2\operatorname{cth}\alpha_m b$$
$$-\beta_m[\beta_m^2-k_m^2(2-\mu)]^2\operatorname{cth}\beta_m b\}=0 \tag{5.15}$$

这里，$\lambda<k_m^2$和$\lambda<k_n^2$。

当$\lambda>k_m^2$和$\lambda>k_n^2$时，式(5.12)～(5.15)分别变成

$$\frac{2P}{aD}\left[-\frac{\operatorname{sh}\alpha_m(b-y_0)}{\operatorname{sh}\alpha_m b}+\frac{\sin\beta_m'(b-y_0)}{\sin\beta_m' b}\right]\sin k_m x_0+\frac{A_m}{D}(\alpha_m\operatorname{cth}\alpha_m b-\beta_m'\operatorname{ctg}\beta_m' b)$$

$$+\frac{4\lambda}{aD}\sum_{n=1,2}^{\infty}\frac{B_{1n}k_m k_n}{K_{mn}}-\frac{4\lambda}{aD}\sum_{n=1,2}^{\infty}\frac{B_{2n}k_m k_n}{K_{mn}}\cos m\pi$$

$$+C_m\left\{\frac{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]}{\operatorname{sh}\alpha_m b}+\frac{\beta_m'[\beta_m'^2+k_m^2(2-\mu)]}{\sin\beta_m' b}\right\}=0 \tag{5.16}$$

$$\frac{2P}{bD}\left[-\frac{\operatorname{sh}\alpha_n(a-x_0)}{\operatorname{sh}\alpha_n a}+\frac{\sin\beta_n'(a-x_0)}{\sin\beta_n' a}\right]\sin k_n y_0+\frac{4\lambda}{bD}\sum_{m=1,2}^{\infty}\frac{A_m k_m k_n}{K_{mn}}$$

$$+\frac{B_{1n}}{D}(\alpha_n \operatorname{cth}\alpha_n a-\beta_n'\operatorname{ctg}\beta_n' a)-\frac{B_{2n}}{D}\left(\frac{\alpha_n}{\operatorname{sh}\alpha_n a}-\frac{\beta_n'}{\sin\beta_n' a}\right)$$

$$-\frac{4\lambda}{b}\sum_{m=1,2}^{\infty}\frac{C_m k_m k_n}{K_{mn}}[k_n^2+k_m^2(2-\mu)]\cos n\pi=0 \tag{5.17}$$

$$\frac{2P}{bD}\left(\frac{\operatorname{sh}\alpha_n x_0}{\operatorname{sh}\alpha_n a}-\frac{\sin\beta_n' x_0}{\sin\beta_n' a}\right)\sin k_n y_0+\frac{4\lambda}{bD}\sum_{m=1,2}^{\infty}\frac{A_m k_m k_n}{K_{mn}}\cos m\pi$$

$$+\frac{B_{1n}}{D}\left(\frac{\alpha_n}{\operatorname{sh}\alpha_n a}-\frac{\beta_n'}{\sin\beta_n' a}\right)-\frac{B_{2n}}{D}(\alpha_n \operatorname{cth}\alpha_n a-\beta_n'\operatorname{ctg}\beta_n' a)$$

$$-\frac{4\lambda}{b}\sum_{m=1,2}^{\infty}\frac{C_m k_m k_n}{K_{mn}}[k_n^2+k_m^2(2-\mu)]\cos m\pi\cos n\pi=0 \tag{5.18}$$

$$\frac{2P}{aD}\left\{-\frac{[\alpha_m^2-k_m^2(2-\mu)]}{\operatorname{sh}\alpha_m b}\operatorname{sh}\alpha_m y_0+\frac{[\beta_m'^2+k_m^2(2-\mu)]}{\sin\beta_m' b}\sin\beta_m' y_0\right\}\sin k_m y_0$$

$$+\frac{A_m}{D}\left\{\frac{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]}{\operatorname{sh}\alpha_m b}+\frac{\beta_m'[\beta_m'^2+k_m^2(2-\mu)]}{\sin\beta_m' b}\right\}$$

$$-\frac{4\lambda}{aD}\sum_{n=1,2}^{\infty}\frac{B_{1n}k_m k_n}{K_{mn}}[k_m^2+k_n^2(2-\mu)]\cos n\pi+\frac{4\lambda}{aD}\sum_{n=1,2}^{\infty}\frac{B_{2n}k_m k_n}{K_{mn}}$$

$$\cdot[k_m^2+k_n^2(2-\mu)]\cos m\pi\cos n\pi+C_m\{\alpha_m[\alpha_m^2-k_m^2(2-\mu)]^2\operatorname{cth}\alpha_m b$$

$$-\beta_m'[\beta_m'^2+k_m^2(2-\mu)]^2\operatorname{ctg}\beta_m' b\}=0 \tag{5.19}$$

从A_m，B_{1n}，B_{2n}和C_m各取15个系数且取$\mu=1/6$和$a/b=1$，我们给出图13表6。

表6 **固定边弯矩**（单位P）**和自由边振幅**（单位Pa^2/D） $a/b=1$

ω/ω_{11}	$x/a(y/b)$ / $M(w)$	0.05	0.15	0.35	0.5	0.7	0.9	0.95
0.0	M_{x0}	0.000647	−0.02006	−0.1066	−0.1497	−0.13490	−0.08149	−0.06926
	M_{y0}	0.001470	−0.01934	−0.1045	−0.1355	−0.08397	−0.004853	0.00147
	w_{yb}	0.000044	0.000508	0.002086	0.002634	0.001712	0.000216	0.000044
0.3	M_{x0}	0.000359	−0.022110	−0.1142	−0.16140	−0.14970	−0.09592	−0.08865
	M_{y0}	0.001633	−0.02097	−0.1122	−0.1451	−0.09023	−0.005342	0.001633
	w_{yb}	0.000062	0.000634	0.002515	0.00316	0.002074	0.000278	0.000062
0.5	M_{x0}	−0.000416	−0.02682	−0.13110	−0.1879	−0.18500	−0.13200	−0.13840
	M_{y0}	0.002043	−0.02459	−0.12920	−0.1664	−0.1042	−0.006413	0.002043
	w_{yb}	0.000106	0.000957	0.003601	0.004489	0.002989	0.000435	0.000106
0.8	M_{x0}	−0.00697	−0.05466	−0.22210	−0.34180	−0.4176	−0.3949	−0.5209
	M_{y0}	0.00512	−0.04353	−0.2211	−0.2824	−0.17911	−0.01179	0.00512
	w_{yb}	0.000457	0.003413	0.01171	0.01438	0.009844	0.001648	0.000457

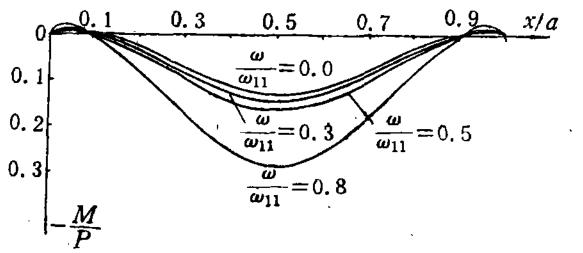

图13a　固定边$y=0$弯矩分布曲线($a/b=1$)

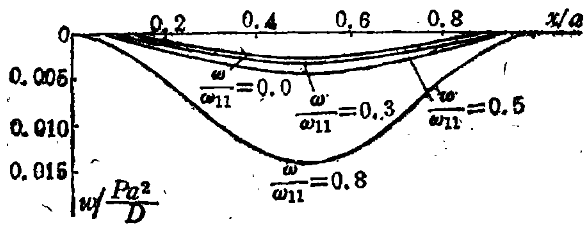

图13b　自由边$y=b$振幅曲线($a/b=1$)

最后，让我们考虑一个三边固定一边简支的矩形板，它是三边固定一边自由矩形板的特殊情况。实际上，令本节所有公式中的$C_m=0$，我们便可得本情况下的所有公式。

对于在简谐均布干扰力作用下的矩形板，我们给出图14表7。

表7　　固定边弯矩（单位qa^2）和中点振幅（单位qa^4/D）　　$a/b=1$

ω/ω_{11} \ $x/a(y/b)$	M	0.1	0.3	0.5	0.55	0.75	0.95	w_{mid}
0.0	M_{x0}	−0.008069	−0.04185	−0.0600	−0.06118	−0.05158	−0.01463	0.001570
	M_{y0}	−0.008034	−0.04131	−0.05503	−0.05415	−0.03399	−0.001594	
0.3	M_{x0}	−0.008342	−0.04502	−0.06530	−0.06663	−0.05591	−0.01583	0.001733
	M_{y0}	−0.008309	−0.04438	−0.05960	−0.05862	−0.03634	−0.001566	
0.5	M_{x0}	−0.008982	−0.05251	−0.07786	−0.07956	−0.06619	−0.01799	0.001878
	M_{y0}	−0.008951	−0.05162	−0.07042	−0.06919	−0.04187	−0.0015	
0.8	M_{x0}	−0.01285	−0.09881	−0.1561	−0.1602	−0.13032	−0.03275	0.004459
	M_{y0}	−0.01284	−0.09640	−0.1374	−0.13464	−0.07597	−0.001066	

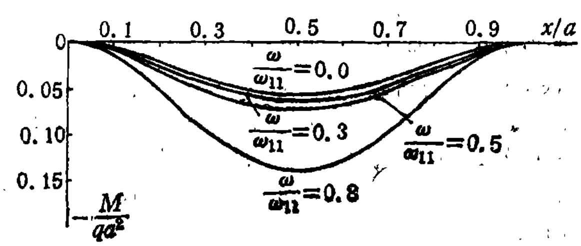

图14a　固定边$y=0$弯矩分布曲线($a/b=1$)

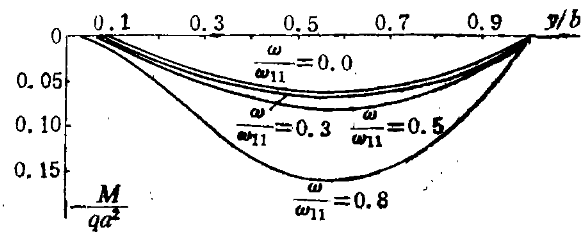

图14b 固定边$x=0$弯矩分布曲线($a/b=1$)

对于在中点一简谐集中干扰力作用下的同样矩形板，列于图15表8。

表8 **固定边弯矩**（单位P）**和中点振幅**（单位Pa^2/D） $a/b=1$

ω/ω_{11}	$x/a(y/b)$ / M	0.1	0.3	0.5	0.55	0.75	0.95	w_{mid}
0.0	M_{x0}	−0.004947	−0.08439	−0.1446	−0.1454	−0.09725	−0.01957	0.006272
	M_{y0}	−0.004961	−0.08325	−0.1339	−0.1302	−0.06097	0.00123	
0.3	M_{x0}	−0.005762	−0.09435	−0.16122	−0.16236	−0.11027	−0.02249	0.006793
	M_{y0}	−0.005778	−0.09295	−0.1485	−0.14453	−0.06834	0.001310	
0.5	M_{x0}	−0.007697	−0.11796	−0.2007	−0.2028	−0.14154	−0.02954	0.008026
	M_{y0}	−0.007722	−0.11592	−0.18307	−0.17831	−0.08578	0.001496	
0.8	M_{x0}	−0.019720	−0.28428	−0.44734	−0.45639	−0.34075	−0.07494	0.015679
	M_{y0}	−0.01980	−0.25768	−0.39589	−0.38627	−0.19356	0.002633	

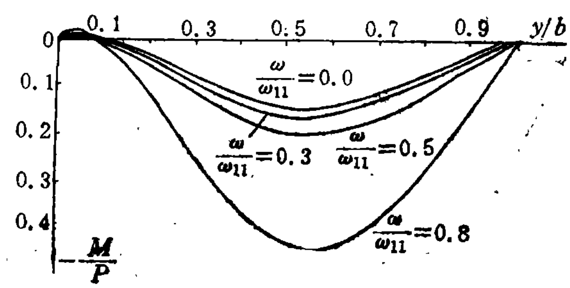

图15a 固定边$x=0$弯矩分布曲线($a/b=1$)

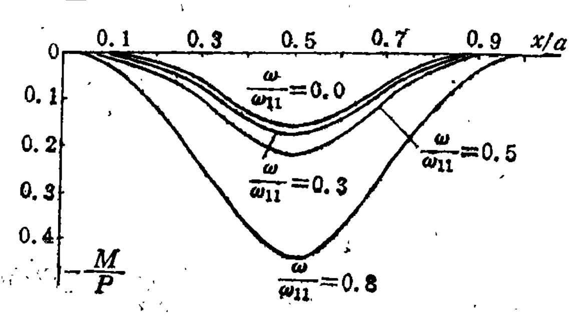

图15b 固定边$y=0$弯矩分布曲线($a/b=1$)

附 录 I

在建立矩形板受迫振动振幅挠曲面方程的过程中，为满足边界条件消除三角级数在边界上出现的第一类间断点以及加快级数的收敛，必须把三角级数转换成双曲函数。应用下述两种不同的方法求解同一梁的挠曲方程，我们将能实现这些转换。

图A-1表示一在弹性基上的简支梁，它受到左端的集中弯矩M_0和拉伸力组合作用。

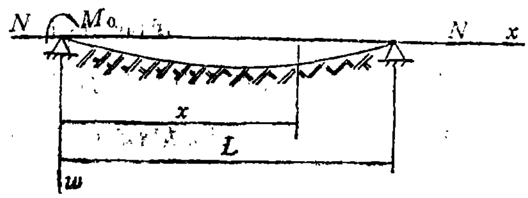

图 A-1

在这种情况下，有微分方程

$$\frac{d^4w}{dx^4}-\frac{N}{EJ}\frac{d^2w}{dx^2}+\frac{K}{EJ}w=0 \qquad (A1.1)$$

记 $2\eta=N/EJ$，$\rho^2=K/EJ$

则方程(A1.1)成为

$$\frac{d^4w}{dx^4}-2\eta\frac{d^2w}{dx^2}+\rho^2w=0 \qquad (A1.2)$$

当$\eta^2>\rho^2$时，方程(A1.2)的解为

$$w(x)=A\,\mathrm{sh}\,\alpha x+B\,\mathrm{ch}\,\alpha x+C\,\mathrm{sh}\,\beta x+D\,\mathrm{ch}\,\beta x \qquad (A1.3)$$

这里，$\alpha=\sqrt{\eta+\sqrt{\eta^2-\rho^2}}$，$\beta=\sqrt{\eta-\sqrt{\eta^2-\rho^2}}$

解(A1.3)必须满足边界条件

$$w(0)=w(l)=w''(l)=0,\ w''(0)=-M_0/EJ \qquad (A1.4)$$

执行边界条件(A1.4)，则(A1.3)成为

$$w(x)=\frac{M_0}{(\alpha^2-\beta^2)EJ}\left[-\frac{\mathrm{sh}\,\alpha(l-x)}{\mathrm{sh}\,\alpha l}+\frac{\mathrm{sh}\,\beta(l-x)}{\mathrm{sh}\,\beta l}\right] \qquad (A1.5)$$

该问题还能用能量法求解。在此情况下，总势能等于

$$\Pi_P=\frac{EJ}{2}\int_0^l\left(\frac{d^2w}{dx^2}\right)^2dx+\frac{N}{2}\int_0^l\left(\frac{dw}{dx}\right)^2dx+\frac{K}{2}\int_0^l w^2dx-M_0\left(\frac{dw}{dx}\right)_{x=0} \qquad (A1.6)$$

假设

$$w(x)=\sum_{m=1,2}^{\infty}a_m\sin\frac{m\pi}{l}x \qquad (A1.7)$$

并应用最小势能原理，我们求得

$$w(x)=\sum_{m=1}^{\infty}\frac{m\pi M_0/l}{(EJ\pi^4/2l^3)(m^4+2\eta m^2l^2/\pi^2+\rho^2l^4/\pi^4)}\sin\frac{m\pi}{l}x \qquad (A1.8)$$

比较(A1.5)和(A1.8)，我们得到

$$\sum_{m=1,2}^{\infty}\frac{m\sin(m\pi x/l)}{m^4+2\eta m^2l^2/\pi^2+\rho^2l^4/\pi^4}=\frac{\pi^3}{2l^2(\alpha^2-\beta^2)}\left[-\frac{\mathrm{sh}\,\alpha(l-x)}{\mathrm{sh}\,\alpha l}+\frac{\mathrm{sh}\,\beta(l-x)}{\mathrm{sh}\,\beta l}\right] \qquad (A1.9)$$

式(A1.9)是基本转换式。一系列其他转换关系可由它导出。

取(A1.9)对x的二次积分并利用端部条件，我们得到

$$\sum_{m=1,2}^{\infty}\frac{\sin(m\pi x/l)}{m(m^4+2\eta m^2l^2/\pi^2+\rho^2l^4/\pi^4)}=\frac{\pi^5}{2l^4}\left\{\frac{1}{\alpha^2-\beta^2}\left[\frac{\mathrm{sh}\,\alpha(l-x)}{\alpha^2\,\mathrm{sh}\,\alpha l}-\frac{\mathrm{sh}\,\beta(l-x)}{\beta^2\,\mathrm{sh}\,\beta l}\right]+\frac{l-x}{\alpha^2\beta^2 l}\right\} \qquad (A1.10)$$

对(A1.9)取x的二阶导数，我们得到

$$\sum_{m=1,2}^{\infty}\frac{m^3\sin(m\pi x/l)}{m^4+2\eta m^2l^2/\pi^2+\rho^2l^4/\pi^4}=\frac{\pi}{2(\alpha^2-\beta^2)}\left[\frac{\alpha^2\,\mathrm{sh}\,\alpha(l-x)}{\mathrm{sh}\,\alpha l}-\frac{\beta^2\,\mathrm{sh}\,\beta(l-x)}{\mathrm{sh}\,\beta l}\right] \qquad (A1.11)$$

对(A1.10)求导，我们得到

$$\sum_{m=1,2}^{\infty}\frac{\cos(m\pi x/l)}{m^4+2\eta m^2l^2/\pi^2+\rho^2l^4/\pi^4}=\frac{\pi^4}{2l^3}\left\{\frac{1}{\alpha^2-\beta^2}\left[-\frac{\mathrm{ch}\,\alpha(l-x)}{\alpha\,\mathrm{sh}\,\alpha l}+\frac{\mathrm{ch}\,\beta(l-x)}{\beta\,\mathrm{sh}\,\beta l}\right]-\frac{1}{\alpha^2\beta^2 l}\right\} \tag{A1.12}$$

将$x=l-x$代入(A1.9)，我们得到

$$\sum_{m=1,2}^{\infty}\frac{-m\cos m\pi\sin(m\pi x/l)}{m^4+2\eta m^2l^2/\pi^2+\rho^2l^4/\pi^4}=\frac{\pi^3}{2l^2(\alpha^2-\beta^2)}\left(-\frac{\mathrm{sh}\,\alpha x}{\mathrm{sh}\,\alpha l}+\frac{\mathrm{sh}\,\beta x}{\mathrm{sh}\,\beta l}\right) \tag{A1.13}$$

将$x=l-x$代入(A1.10)，我们得到

$$\sum_{m=1,2}^{\infty}\frac{-\cos m\pi\sin(m\pi x/l)}{m(m^4+2\eta m^2l^2/\pi^2+\rho^2l^4/\pi^4)}=\frac{\pi^5}{2l^4}\left\{\frac{1}{\alpha^2-\beta^2}\left[\frac{\mathrm{sh}\alpha x}{\alpha^2\,\mathrm{sh}\,\alpha l}-\frac{\mathrm{sh}\beta x}{\beta^2\mathrm{sh}\beta l}\right]+\frac{x}{\alpha^2\beta^2 l}\right\} \tag{A1.14}$$

将$x=l-x$代入(A1.11)，我们得到

$$\sum_{m=1,2}^{\infty}\frac{-m^3\cos m\pi\sin(m\pi x/l)}{m^4+2\eta m^2l^2/\pi^2+\rho^2l^4/\pi^4}=\frac{\pi}{2(\alpha^2-\beta^2)}\left(\frac{\alpha^2\,\mathrm{sh}\,\alpha x}{\mathrm{sh}\,\alpha l}-\frac{\beta^2\,\mathrm{sh}\,\beta x}{\mathrm{sh}\,\beta l}\right) \tag{A1.15}$$

将(A1.9)与(A1.13)相加，我们得到

$$\sum_{m=1,3}^{\infty}\frac{m\sin(m\pi x/l)}{m^4+2\eta m^2l^2/\pi^2+\rho^2l^4/\pi^4}=\frac{\pi^3}{4l^2(\alpha^2-\beta^2)}\left[-\frac{\mathrm{ch}\,\alpha(l/2-x)}{\mathrm{ch}\,\alpha(l/2)}+\frac{\mathrm{ch}\,\beta(l/2-x)}{\mathrm{ch}\,\beta(l/2)}\right] \tag{A1.16}$$

将(A1.10)与(A1.14)相加，我们得到

$$\sum_{m=1,3}^{\infty}\frac{\sin(m\pi x/l)}{m(m^4+2\eta m^2l^2/\pi^2+\rho^2l^4/\pi^4)}=\frac{\pi^5}{4l^4}\left\{\frac{1}{\alpha^2-\beta^2}\left[\frac{\mathrm{ch}\,\alpha(l/2-x)}{\alpha^2\mathrm{ch}\alpha(l/2)}-\frac{\mathrm{ch}\,\beta(l/2-x)}{\beta^2\,\mathrm{ch}\,\beta(l/2)}\right]+\frac{1}{\alpha^2\beta^2}\right\} \tag{A1.17}$$

将(A1.11)与(A1.15)相加，我们得到

$$\sum_{m=1,3}^{\infty}\frac{m^3\sin(m\pi x/l)}{m^4+2\eta m^2l^2/\pi^2+\rho^2l^4/\pi^4}=\frac{\pi}{4(\alpha^2-\beta^2)}\left[\frac{\alpha^2\,\mathrm{ch}\,\alpha(l/2-x)}{\mathrm{ch}\,\alpha(l/2)}-\frac{\beta^2\,\mathrm{ch}\,\beta(l/2-x)}{\mathrm{ch}\,\beta(l/2)}\right] \tag{A1.18}$$

附 录 Ⅱ

当我们转换三角级数为双曲函数时，必须处理$\lambda<k_m^2$和$\lambda<k_n^2$的情况。为此，让我们考虑图A-2所示梁的受迫振动。

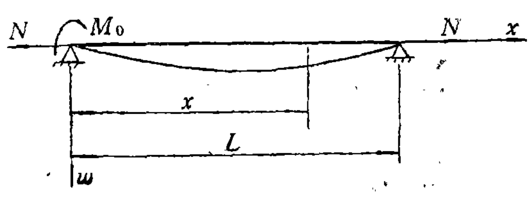

图 A-2

假设$M=M_0\sin\omega t$并且忽略阻尼，我们得到$W(x,t)=w(x)\sin\omega t$和微分方程

$$\frac{d^4w}{dx^4}-\frac{N}{EJ}\frac{d^2w}{dx^2}-\frac{\omega^2\rho'}{EJ}w=0 \tag{A2.1}$$

令 $2\eta_1=N/EJ$，$\rho_1^2=\omega^2\rho'/EJ$

则(A2.1)成为

$$\frac{d^4w}{dx^4}-2\eta_1\frac{d^2w}{dx^2}-\rho^2w=0 \tag{A2.2}$$

其解为

$$w(x)=A_1\,\mathrm{sh}\,\alpha_1x+B_1\,\mathrm{ch}\alpha_1x+C_1\sin\beta_1x+D_1\cos\beta_1x \tag{A2.3}$$

这里

$$\alpha_1=\sqrt{\eta_1+\sqrt{\eta_1^2+\rho_1^2}},\quad \beta_1=\sqrt{\sqrt{\eta_1^2+\rho_1^2}-\eta_1}$$

(A2.3)必须满足边界条件

$$w(0)=w(l)=w''(l)=0,\quad w''(0)=-M_0/EJ \tag{A2.4}$$

执行边界条件(A2.4)，我们得到

$$w(x)=\frac{M_0}{(\alpha_1^2+\beta_1^2)EJ}\left[-\frac{\mathrm{sh}\,\alpha_1(l-x)}{\mathrm{sh}\,\alpha_1l}+\frac{\sin\beta_1(l-x)}{\sin\beta_1l}\right] \tag{A2.5}$$

我们还可以应用能量法解此问题。总势能为

$$\Pi_P=\frac{EJ}{2}\int_0^l\left(\frac{d^2w}{dx^2}\right)^2dx-\frac{N}{2}\int_0^l\left(\frac{dw}{dx}\right)^2dx-\frac{\omega^2\rho'}{2}\int_0^l w^2dx-M_0\left(\frac{dw}{dx}\right)_{x=0} \tag{A2.6}$$

应用最小势能原理，我们得

$$w(x)=\sum_{m=1,2}^{\infty}\frac{m\pi M_0/l}{(EJ\pi^4/2l^2)(m^4+2\eta_1m^2l^2/\pi^2-\rho_1^2l^4/\pi^4)}\sin\frac{m\pi}{l}x \tag{A2.7}$$

让(A2.5)等于(A2.7)，我们得

$$\sum_{m=1,2}^{\infty}\frac{m\sin(m\pi x/l)}{m^4+2\eta_1m^2l^2/\pi^2-\rho_1^2l^4/\pi^4}=\frac{\pi^3}{2l^2(\alpha_1^2+\beta_1^2)}\left[-\frac{\mathrm{sh}\,\alpha_1(l-x)}{\mathrm{sh}\,\alpha_1l}+\frac{\sin\beta_1(l-x)}{\sin\beta_1l}\right] \tag{A2.8}$$

应用与附录1相同的方法，我们可得下面一系列转换关系

$$\sum_{m=1,2}^{\infty}\frac{\sin(m\pi x/l)}{m(m^4+2\eta_1m^2l^2/\pi^2-\rho_1^2l^4/\pi^4)}=\frac{\pi^5}{2l^4}\left\{\frac{1}{\alpha^2+\beta_1^2}\left[\frac{\mathrm{sh}\,\alpha_1(l-x)}{\alpha_1^2\,\mathrm{sh}\,\alpha_1l}+\frac{\sin\beta_1(l-x)}{\beta_1^2\sin\beta_1l}\right]-\frac{l-x}{\alpha_1^2\beta_1^2l}\right\} \tag{A2.9}$$

$$\sum_{m=1,2}^{\infty}\frac{m^3\sin(m\pi x/l)}{m^4+2\eta_1m^2l^2/\pi^2-\rho_1^2l^4/\pi^4}=\frac{\pi}{2(\alpha_1^2+\beta_1^2)}\left[\frac{\alpha_1^2\,\mathrm{sh}\,\alpha_1(l-x)}{\mathrm{sh}\,\alpha_1l}+\frac{\beta_1^2\sin\beta_1(l-x)}{\sin\beta_1l}\right] \tag{A2.10}$$

$$\sum_{m=1,2}^{\infty}\frac{\cos(m\pi x/l)}{m^4+2\eta_1m^2l^2/\pi^2-\rho_1^2l^4/\pi^4}=\frac{\pi^4}{2l^3}\left\{\frac{1}{\alpha_1^2+\beta_1^2}\left[-\frac{\mathrm{ch}\,\alpha_1(l-x)}{\alpha_1\,\mathrm{sh}\,\alpha_1l}-\frac{\cos\beta_1(l-x)}{\beta_1\sin\beta_1l}\right]+\frac{1}{\alpha_1^2\beta_1^2l}\right\} \tag{A2.11}$$

$$\sum_{m=1,2}^{\infty}\frac{-m\cos m\pi\sin(m\pi x/l)}{m^4+2\eta_1m^2l^2/\pi^2-\rho_1^2l^4/\pi^4}=\frac{\pi^3}{2l(\alpha_1^2+\beta_1^2)}\left(-\frac{\mathrm{sh}\,\alpha_1x}{\mathrm{sh}\,\alpha_1l}+\frac{\sin\beta_1x}{\sin\beta_1l}\right) \tag{A2.12}$$

$$\sum_{m=1,2}^{\infty}\frac{-\cos m\pi\sin(m\pi x/l)}{m(m^4+2\eta_1m^2l^2/\pi^2-\rho_1^2l^4/\pi^4)}=\frac{\pi^5}{2l^4}\left[\frac{1}{\alpha_1^2+\beta_1^2}\left(\frac{\mathrm{sh}\,\alpha_1x}{\alpha_1^2\,\mathrm{sh}\,\alpha_1l}+\frac{\sin\beta_1x}{\beta_1^2\sin\beta_1l}\right)-\frac{x}{\alpha_1^2\beta_1^2l}\right] \tag{A2.13}$$

$$\sum_{m=1,2}^{\infty}\frac{-m^3\cos m\pi\sin(m\pi x/l)}{m^4+2\eta_1 m^2l^2/\pi^2-\rho_1^2l^4/\pi^4}=\frac{\pi}{2(\alpha_1^2+\beta_1^2)}\left(\frac{\alpha_1^2\,\mathrm{sh}\,\alpha_1 x}{\mathrm{sh}\,\alpha_1 l}+\frac{\beta_1^2\sin\beta_1 x}{\sin\beta_1 l}\right) \tag{A2.14}$$

$$\sum_{m=1,3}^{\infty}\frac{m\sin(m\pi x/l)}{m^4+2\eta_1 m^2l^2/\pi^2-\rho_1^2l^4/\pi^4}=\frac{\pi^3}{4l^2(\alpha_1^2+\beta_1^2)}\left[-\frac{\mathrm{ch}\,\alpha_1(l/2-x)}{\mathrm{ch}\,\alpha_1(l/2)}+\frac{\cos\beta_1(l/2-x)}{\cos\beta_1(l/2)}\right] \tag{A2.15}$$

$$\sum_{m=1,3}^{\infty}\frac{\sin(m\pi x/l)}{m(m^4+2\eta_1 m^2l^2/\pi^2-\rho_1^2l^4/\pi^4)}=\frac{\pi^5}{4l^4}\left\{\frac{1}{\alpha_1^2+\beta_1^2}\left[\frac{\mathrm{ch}\,\alpha_1(l/2-x)}{\alpha_1^2\,\mathrm{ch}\,\alpha_1(l/2)}+\frac{\cos\beta_1(l/2-x)}{\beta_1^2\cos\beta_1(l/2)}\right]-\frac{1}{\alpha_1^2\beta_1^2}\right\} \tag{A2.16}$$

$$\sum_{m=1,3}^{\infty}\frac{m^3\sin(m\pi x/l)}{m^4+2\eta_1 m^2l^2/\pi^2-\rho_1^2l^4/\pi^4}=\frac{\pi}{4(\alpha_1^2+\beta_1^2)}\left[\frac{\alpha_1^2\,\mathrm{ch}\,\alpha_1(l/2-x)}{\mathrm{ch}\,\alpha_1(l/2)}+\frac{\beta_1^2\cos\beta_1(l/2-x)}{\cos\beta_1(l/2)}\right] \tag{A2.17}$$

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The Method of the Reciprocal Theorem of Forced Vibration for the Elastic Thin Rectangular Plates (Ⅰ)——Rectangular Plates with Four Clamped Edges and with Three Clamped Edges

Fu Bao-lian

(Yanshan University, Qinhuangdao)

Li Nong

(Luoyang Technology Institute, Luoyang)

Abstract

In this paper the method of the reciprocal theorem (MRT) is extended to solve the steady state responses of rectangular plates under harmonic disturbing forces. A series of the closed solutions of rectangular plates with various boundary conditions are given and the tables and figures which have practical value are provided.

MRT is a simple, convenient and general method for solving the steady state responses of rectangular plates under various harmonic disturbing forces.

The paper contains three parts: (Ⅰ) rectangular plates with four clamped edges and with three clamped edges; (Ⅱ) rectangular plates with two adjacent clamped edges; (Ⅲ) cantilever plates.

We are going to publish them one after another.