文章编号: 1000_0887(2004) 07_0735_06

带强耗散项的拟线性波方程的数值解*

阿特青 古勒

(玉珍翠延大学 数学系, 凡城 65080, 土耳其)

(周哲玮推荐)

摘要: 研究了一类拟线性波方程的数值解• 构造了带强耗散项的拟线性波方程的三级差分格式, 并证明其收敛性, 估计了差分解的误差• 最后给出数值例子•

关 键 词: 周期问题; 拟线性波方程; 差分格式; 数值解

中图分类号: O175.38 文献标识码: A

引 言

本文讨论边值问题

$$Lu \equiv \frac{\partial^2 u}{\partial t^2} + L_1\left[\frac{\partial u}{\partial t}\right] + L_0 u + f(x, t, u(x, t)) = 0,$$

$$(x, t) \in D = (0, l) \times (0, T], \tag{1}$$

$$u(0, t) = u(l, t) = 0, \tag{2}$$

$$u(x, 0) = u(x, T), \tag{3}$$

$$\frac{\partial u}{\partial t}(x, 0) = \frac{\partial u}{\partial t}(x, T), \tag{4}$$

其中

$$L_1\left[\frac{\partial u}{\partial t}\right] = -\frac{\partial}{\partial x}\left(a(x, t)\frac{\partial^2 u}{\partial t \partial x}\right) + b(x, t)\frac{\partial u}{\partial t},$$

$$L_0 u = -\frac{\partial}{\partial x}\left(c(x, t)\frac{\partial u}{\partial x}\right),$$

a, b, c, f 为D 中充分光滑的函数• 函数$f(x, t, u(x, t))$ 及对 x, t, u 的一阶和二阶导数在$D \times (-\infty, +\infty)$ 中有界且连续•

这类方程出现在数学物理和流体力学等很多领域• 应用于通信线路、等离子气体中的电子等离子波、离子声波和其他物理模型的研究(参见 Bullough1980, Lonngren1978, Tkezi1978 的论述) •

本文提出了该问题的三级差分格式• 构造了带强耗散项波方程的差分格式• 进行了数学研究, 证明其近似误差收敛于 $O(h^2 + \tau^2)$• 估计了差分解的误差• 最后, 在一个数值例子上给出了“理论”结果•

多位数学家(Lagnese 1972[1]、Leopold 1985[2]、Webb 1980[3]、Lebedev 1957[4]) 研究了这类问

* 收稿日期: 2002_11_24
本文原文为英文, 由吴承平 译, 张禄坤 校.

题精确解的存在性、唯一性和稳定性•

Amirliyer 1988[5] 用较简单的模型研究了这类线性方程的数值解•

1 差分格式

在问题(1)~(4)中,我们假设 $0 < a_* \leqslant a(x,t) \leqslant a^*$, $0 < b_* \leqslant b(x,t) \leqslant b^*$, $0 \leqslant c_* \leqslant c(x,t) \leqslant c^*$ 且 $a_* \leqslant \partial a(x,t)/\partial t \leqslant a^*$, $b_* \leqslant \partial b(x,t)/\partial t \leqslant b^*$, $c_* \leqslant \partial c(x,t)/\partial t \leqslant c^*$•

在区域 D 中建立网格 $\omega_{h\tau} = \omega_h \times \omega_\tau$, 使

$\omega_h = \{x_i = ih,\ i = 1, 2, \cdots, N-1,\ h = l/N\}$,

$\omega_\tau = \{t_j = j\tau, j = 1, 2, \cdots, M-1,\ \tau = T/M\}$

且 $\omega_h = \omega_h \cup \{x = 0, l\}$, $\omega_\tau^+ = \omega_\tau \cup \{t = T\}$•

我们分两步建立差分格式• 首先取基函数

$$\varphi_i(x) = \begin{cases} \varphi_i^{(1)}(x) \equiv \dfrac{x - x_{i-1}}{h} & (x_{i-1} < x < x_i) \\ \varphi_i^{(2)}(x) \equiv \dfrac{x_{i+1} - x}{h} & (x_i < x < x_{i+1}) \\ 0 & (x \notin (x_{i-1}, x_{i+1})) \end{cases} \qquad (i = 1, 2, \cdots, N-1),$$

接着,用于

$$\varphi_j(t) = \begin{cases} \varphi_j^{(1)}(t) \equiv \dfrac{t - t_{j-1}}{\tau} & (t_{j-1} < t < t_j) \\ \varphi_j^{(2)}(t) \equiv \dfrac{t_{j+1} - t}{\tau} & (t_j < t < t_{j-1}) \\ 0 & (t \notin (t_{j-1}, t_{j+1})) \end{cases} \qquad (j = 1, 2, \cdots, M-1),$$

于是,得正确的离散差分关系

$$\begin{cases} Lu_i^j \equiv u_{tt,i}^j - (a(x_{i-0.5}, t_j)(u_{\overset{\circ}{t}x}^j))_{x,i} + b(x_i, t_j)u_{\overset{\circ}{t},i}^j - (c(x_{i-0.5}, t_j)u_x^j)_{x,i} + \\ \qquad f(x_i, t_j, u(x_i, t_j)) + R_i^j \equiv 0 \qquad (i = 1, 2, \cdots, N-1; j = 1, 2, \cdots, M-1), \\ lu_i^j + f(x_i, t_j, u(x_i, t_j)) + R_i^j \equiv 0, \end{cases} \tag{5}$$

$$\begin{cases} lu_i^j \equiv u_{tt,i}^j - (a(x_{i-0.5}, t_j)(u_{\overset{\circ}{t},x}^j))_{x,i} + b(x_i, t_j)u_{\overset{\circ}{t},i}^j - (c(x_{i-0.5}, t_j)u_x^j)_{x,i}, \\ R_i^j = (R^{(0)})_{x,i}^j + (R^{(1)})_i^j• \end{cases} \tag{6}$$

若问题(1)~(4)的初始值充分光滑,则 $R^{(0)}$ 和 $R^{(1)}$ 的局部误差为 $O(h^2 + \tau^2)$ 形式•

现在我们取条件(4)的近似• 除上述基函数外,我们取如下基函数

$$\varphi_0(t) = \begin{cases} \varphi_0^{(1)}(t) \equiv \dfrac{t_1 - t}{\tau} & (t_0 < t < t_1) \\ \varphi_0^{(2)}(t) \equiv \dfrac{t - t_{M-1}}{\tau} & (t_{M-1} < t < t_M) \\ 0 & (t \notin (t_0, t_1) \cup (t_{M-1}, t_M)) \end{cases}$$

关系式(5)乘以 φ_0 并在区间$(0, T)$积分,则由条件(4)可得点 $t = t_M$ 的差分关系

$$u_0^M - (a(x_{i-0.5}, t_M)u_{\overset{\circ}{t},x}^M)_{x,i} + b(x_i, t_M)u_{\overset{\circ}{t},i}^M - (c(x_{i-0.5}, t_M)u_x^M)_{x,i} + \\ f(x_i, t_M, u(x_i, t_M)) + R_i^M \equiv 0• \tag{7}$$

$$R_i^M = (R^{M(0)})_{x,i} + (R^{M(1)})_i• \tag{8}$$

在充分光滑的条件下，$R_i^{M(0)}$ 和 $R_i^{M(1)}$ 的误差阶为 $O(h^2+\tau^2)$•

最后根据关系式(6)和(7)得到问题(1)～(4)的差分格式为

$$\begin{cases} ly \equiv y_{tt}-(a(x-h/2,t)y_{\overset{0}{tx}})_x+b(x,t)y_{\overset{0}{t}}- \\ \qquad (c(x-h/2,t)y_{\bar{x}})_x+f(x,t,y(x,t))\equiv 0 \\ \qquad\qquad ((x,t)\in\omega_h\times\omega_\tau^+), \\ y(0,t)=y(l,t)=0 \quad (t\in\omega_\tau), \\ y(x,0)=y(x,T) \quad (x\in\omega_h), \\ y(x,\tau)=y(x,T+\tau) \quad (x\in\omega_h), \end{cases} \tag{9}$$

2 近似解的误差估计

当 $z=y-u$ 时，该差分问题的误差可以写为

$$lz+f(x_i,t_j,y(x_i,t_j))-f(x_i,t_j,u(x_i,t_j))-R_i^j\equiv 0 \qquad ((x,t)\in\omega_h\times\omega_\tau^+),$$

$$z(0,t)=z(l,t)=0 \quad (t\in\omega_\tau),$$

$$z(x,0)=z(x,T) \quad (x\in\omega_h),$$

$$z(x,\tau)=z(x,T+\tau) \quad (x\in\omega_h),$$

其中 R 的近似误差由式(6)和式(8)给出•

引理 2.1 若以下条件成立

$$\lambda_0\left(c_*+\frac{l^2}{8}d_*-\frac{1}{4}\aleph_0-\frac{l^2}{32}\aleph_1\right)-\frac{1}{2}\left(c^*+\frac{l^2}{32\mu_1}d^*\right)>0, \tag{10}$$

则差分问题的误差估计

$$\|z_t\|^2+\|\hat{z}_t\|^2+\|z\|^2\leqslant C\tau\sum_{i=1}^{M}(\|R_i^{(0)}\|^2+\|R_i^{(1)}\|^2)\exp(-C_1t_{M-i}) \qquad (t\in\omega_\tau^+) \tag{11}$$

为真，且

$$\lambda_0<\frac{4}{l^2}a_*+\frac{b_*}{2},\quad \aleph_0=\max(a^*,3a^*),\quad \aleph_1=\max(b^*,3b^*),$$

其中 C 和 C_1 为与 h 和 τ 无关的正常数•

证明 先取等式

$$(lz,z_t+\lambda z)=((R_i^{(0)})_x+R_i^{(1)},z_t+\lambda z), \tag{12}$$

其中 $\lambda>0$ 为待定实参数• 由平均值定理可得

$$f(x_i,t_j,y(x_i,t_j))-f(x_i,t_j,u(x_i,t_j))\equiv\frac{\partial f}{\partial u}(x_i,t_j,u(x_i,t_j))z(x_i,t_j),$$

$$u=y+\theta(u-y) \qquad (0<\theta<1)•$$

特别地设

$$\frac{\partial f}{\partial u}(x,t,u)=d(x,t),\quad d_*\leqslant d(x,t)\leqslant d^*•$$

由关系式(12)，经过若干变换，可得如下形式的不等式

$$\delta_t\leqslant C_1\delta+\rho,$$

其中

$$\delta(t)=\left(\left\{\frac{1}{2}+\frac{3\tau}{4}b-\frac{\lambda}{4}\tau^2b-(\mu_1d^*+\mu_2)\tau\right\}z_t,z_t\right)+$$

$$\left(\left\{\frac{3\tau}{4}a-\frac{\lambda}{4}\tau^2 a-\frac{\tau^2}{2}-\mu_4\tau\right\}z_t, z_{xt}\right)+\lambda(z_t, z)+\left(\left\{\frac{1}{2}c+\frac{\lambda}{4}a\right\}\hat{z}_x, \hat{z}_x\right)+$$
$$\left(\left[\frac{\lambda a}{4}-\frac{\tau}{2}\left\{\frac{1}{2}c_t+\frac{\lambda}{4}a_t-\lambda c+\lambda\mu_4\right\}\right]z_x, z_x\right)+$$
$$\left(\frac{\lambda}{4}b\hat{z}, \hat{z}\right)+\left(\left[\frac{\lambda}{4}b-\frac{\tau}{2}\left\{\frac{\lambda}{4}b_t-\lambda d^*+\lambda\mu_3+\frac{1}{4\mu_1}d^*\right\}\right]z, z\right)$$

为网格函数•

$$C_1=\min\left\{\frac{D_1}{B_1}, \frac{D_2}{B_2}, \frac{D_3}{B_3}\right\}>0,$$
$$\rho=\left(\frac{1}{4\mu_2}+\frac{\lambda}{4\mu_3}\right)\|R_i^{(1)}\|^2+\frac{1+\lambda}{4\mu_4}\|R_i^{(0)}\|^2,$$
$$D_1=\frac{l^2}{8}\min\left(0, b^*+\frac{\tau}{4}b_*-\lambda+\frac{\lambda}{4}\tau^2 b_*+\mu_1 d^*+\mu_2\right)+$$
$$a_*+\frac{\tau}{4}a_*-\frac{\tau}{2}c^*+\frac{\lambda}{4}\tau^2 a_*-\mu_4,$$
$$D_2=\frac{l^2}{8}\min\left(0, \frac{1}{2}\left\{-\frac{\lambda}{4}b^*+\lambda d^*-\lambda\mu_3-\frac{1}{4\mu_1}d^*\right\}\right)+$$
$$\frac{1}{2}\left\{-\frac{1}{2}c^*-\frac{\lambda}{4}a^*+\lambda c_*-\lambda\mu_4\right),$$
$$D_3=\frac{l^2}{8}\min\left(0, \frac{1}{2}\left\{-\frac{3\lambda}{4}b^*+\lambda d^*-\lambda\mu_3-\frac{1}{4\mu_1}d^*\right\}\right)+$$
$$\frac{1}{2}\left\{-\frac{1}{2}c^*-\frac{3\lambda}{4}a^*+\lambda c_*-\lambda\mu_4\right),$$
$$B_1=\frac{l^2}{8}\max\left(0, \frac{1}{2}+\frac{3\tau}{4}b^*-\frac{\lambda}{4}\tau^2 b_*-(\mu_1 d^*+\mu_2)\tau+\lambda\mu_5\right)+$$
$$\frac{3\tau}{4}a^*-\frac{\lambda}{4}\tau^2 a_*-\frac{\tau^2}{2}c_*-\mu_4\tau,$$
$$B_2=\frac{l^2}{8}\max\left(0, \frac{\lambda b^*}{4}\right)+\frac{c_*}{2}+\frac{\lambda}{4}a^*,$$
$$B_3=\frac{l^2}{8}\max\left(0, \left[\frac{\lambda b^*}{4}+\frac{\lambda}{4\mu_5}-\frac{\tau}{2}\left\{\frac{\lambda}{4}b_*-\lambda d^*+\lambda\mu_3+\frac{1}{4\mu_1}d^*\right\}\right]\right)+$$
$$\frac{\lambda a^*}{4}-\frac{\tau}{2}\left\{\frac{c_*}{2}+\frac{\lambda}{4}a_*-\lambda c^*+\lambda\mu_4\right\}•$$

由条件(10)对充分小的 μ_i, $i=\overline{1, 5}$, 可得

$$\delta(t)\geqslant C(\|z_t\|^2+\|\hat{z}_x\|^2+\|z\|^2)\qquad(C>0)• \tag{13}$$

对带周期条件的差分不等式应用等价的差分替换, 可得不等式

$$\delta_K\leqslant[1-\exp(-C_1T)]^{-1}\left\{\tau\sum_{i=1}^{M}|\rho_i|\exp(-C_1 t_{M-i})\right\}\exp(-C_1 t_K)+$$
$$\tau\sum_{i=1}^{M}|\rho_i|\exp(-C_1 t_{K-i})\qquad(K=1, 2, \cdots, M)•$$

考虑到式(13), 引理证毕• □

于是可以得如下定理•

定理 2.1 设 u 为 $C^5(D)$ 的一个单元且满足引理 2.1 的条件, 则差分问题(7)的解在网格 $\omega_h\times\omega_\tau^+$ 中收敛于(4)的解, 收敛速度的估计为

$$\|z_t\|+\|\hat{z}_x\|+\|z\|\leqslant C(h^2+\tau^2)\qquad(t\in\omega_\tau^+)• \tag{14}$$

证明　由式(11)可直接得出证明并得到局部误差. 实际上由式(11)的右边可得如下不等式

$$\tau\sum_{i=1}^{M}(\|R_j^{(0)}\|+\|R_j^{(1)}\|)\exp(-C_1t_{M-j})\leqslant$$

$$\tau\sum_{j=1}^{M-1}(\|R_j^{(0)}\|+\|R_j^{(1)}\|)+\tau(\|R^{M(0)}\|+\|R^{M(1)}\|).$$

该不等式的局部误差为

$$|R^{(0)}|,\ |R^{(1)}|\leqslant C(h^2+\tau^2)\qquad((x,t)\in\omega_h\times\omega_\tau),$$

$$|R^{M(0)}|,\ |R^{M(1)}|\leqslant C(h^2+\tau)\qquad(x\in\omega_h),$$

显然定理成立. □

3 数值例子

考虑如下例子的"理论"结果

$$f(x,t,u(x,t))\equiv\frac{xt}{1+(\sin\pi x\,e^{\sin2\pi t})^2}-\frac{xt}{1+\mu^2}+$$

$$(4\cos^2 2\pi t-4\sin2\pi t+2\pi\cos2\pi t-1)\pi^2\sin\pi x\,e^{\sin2\pi t},$$

$$a(x,t)=1,\ b(x,t)=0,\ c(x,t)=1,\ T=1,\ l=1.$$

近似问题的精确解为

$$u(x,t)=\sin\pi x\,e^{\sin2\pi t}.$$

对式(9)应用如下迭代步骤

$$y_{tt,i}^{(n+1)}-(Ay_{\overset{0}{t}x}^{(n+1)})_x+By_{\overset{0}{t}}^{(n+1)}-(Cy_{\bar{x}}^{(n+1)})_x=f,$$

$$y^{(n+1)}(0,t)=y^{(n+1)}(l,t)=0,$$

$$y^{(n+1)}(x,0)=y^{(n)}(x,T),$$

$$y^{(n+1)}(x,\tau)=y^{(n)}(x,T+\tau),$$

其中, $n=0,1,2,\cdots$; $y^{(0)}(x,T)$ 和 $y^{(0)}(x,T+\tau)$ 为任意函数.

计算结果列于表1和表2.

表1

$H=0.10$ (X,T)	$T_0=0.20$ $U(I,J)$	第二步 $Y(I,J)$	误　差 R
(0.20, 0.20)	1.521 448 59	1.866 095 29	0.226 525 37
(0.40, 0.40)	1.711 899 31	1.825 177 86	0.066 171 27
(0.60, 0.60)	0.528 365 48	0.187 618 23	0.644 908 23
(0.80, 0.80)	0.227 080 63	0.257 516 17	0.134 029 68

表2

$H=0.20$ (X,T)	$T_0=0.20$ $U(I,J)$	第四步 $Y(I,J)$	误　差 R
(0.20, 0.20)	1.521 448 59	1.739 810 95	0.143 522 66
(0.40, 0.40)	1.711 899 31	2.029 418 00	0.185 477 43
(0.60, 0.60)	0.528 365 48	0.303 254 45	0.462 051 74
(0.80, 0.80)	0.227 080 63	0.209 415 19	0.077 793 70

[参 考 文 献]

[1] Lagnese J L. General boundary_value problems for differential equations of Sobolev type[J]. SIAM, J Math Anal, 1972, **3**: 105—119.

[2] Leopold H. Periodic solutions to a one_dimensional strongly nonlinear wave equation with strong dissipation[J]. Czechoslovak Math J, 1985, **35**(110): 278—293.

[3] Webb G F. Existence and asymptotic behavior for a strongly damped nonlinear wave equation[J]. Canad J Math, 1980, **32**: 631—643.

[4] Lebedev V I. The method of difference for the equations of Sobolev type[J]. Dokl Acad Sci USSR, 1957, **114**: 1166—1169.

[5] Amiraliyev G M. Towards the numerical solution of the perodical on time problem for pseudo_parabolic equation[J]. Numerical Methods of Analysis BAKU State University, 1988, 3—8.

[6] Amiraliyev G M. On difference schemes for problems of the theory of dispersive waves[J]. Soviet Math Dokl, 1991, **42**: 235—238.

[7] Amiraliyev G M. Investigation of the difference schemes for the quasi_linear Sobolev equations[J]. Differential Equations, 1987, **23**: 1453—1455. (in Russian)

On the Numerical Solution of Quasilinear Wave Equation With Strong Dissipative Term

Aytekin G lle
(Department of Mathematics, Y z nc Yil University, Van 65080, Turkey)

Abstract: The numerical solution for a type of quasilinear wave equation is studied. The three_level difference scheme for quasi_linear waver equation with strong dissipative term is constructed and the convergence is proved. The error of the difference solution is estimated. The theoretical results are controlled on a numerical example.

Key words: periodical problem; quasilinear wave equation; difference scheme; numerical solution