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变厚度正交各向异性矩形板非线性非对称弯曲问题的基本方程*

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摘要: 在不计体力, 考虑了薄膜力引起在 z 方向的分力, 导出了厚度线性变化的正交各向异性矩形板非线性非对称弯曲问题的本构方程; 在引进无量纲变量和引入三个小参数的条件下, 给出了挠度函数 $W(x, y)$ 和应力函数 $\Phi(x, y)$ 的无量纲化的支配方程形式•

关 键 词: 变厚度正交各向异性矩形板; 非线性非对称弯曲; 平衡方程; 协调方程; 本构方程; 无量纲化方程;

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引 言

以前对各向同性或正交各向异性板的非线性非对称弯曲问题的研究仅限于对等厚度板的研究[1~5], 而对于变厚度正交各向异性矩形板的非线性非对称弯曲问题的研究在国内外尚未见到• 然而在许多实际应用中, 发生非线性非对称弯曲的正交各向异性矩形板的厚度是变化的, 理论和实践都需要对变厚度正交各向异性矩形板的非线性非对称弯曲问题开展研究•

在这篇文章里, 不计体力, 考虑了薄膜力 N_x, N_y 和 N_{xy} 引起的在 z 方向分力, 给出厚度线性变化正交各向异性矩形板的非线性非对称弯曲问题的本构方程; 在引进无量纲变量和引入三个小参数的条件下, 给出了挠度 $W(x, y)$ 和应力函数 $\Phi(x, y)$ 的无量纲化的支配方程形式•

1 平衡方程

在非线性非对称弯曲情况下, 不计体力, 考虑了薄膜力 N_x, N_y 和 N_{xy} 因转角引起在 z 方向的分力, 且忽略高阶小量后, 沿 z 方向的力平衡条件为:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q(x, y) + N_x \frac{\partial^2 W}{\partial x^2} + N_x \frac{\partial^2 W}{\partial y^2} + 2N_{xy} \frac{\partial^2 W}{\partial x \partial y} = 0, \tag{1}$$

x 轴和 y 轴方向的力矩平衡条件为:

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0, \tag{2}$$

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$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=0. \tag{3}$$

$q(x, y)$ 为非均布横向载荷. $W(x, y)$ 为挠度函数. Q_x、Q_y 为横向剪力; M_x 和 M_y 为弯矩, M_{xy} 为扭矩, 它们分别为:

$$M_x=-D_1\left(\frac{\partial^2 W}{\partial x^2}+\mu_{21}\frac{\partial^2 W}{\partial y^2}\right), \tag{4}$$

$$M_y=-D_2\left(\frac{\partial^2 W}{\partial y^2}+\mu_{12}\frac{\partial^2 W}{\partial x^2}\right), \tag{5}$$

$$M_{xy}=-2D_k\frac{\partial^2 W}{\partial x\partial y}, \tag{6}$$

其中

$$\begin{cases} D_1=\dfrac{E_1h^3(x, y)}{12(1-\mu_{12}\mu_{21})},\ D_2=\dfrac{E_2h^3(x, y)}{12(1-\mu_{12}\mu_{21})},\ D_k=\dfrac{Gh^3(x, y)}{12}, \\ D_3=\mu_{12}D_2+2D_k=\mu_{21}D_1+2D_k, \end{cases} \tag{7}$$

$h(x, y)$ 是矩形薄板的厚度, E_1 和 E_2 分别为 x 方向和 y 方向的弹性模量; μ_{12} 和 μ_{21} 是 x 方向和 y 方向的泊松比; G 是板的剪切模量; D_1 和 D_2 分别是 x 方向和 y 方向的抗弯刚度, D_3 是折合刚度, D_k 是抗扭刚度.

将(4)～(7)代入(2)和(3), 然后再代入(1)得

$$\begin{aligned}
&\frac{E_1h^3(x, y)}{12(1-\mu_{12}\mu_{21})}\left(\frac{\partial^4 W}{\partial x^4}+\mu_{21}\frac{\partial^4 W}{\partial x^2\partial y^2}\right)+\frac{E_1h^2(x, y)}{2(1-\mu_{12}\mu_{21})}\frac{\partial h(x, y)}{\partial x}\left(\frac{\partial^3 W}{\partial x^3}+\mu_{21}\frac{\partial^3 W}{\partial x\partial y^2}\right)+ \\
&\quad\left[\frac{E_1h(x, y)}{2(1-\mu_{12}\mu_{21})}\left(\frac{\partial h(x, y)}{\partial x}\right)^2+\frac{E_1h^2(x, y)}{4(1-\mu_{12}\mu_{21})}\frac{\partial^2 h(x, y)}{\partial x^2}\right]\left(\frac{\partial^2 W}{\partial x^2}+\mu_{21}\frac{\partial^2 W}{\partial y^2}\right)+ \\
&\quad\frac{G}{3}h^3(x, y)\frac{\partial^4 W}{\partial x^2\partial y^2}+Gh^2(x, y)\left(\frac{\partial h(x, y)}{\partial y}\frac{\partial^3 W}{\partial x^2\partial y}+\frac{\partial h(x, y)}{\partial x}\frac{\partial^2 W}{\partial x\partial y^2}\right)+ \\
&\quad 2\left[Gh(x, y)\frac{\partial h(x, y)}{\partial x}\frac{\partial h(x, y)}{\partial y}+\frac{G}{2}h^2(x, y)\frac{\partial^2 h(x, y)}{\partial x\partial y}\right]\frac{\partial^2 W}{\partial x\partial y}+ \\
&\quad\frac{E_2h^3(x, y)}{12(1-\mu_{12}\mu_{21})}\left(\frac{\partial^4 W}{\partial y^4}+\mu_{12}\frac{\partial^4 W}{\partial x^2\partial y^2}\right)+\frac{E_2h^2(x, y)}{2(1-\mu_{12}\mu_{21})}\times \\
&\quad\frac{\partial h(x, y)}{\partial y}\left(\frac{\partial^3 W}{\partial y^3}+\mu_{12}\frac{\partial^3 W}{\partial x^2\partial y}\right)+\left[\frac{E_2h(x, y)}{2(1-\mu_{12}\mu_{21})}\left(\frac{\partial h(x, y)}{\partial y}\right)^2+\right. \\
&\quad\left.\frac{E_2h^2(x, y)}{4(1-\mu_{12}\mu_{21})}\frac{\partial^2 h(x, y)}{\partial y^2}\right]\left(\frac{\partial^2 W}{\partial y^2}+\mu_{12}\frac{\partial^2 W}{\partial x^2}\right)- \\
&\quad q(x, y)-N_x\frac{\partial^2 W}{\partial x^2}-N_y\frac{\partial^2 W}{\partial y^2}-2N_{xy}\frac{\partial^2 W}{\partial x\partial y}=0.
\end{aligned} \tag{8}$$

为简单起见, 仅讨论 $h(x, y)$ 只随 x 变化的情况, 即 $h(x, y)\equiv h(x)$, 则(8)变为

$$\begin{aligned}
&\frac{E_1h^3(x, y)}{12(1-\mu_{12}\mu_{21})}\frac{\partial^4 W}{\partial x^4}+\left(\frac{E_1\mu_{21}}{6(1-\mu_{12}\mu_{21})}+\frac{G}{3}\right)h^3(x, y)\frac{\partial^4 W}{\partial x^2\partial y^2}+ \\
&\quad\frac{E_2h^3(x, y)}{12(1-\mu_{12}\mu_{21})}\frac{\partial^4 W}{\partial y^4}+\frac{E_1h^2(x)}{2(1-\mu_{12}\mu_{21})}\frac{dh(x)}{dx}\frac{\partial^3 W}{\partial x^3}+ \\
&\quad\left(\frac{E_1\mu_{21}}{2(1-\mu_{12}\mu_{21})}+G\right)h^2(x, y)\frac{dh(x)}{dx}\frac{\partial^3 W}{\partial x\partial y^2}+ \\
&\quad\frac{E_1h(x)}{2(1-\mu_{12}\mu_{21})}\left[\left(\frac{dh(x)}{dx}\right)^2+\frac{h(x)}{2}\frac{d^2h(x)}{dx^2}\right]\times
\end{aligned}$$

$$h(x)\left(\frac{\partial^2 W}{\partial x^2}+\mu_{21}\frac{\partial^2 W}{\partial y^2}\right)-q(x,y)-h(x)\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 \Phi}{\partial y^2}-$$
$$h(x)\frac{\partial^2 W}{\partial y^2}\frac{\partial^2 \Phi}{\partial x^2}+2h(x)\frac{\partial^2 W}{\partial x\partial y}\frac{\partial^2 \Phi}{\partial x\partial y}=0. \tag{9}$$

此处取[6]

$$N_x=h(x)\frac{\partial^2 \Phi}{\partial y^2},\ N_y=h(x)\frac{\partial^2 \Phi}{\partial y^2},\ N_{xy}=-h(x)\frac{\partial^2 \Phi}{\partial x\partial y}, \tag{10}$$

$\Phi(x,y)$ 是应力函数.

2　协调方程

非对称大挠度问题的中面变形几何方程为:

$$\begin{cases}\varepsilon_x^0=\dfrac{\partial u}{\partial x}+\dfrac{1}{2}\left(\dfrac{\partial W}{\partial x}\right)^2,\ \varepsilon_y^0=\dfrac{\partial v}{\partial y}+\dfrac{1}{2}\left(\dfrac{\partial W}{\partial y}\right)^2,\\ \gamma_{xy}^0=\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}+\dfrac{\partial W}{\partial x}\dfrac{\partial W}{\partial y}.\end{cases} \tag{11}$$

其中 ε_x^0, ε_y^0, γ_{xy}^0 为中面内任意一点的应变分量, u, v 分别为沿 x 轴和 y 轴的位移分量.

薄膜力 N_x, N_y, N_{xy} 与应变分量的关系为:

$$\begin{cases}\varepsilon_x^0=\dfrac{1}{E_1h(x)}(N_x-\mu_{12}N_y),\ \varepsilon_y^0=\dfrac{1}{E_2h(x)}(N_y-\mu_{21}N_x),\\ \gamma_{xy}^0=\dfrac{1}{Gh(x)}N_{xy}.\end{cases} \tag{12}$$

由(11)得协调方程

$$\frac{\partial^2 \varepsilon_x^0}{\partial y^2}+\frac{\partial^2 \varepsilon_y^0}{\partial x^2}-\frac{\partial^2 \gamma_{xy}^0}{\partial x\partial y}=\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}. \tag{13}$$

将(12)代入上式, 可得用挠度函数 $W(x,y)$ 和应力函数 $\Phi(x,y)$ 表示的协调方程

$$\frac{1}{E_1}\frac{\partial^4 \Phi}{\partial y^4}+\left(\frac{1}{G}-\frac{\mu_{12}}{E_1}-\frac{\mu_{21}}{E_2}\right)\frac{\partial^4 \Phi}{\partial x^2\partial y^2}+\frac{1}{E_2}\frac{\partial^4 \Phi}{\partial x^4}=$$
$$\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}. \tag{14}$$

方程(9)和(14)构成了变厚度正交各向异性矩形薄板非线性非对称弯曲问题的基本方程.

假设板的厚度变化规律为

$$h(x)=h_0\left[1+\varepsilon\gamma\left(\frac{x}{a}-\frac{1}{2}\right)\right], \tag{15}$$

其中 h_0 为板中心的厚度, a 为正交各向异性薄板的长度, $|\gamma|<1$,

$$\varepsilon=\begin{cases}-1 & (0\leqslant x\leqslant a/2),\\ 1 & (a/2\leqslant x\leqslant a).\end{cases} \tag{16}$$

3　无量纲化的基本方程

引入无量纲变量

$$W=\frac{W}{a},\ x=\frac{x}{a},\ y=\frac{y}{a},\ \Phi=\frac{\Phi}{E_1a^2},\ q=\frac{aq}{h_0E_1}. \tag{17}$$

将方程(9)、(14)无量纲化(略去符号"～"), 则得

$$\Gamma^3\left[\varepsilon_1^2\frac{\partial^4 W}{\partial x^4}+\varepsilon_2^2\frac{\partial^4 W}{\partial x^2\partial y^2}+\delta_2\varepsilon_1^2\frac{\partial^4 W}{\partial y^4}\right]+6\Gamma^2\left(\varepsilon_1^2\frac{\partial^3 W}{\partial x^3}+\right.$$

$$\varepsilon\gamma\varepsilon_2^2\frac{\partial^3 W}{\partial x\partial y^2}\Bigg)+6\gamma^2\varepsilon_1^2\Gamma\left(\frac{\partial^2 W}{\partial x^2}+\mu_{21}\frac{\partial^2 W}{\partial y^2}\right)-$$

$$\Gamma\left[\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 \Phi}{\partial y^2}+\frac{\partial^2 W}{\partial y^2}\frac{\partial^2 \Phi}{\partial x^2}-2\frac{\partial^2 W}{\partial x\partial y}\frac{\partial^2 \Phi}{\partial x\partial y}\right]=q(x,y), \tag{18}$$

$$\delta_2\frac{\partial^4 \Phi}{\partial y^4}+\delta_1\frac{\partial^4 \Phi}{\partial x^2\partial y^2}+\frac{\partial^4 \Phi}{\partial x^4}-\delta_2\left[\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\right]=0, \tag{19}$$

式中，$\Gamma=1+\varepsilon\gamma(x-0.5)$•

若假定 $G<E_2<E_1$，则其中

$$\varepsilon_1^2=\frac{h_0^2}{12(1-\mu_{12}\mu_{21})a^2}\ll 1,\quad \varepsilon_3^2=\frac{E_2}{E_1}\varepsilon_1^2\ll 1$$

$$\varepsilon_2^2=2\left(\varepsilon_1^2\mu_{21}+\frac{1}{6}\frac{G}{E_1}\frac{h_0^2}{a^2}\right)\ll 1,\quad \delta_1=E_2\left(\frac{1}{G}-\frac{\mu_{12}}{E_1}\frac{\mu_{21}}{E_2}\right),\quad \delta_2=\frac{E_2}{E_1}<1•$$

[参　考　文　献]

[1] 秦圣立,黄家寅. 圆柱型正交各向异性圆形薄板的非线性非对称弯曲问题(Ⅰ)[J]. 应用数学和力学, 1994, **15**(8): 741—757.

[2] 黄家寅,秦圣立,夏云杰,等. 圆柱型正交各向异性圆形薄板的非线性非对称弯曲问题(Ⅱ)[J]. 应用数学和力学, 1995, **16**(11): 1003—1016.

[3] 黄家寅,秦圣立. 四边固支正交各向异性矩形板的非线性非对称弯曲问题[J]. 应用数学和力学, 1996, **17**(3): 213—238.

[4] 秦圣立,张爱淑. 关于具有初始挠度的圆形薄板的跳跃问题[J]. 应用数学和力学, 1987, **8**(5): 433—443.

[5] 周次青. 正交各向异性板的非线性弯曲[J]. 应用数学和力学, 1984, **5**(3): 419—436.

[6] 何福保.《板壳理论》[M]. 西安: 西安交通大学出版社, 1993.

Basic Equations of the Problem of the Nonlinear Unsymmetrical Bending for Orthotropic Rectangular Thin Plate With Variable Thickness

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Abstract: Under the case of ignoring the body forces and considering components caused by diversion of membrane in vertical direction (z direction), the constitutive equations of the problem of the non_linear unsymmetrical bending for orthotropic rectangular thin plate with variable thickness are given; then introducing the dimensionless variables and three small parameters, the dimensionaless governing equations of the deflection function and stress function are given.

Key words: orthotropic rectangular thin plate with variable thickness; nonlinear unsymmetrical bending; equilibrium equation; compatibility equation; constitutive equation; dimensionless equation