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关于利用一种自变量变换求解幂硬化材料Ⅲ型裂纹问题的有效性的探讨

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摘要

利用物理平面与应变平面以及物理平面与应力平面的变换，可得到幂硬化材料Ⅲ型裂纹尖端附近渐近解的解析式。本文讨论了此变换的有效性。分析结果表明：除幂硬化材料的极限情况——理想塑性外，此变换有效。

在求解幂硬化材料Ⅲ型裂纹尖端附近的渐近解时，通常需要用数值方法求解一个二阶常微分方程，如将自变量x,y变换成γ_x,γ_y以及将自变量x,y变换成τ_x,τ_y，则可求出解析表达式。

Ⅲ型问题的变形协调条件是

$$\frac{\partial\gamma_x}{\partial y}-\frac{\partial\gamma_y}{\partial x}=0 \tag{1}$$

平衡方程是

$$\frac{\partial\tau_x}{\partial x}+\frac{\partial\tau_y}{\partial y}=0 \tag{2}$$

幂硬化材料的应力应变关系为

$$\frac{\gamma_x}{\gamma_0}=\left(\frac{\tau_x}{\tau_0}\right)^n,\quad \frac{\gamma_y}{\gamma_0}=\left(\frac{\tau_y}{\tau_0}\right)^n \tag{3}$$

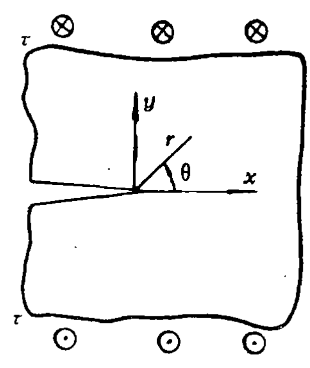

图 1

其中τ_0和γ_0分别是材料的屈服极限和屈服应变，n是材料硬化指数N的倒数。

在物理平面上，应变是自变量x,y的函数

$$\gamma_x=\gamma_x(x,y),\quad \gamma_y=\gamma_y(x,y) \tag{4}$$

如将γ_x,γ_y作为自变量，即设

$$x=x(\gamma_x,\gamma_y) \tag{5}$$

$$y=y(\gamma_x,\gamma_y) \tag{6}$$

可将物理平面变换成应变平面。

由（5）式，我们有

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* 李灏推荐。

$$\frac{dx}{dx}=\frac{\partial x}{\partial \gamma_x}\cdot\frac{\partial \gamma_x}{\partial x}+\frac{\partial x}{\partial \gamma_y}\cdot\frac{\partial \gamma_y}{\partial x}=1 \tag{a}$$

$$\frac{dx}{dy}=\frac{\partial x}{\partial \gamma_x}\cdot\frac{\partial \gamma_x}{\partial y}+\frac{\partial x}{\partial \gamma_y}\cdot\frac{\partial \gamma_y}{\partial y}=0 \tag{b}$$

由(6)式，我们有

$$\frac{dy}{dx}=\frac{\partial y}{\partial \gamma_x}\cdot\frac{\partial \gamma_x}{\partial x}+\frac{\partial y}{\partial \gamma_y}\cdot\frac{\partial \gamma_y}{\partial x}=0 \tag{c}$$

$$\frac{dy}{dy}=\frac{\partial y}{\partial \gamma_x}\cdot\frac{\partial \gamma_x}{\partial y}+\frac{\partial y}{\partial \gamma_y}\cdot\frac{\partial \gamma_y}{\partial y}=1 \tag{d}$$

由(a)式和(b)式，我们可得

$$\frac{\partial x}{\partial \gamma_y}=\frac{\begin{vmatrix}\dfrac{\partial \gamma_x}{\partial x} & 1\\ \dfrac{\partial \gamma_x}{\partial y} & 0\end{vmatrix}}{\begin{vmatrix}\dfrac{\partial \gamma_x}{\partial x} & \dfrac{\partial \gamma_y}{\partial x}\\ \dfrac{\partial \gamma_x}{\partial y} & \dfrac{\partial \gamma_y}{\partial y}\end{vmatrix}}=-\frac{\partial \gamma_x}{\partial y}/\Delta,$$

即

$$\frac{\partial \gamma_x}{\partial y}=-\Delta\frac{\partial x}{\partial \gamma_y} \tag{e}$$

设Jacobi行列式

$$\Delta=\begin{vmatrix}\dfrac{\partial \gamma_x}{\partial x} & \dfrac{\partial \gamma_y}{\partial x}\\ \dfrac{\partial \gamma_x}{\partial y} & \dfrac{\partial \gamma_y}{\partial y}\end{vmatrix}\neq 0 \tag{7}$$

由(c)式和(d)式，我们得

$$\frac{\partial y}{\partial \gamma_x}=\frac{\begin{vmatrix}0 & \dfrac{\partial \gamma_y}{\partial x}\\ 1 & \dfrac{\partial \gamma_y}{\partial y}\end{vmatrix}}{\begin{vmatrix}\dfrac{\partial \gamma_x}{\partial x} & \dfrac{\partial \gamma_y}{\partial x}\\ \dfrac{\partial \gamma_x}{\partial y} & \dfrac{\partial \gamma_y}{\partial y}\end{vmatrix}}=-\frac{\partial \gamma_y}{\partial x}/\Delta$$

即

$$\frac{\partial \gamma_y}{\partial x}=-\Delta\frac{\partial y}{\partial \gamma_x} \tag{f}$$

将(e)式和(f)式代入(1)式，可得

$$\frac{\partial x}{\partial \gamma_y}-\frac{\partial y}{\partial \gamma_x}=0 \tag{8}$$

设$x=x(\tau_x,\ \tau_y)$和$y=y(\tau_x,\ \tau_y)$，则又可将物理平面变换成应力平面，用与以上类似的方法可得

$$\frac{\partial x}{\partial \tau_x}+\frac{\partial y}{\partial \tau_y}=0 \tag{9}$$

条件是其Jacobi行列式不等于零。现在我们来详细讨论Jacobi行列式等于零的情况，从而找出在此变换中丢掉的那部分解。

由Jacobi行列式等于零，我们有

$$\Delta=\begin{vmatrix} \frac{\partial \gamma_x}{\partial x} & \frac{\partial \gamma_y}{\partial x} \\ \frac{\partial \gamma_x}{\partial y} & \frac{\partial \gamma_y}{\partial y} \end{vmatrix}$$

$$=\frac{\partial \gamma_x}{\partial x}\cdot\frac{\partial \gamma_y}{\partial y}-\frac{\partial \gamma_x}{\partial y}\cdot\frac{\partial \gamma_y}{\partial x}$$

$$=\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2=0 \tag{10}$$

其中w是质点沿z轴方向的位移。将(10)式写成极坐标形式

$$\frac{1}{r}\cdot\frac{\partial^2 w}{\partial r^2}\cdot\frac{\partial w}{\partial r}-\frac{1}{r^2}\left(\frac{\partial^2 w}{\partial r\partial \theta}\right)^2+\frac{1}{r^2}\cdot\frac{\partial^2 w}{\partial r^2}\cdot\frac{\partial^2 w}{\partial \theta^2}$$

$$+\frac{2}{r^3}\cdot\frac{\partial^2 w}{\partial r\partial \theta}\cdot\frac{\partial w}{\partial \theta}-\frac{1}{r^4}\left(\frac{\partial w}{\partial \theta}\right)^2=0$$

(10)′

对于理想塑性材料，有关文献给出弹性区位移$w=-\frac{\tau_0}{G}x+F\left(\frac{\pi}{2}\right)$，塑性区位移仅是$\theta$的函数$w=F(\theta)$，其中$G$是材料的剪切弹性模量，$F\left(\frac{\pi}{2}\right)$是常数。显而易见，弹性区位移满足方程(10)。将$w=F(\theta)$代入(10)′式，有

$$-\frac{1}{r^4}[F'(\theta)]^2=0$$

显然，此式不是恒等式。

对于幂硬化材料，即n为有限数时，可设

$$w=w(r,\theta)=\gamma_0 A^n r^{\frac{1}{n+1}}S(\theta),$$

其中A为常数，$S(\theta)$为角分布函数。代入方程(10)′并加以整理后得

$$SS''+n(S')^2+\frac{1}{n+1}S^2=0 \tag{11}$$

式中$S'=dS(\theta)/d\theta$。设

$$S(\theta)=e^u,$$

我们有

$$S'=u'e^u$$
$$S''=u''e^u+(u')^2e^u$$

代入方程(11)得

$$e^{u}[u''e^{u}+(u')^{2}e^{u}]+n(u')^{2}e^{2u}+\frac{1}{n+1}e^{2u}=0$$

约去因子$e^{2u}\neq0$，得

$$u''+(u')^{2}+n(u')^{2}+\frac{1}{n+1}=0$$

设$u'=T$，则方程化为

$$T'+(n+1)T^{2}+\frac{1}{n+1}=0,$$

$$\frac{dT}{(n+1)T^{2}+\frac{1}{n+1}}=-d\theta$$

积分得

$$\mathrm{tg}^{-1}[(n+1)T]=-\theta-c_1,$$

$$T=-\frac{1}{n+1}\mathrm{tg}(\theta+c_1)$$

即

$$u'=-\frac{1}{n+1}\mathrm{tg}(\theta+c_1)$$

再次积分得

$$u=\frac{1}{n+1}\ln\cos(\theta+c_1)+\frac{1}{n+1}\ln c_2$$

因此方程(11)的通解为

$$S(\theta)=\exp\left[\frac{1}{n+1}\ln c_2\cos(\theta+c_1)\right]$$

$$=[c_2\cos(\theta+c_1)]^{\frac{1}{n+1}},$$

其中c_1和c_2是积分常数。于是位移可表示为

$$w(r,\theta)=\gamma_0A^{n}r^{\frac{1}{n+1}}[c_2\cos(\theta+c_1)]^{\frac{1}{n+1}}$$

$\theta=0$处，反对称性条件是

$$w(r,0)=0 \tag{12}$$

即

$$\gamma_0A^{n}r^{\frac{1}{n+1}}(c_2\cos c_1)^{\frac{1}{n+1}}=0,$$

因此

$$c_1=\frac{m\pi}{2}\qquad(m=1,3,5,\cdots)$$

则

$$\cos(\theta+c_1)=\cos\left(\theta+\frac{m\pi}{2}\right)$$

$$=\begin{cases}-\sin\theta & (m=1,5,9,\cdots)\\ \sin\theta & (m=3,7,11,\cdots)\end{cases}$$

因两种情况只相差一个符号，所以可设满足方程(10)′和条件(12)的位移有如下形式

$$w(r,\theta)=cr^{\frac{1}{n+1}}(\sin\theta)^{\frac{1}{n+1}} \tag{13}$$

其中 $c=\gamma_0 A^n c_2^{\frac{1}{n+1}}$.

在极坐标下，应变分量为

$$\gamma_r=\frac{\partial w}{\partial r},\quad \gamma_\theta=\frac{1}{r}\cdot\frac{\partial w}{\partial\theta} \tag{14}$$

应力应变关系为

$$\gamma_r=\gamma_0\left(\frac{\tau_r}{\tau_0}\right)^n,\quad \gamma_\theta=\gamma_0\left(\frac{\tau_\theta}{\tau_0}\right)^n$$

或写成

$$\tau_r=\tau_0\left(\frac{\gamma_r}{\gamma_0}\right)^{\frac{1}{n}},\quad \tau_\theta=\tau_0\left(\frac{\gamma_\theta}{\gamma_0}\right)^{\frac{1}{n}} \tag{15}$$

平衡方程为

$$\frac{\partial(r\tau_r)}{\partial r}+\frac{\partial\tau_\theta}{\partial\theta}=0 \tag{16}$$

将(13)式代入(14)式，得

$$\gamma_r=\frac{\partial w}{\partial r}=\frac{c}{n+1}r^{\frac{-n}{n+1}}(\sin\theta)^{\frac{1}{n+1}},$$

$$\gamma_\theta=\frac{1}{r}\cdot\frac{\partial w}{\partial\theta}==\frac{c}{n+1}r^{\frac{-n}{n+1}}(\sin\theta)^{\frac{-n}{n+1}}\cos\theta$$

代入(15)式，得

$$\tau_r=\tau_0\left[\frac{c}{(n+1)\gamma_0}\right]^{\frac{1}{n}}r^{-\frac{1}{n+1}}(\sin\theta)^{\frac{1}{n(n+1)}},$$

$$\tau_\theta=\tau_0\left[\frac{c}{(n+1)\gamma_0}\right]^{\frac{1}{n}}r^{-\frac{1}{n+1}}(\sin\theta)^{-\frac{1}{n+1}}(\cos\theta)^{\frac{1}{n}}$$

将它们代入(16)式，得

$$\tau_0\left[\frac{c}{(n+1)\gamma_0}\right]^{\frac{1}{n}}\frac{\partial}{\partial r}[r^{(-\frac{1}{n+1}+1)}(\sin\theta)^{\frac{1}{n(n+1)}}]$$

$$+\tau_0\left[\frac{c}{(n+1)\gamma_0}\right]^{\frac{1}{n}}\frac{\partial}{\partial\theta}[r^{-\frac{1}{n+1}}(\sin\theta)^{-\frac{1}{n+1}}(\cos\theta)^{\frac{1}{n}}]=0$$

整理得

$$n(\mathrm{tg}\theta)^{\frac{1}{n}}-\mathrm{ctg}\,\theta-\frac{n+1}{n}\mathrm{tg}\theta=0$$

显而易见，此式不是恒等式。至此可见使Jacobi行列式等于零的位移并不满足平衡方程。我们的结论是：对于幂硬化材料的Ⅲ型裂纹尖端附近渐近解，可在应变平面和应力平面上求出解析表达式。但对于$n\to\infty$的极限情况，在弹性区，本变换的Jacobi行列式等于零，这时应另行求解。

A Discussion about the Effectiveness on Using a Variate-Transformation to Find out Solutions of Mode Ⅲ Crack Problems in Power Hardening Media

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Abstract

By the use of the transformations of physical plane to strain plane and physical plane to stress plane, an analytic expression of the asymptotic solution near a mode Ⅲ Crack tip in a power hardening medium can be obtained. In this paper the effectiveness of the transformation is discussed. Analytical results show that the transformation is effective except for a special limit case of power hardening media — the ideal plastic materials.