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四边固定变厚度正交各向异性矩形板的非线性非对称弯曲问题的一致有效渐近解*

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摘要: 利用“ 修正的两变量法” 和“ 混合摄动法” , 引进 4 个小参数, 对厚度线性变化的正交各向异性矩形板的非线性非对称弯曲问题进行了研究, 得到了挠度函数 $W(x, y)$ 和应力函数 $\Phi(x, y)$ 对 ε_1 为 N 阶及对 ε_2 为 M 阶的一致有效渐近解•

关 键 词: 变厚度正交各向异性矩形板; 四边固定; 非线性非对称弯曲; 修正的两变量法; 混合摄动法; 一致有效渐近解
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引 言

在上个世纪的 80 年代初, 江福汝先生在研究非线性而非对称弯曲问题中开创性地提出了“ 修正的两变量法”[1], 这对于研究非线性非对称弯曲问题是十分重要的• 在文献[2] ~ [6]中对等厚度各向同性或正交各向异性板的非对称弯曲问题进行了研究, 而对变厚度正交各向异性圆型薄板的非线性非对称弯曲问题进行了研究• 在本文中利用“ 修正的两变量法” 和“ 混合摄动法”[7] 对变厚度正交各向异性矩形板的非线性非对称弯曲问题进行研究• 在厚度沿 x 方向变化的情况下, 引进四个小参数, 得到了挠度函数 $W(x, y)$ 和应力函数 $\Phi(x, y)$ 对 ε_1 为 N 阶及对 ε_2 为 M 阶的一致有效渐近解•

1 递推方程和递推边界条件

文献[8]中, 给出了厚度线性变化正交各向异性矩形板非线性非对称弯曲问题的无量纲的本构方程• 设非线性非对称弯曲正交各向异性矩形板的四边固定, 则无量纲化边界条件为

$$W|_{x=0} = \left.\frac{\partial W}{\partial x}\right|_{x=0} = W|_{x=1} = \left.\frac{\partial W}{\partial x}\right|_{x=1} = 0, \tag{1a}$$

$$W|_{y=0} = \left.\frac{\partial W}{\partial y}\right|_{y=0} = W|_{y=b/a} = \left.\frac{\partial W}{\partial y}\right|_{y=b/a} = 0, \tag{1b}$$

$$W|_{(0,0)} = \left.\frac{\partial W}{\partial x}\right|_{(0,0)} = \left.\frac{\partial W}{\partial y}\right|_{(0,0)} = W|_{(1,0)} = 0, \tag{1c}$$

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$$\left.\frac{\partial W}{\partial x}\right|_{(1,0)}=\left.\frac{\partial W}{\partial y}\right|_{(1,0)}=W|_{(0,b/a)}=\left.\frac{\partial W}{\partial x}\right|_{(0,b/a)}=0, \tag{1d}$$

$$\left.\frac{\partial W}{\partial y}\right|_{(0,b/a)}=W|_{(1,b/a)}=\left.\frac{\partial W}{\partial x}\right|_{(1,b/a)}=\left.\frac{\partial W}{\partial y}\right|_{(1,b/a)}=0, \tag{1e}$$

$$\left.\left(\frac{\partial^2\Phi}{\partial x^2}-\mu_{21}\frac{\partial^2\Phi}{\partial y^2}\right)\right|_{x=0}=0,\quad \left.\left(\frac{\partial^2\Phi}{\partial x^2}-\mu_{21}\frac{\partial^2\Phi}{\partial y^2}\right)\right|_{x=1}=0, \tag{1f}$$

$$\left.\left(\frac{\partial^2\Phi}{\partial y^2}-\mu_{12}\frac{\partial^2\Phi}{\partial x^2}\right)\right|_{y=0}=0,\quad \left.\left(\frac{\partial^2\Phi}{\partial y^2}-\mu_{12}\frac{\partial^2\Phi}{\partial x^2}\right)\right|_{y=b/a}=0\cdot \tag{1g}$$

$\varepsilon,\ \delta,\ \varepsilon_1,\ \varepsilon_2,\ \delta_1,\ \delta_2$ 的定义见文献[8]•

设挠度函数 $W(x,y)$ 和应力函数 $\Phi(x,y)$ 对 ε_1 为 N 阶和对 ε_2 为 M 阶的展开式为

$$\begin{aligned}W(x,y,\varepsilon_1,\varepsilon_2)=&\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m w_{nm}(x,y)+\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\varepsilon_2^{m+\alpha_2}v_{nm}^{(1)}(\xi,\eta,y)+\\&\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\varepsilon_2^{n+\alpha_4}v_{nm}^{(2)}(\xi,\eta,y)+\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\varepsilon_2^{m+\alpha_6}v_{nm}^{(3)}(x,\alpha,\beta)+\\&\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\varepsilon_2^{m+\alpha_8}v_{nm}^{(4)}(x,\alpha,\beta),\end{aligned} \tag{2}$$

$$\begin{aligned}\Phi(x,y,\varepsilon_1,\varepsilon_2)=&\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m\varphi_{nm}(x,y)+\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_1}\varepsilon_2^{m+\beta_2}\psi_{nm}^{(1)}(\xi,\eta,y)+\\&\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_3}\varepsilon_2^{m+\beta_4}\psi_{nm}^{(2)}(\xi,\eta,y)+\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_5}\varepsilon_2^{m+\beta_6}\psi_{nm}^{(3)}(x,\alpha,\beta)+\\&\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_7}\varepsilon_2^{m+\beta_8}\psi_{nm}^{(4)}(x,\alpha,\beta)\end{aligned} \tag{3}$$

式中 $v_{nm}^{(1)},\ \psi_{nm}^{(1)}$ 和 $v_{nm}^{(2)},\ \psi_{nm}^{(2)}$ 分别是 $x=0$ 和 $x=1$ 邻域的边界层型函数; $v_{nm}^{(3)},\ \psi_{nm}^{(3)}$ 和 $v_{nm}^{(4)},\ \psi_{nm}^{(4)}$ 分别是 $y=0$ 和 $y=b/a$ 邻域的边界层型函数• $\alpha_1,\ \dots,\ \alpha_8;\ \beta_1,\ \dots,\ \beta_8$ 则为待定系数•

将微分算子[6]、挠度函数(2)和应力函数(3)代入本构方程得:

$$\begin{aligned}&\Bigg\{\varepsilon_1^2\Gamma^3\left[\frac{\partial^4}{\partial x^4}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m w_{nm}\right)+\delta_2\frac{\partial^4}{\partial y^4}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m w_{nm}\right)\right]+\\&\varepsilon_2^2\Gamma^3\frac{\partial^4}{\partial x^2\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m w_{nm}\right)+6\gamma\Gamma^2(\varepsilon_1^2+\varepsilon_2^2)\left[\frac{\partial^3}{\partial x^3}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m w_{nm}\right)+\right.\\&\left.\frac{\partial^3}{\partial x\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m w_{nm}\right)\right]+6\varepsilon_1^2\gamma^2\Gamma(1+\mu_{21})\left[\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m w_{nm}\right)+\right.\\&\left.\frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m w_{nm}\right)\right]-\Gamma\left[\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m w_{nm}\right)\frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m\varphi_{nm}\right)+\right.\\&\frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m w_{nm}\right)\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m\varphi_{nm}\right)-\\&\left.2\frac{\partial^2}{\partial x\partial y}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m w_{nm}\right)\frac{\partial^2}{\partial x\partial y}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^n\varepsilon_2^m\varphi_{nm}\right)\right]\Bigg\}+\\&\Bigg\{\varepsilon_1^2\Gamma^3\left[\varepsilon_1^{-4}\left(\sum_{i=0}^{4}\varepsilon_1^i A_{4,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\varepsilon_2^{m+\alpha_2}v_{nm}^{(1)}\right)+\right.\\&\varepsilon_1^{-4}\left(\sum_{i=0}^{4}\varepsilon_1^i A_{4,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\varepsilon_2^{m+\alpha_4}v_{nm}^{(2)}\right)+\\&\left.\frac{\partial^4}{\partial x^4}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\varepsilon_2^{m+\alpha_6}v_{nm}^{(3)}+\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\varepsilon_2^{m+\alpha_8}v_{nm}^{(4)}\right)\right]+\end{aligned}$$

$$\varepsilon_2^2\Gamma^3\Bigg[\varepsilon_1^{-2}\Big(\sum_{i=0}^{2}\varepsilon_1^iA_{2,i}\Big)\frac{\partial^2}{\partial y^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\varepsilon_2^{m+\alpha_2}v_{nm}^{(1)}\Big)+$$

$$\varepsilon_1^{-2}\Big(\sum_{i=0}^{2}\varepsilon_1^iA_{2,i}\Big)\frac{\partial^2}{\partial y^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\varepsilon_2^{m+\alpha_4}v_{nm}^{(2)}\Big)+$$

$$\frac{\partial^2}{\partial x^2}\varepsilon_1^{-2}\Big(\sum_{i=0}^{2}\varepsilon_1^iB_{2,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\varepsilon_2^{m+\alpha_6}v_{nm}^{(3)}\Big)+$$

$$\frac{\partial^2}{\partial x^2}\varepsilon_1^{-2}\Big(\sum_{i=0}^{2}\varepsilon_1^iB_{2,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\varepsilon_2^{m+\alpha_8}v_{nm}^{(4)}\Big)\Bigg]+$$

$$\varepsilon_1^2\delta_2\Gamma^3\Bigg[\frac{\partial^4}{\partial y^4}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\varepsilon_2^{m+\alpha_2}v_{nm}^{(1)}\Big)+\frac{\partial^4}{\partial y^4}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\varepsilon_2^{m+\alpha_4}v_{nm}^{(2)}\Big)+$$

$$\varepsilon_1^{-4}\Big(\sum_{i=0}^{4}\varepsilon_1^iB_{4,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\varepsilon_2^{m+\alpha_6}v_{nm}^{(3)}\Big)+$$

$$\varepsilon_1^{-4}\Big(\sum_{i=0}^{4}\varepsilon_1^iB_{4,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\varepsilon_2^{m+\alpha_8}v_{nm}^{(4)}\Big)\Bigg]+$$

$$6\varepsilon_1^2\varepsilon\gamma\Gamma^2\Bigg[\varepsilon_1^{-3}\Big(\sum_{i=0}^{3}\varepsilon_1^iA_{3,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\varepsilon_2^{m+\alpha_2}v_{nm}^{(1)}\Big)+$$

$$\varepsilon_1^{-3}\Big(\sum_{i=0}^{3}\varepsilon_1^iA_{3,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\varepsilon_2^{m+\alpha_4}v_{nm}^{(2)}\Big)+$$

$$\frac{\partial^3}{\partial x^3}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\varepsilon_2^{m+\alpha_6}v_{nm}^{(3)}\Big)+\frac{\partial^3}{\partial x^3}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\varepsilon_2^{m+\alpha_8}v_{nm}^{(4)}\Big)\Bigg]+$$

$$6\varepsilon_2^2\varepsilon\gamma\Gamma^2\Bigg[\varepsilon_1^{-1}\Big(\sum_{i=0}^{1}\varepsilon_1^iA_{1,i}\Big)\frac{\partial^2}{\partial y^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\varepsilon_2^{m+\alpha_2}v_{nm}^{(1)}\Big)+$$

$$\varepsilon_1^{-1}\Big(\sum_{i=0}^{1}\varepsilon_1^iA_{1,i}\Big)\frac{\partial^2}{\partial y^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\varepsilon_2^{m+\alpha_4}v_{nm}^{(2)}\Big)+$$

$$\frac{\partial}{\partial x}\varepsilon_1^{-2}\Big(\sum_{i=0}^{2}\varepsilon_1^iB_{2,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\varepsilon_2^{m+\alpha_6}v_{nm}^{(3)}\Big)+$$

$$\frac{\partial}{\partial x}\varepsilon_1^{-2}\Big(\sum_{i=0}^{2}\varepsilon_1^iB_{2,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\varepsilon_2^{m+\alpha_8}v_{nm}^{(4)}\Big)\Bigg]+$$

$$6\varepsilon_1^2\gamma^2\Gamma\Bigg[\varepsilon_1^{-2}\Big(\sum_{i=0}^{2}\varepsilon_1^iA_{2,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\varepsilon_2^{m+\alpha_2}v_{nm}^{(1)}\Big)+$$

$$\varepsilon_1^{-2}\Big(\sum_{i=0}^{2}\varepsilon_1^iA_{2,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\varepsilon_2^{m+\alpha_4}v_{nm}^{(2)}\Big)+$$

$$\frac{\partial^2}{\partial x^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\varepsilon_2^{m+\alpha_6}v_{nm}^{(3)}\Big)+\frac{\partial^2}{\partial x^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\varepsilon_2^{m+\alpha_8}v_{nm}^{(4)}\Big)\Bigg]+$$

$$6\varepsilon_1^2\gamma^2\mu_{21}\Gamma\Bigg[\frac{\partial^2}{\partial y^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\varepsilon_2^{m+\alpha_2}v_{nm}^{(1)}\Big)+\frac{\partial^2}{\partial y^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\varepsilon_2^{m+\alpha_4}v_{nm}^{(2)}\Big)\Bigg]-$$

$$\varepsilon_1^{-2}\Big(\sum_{i=0}^{2}\varepsilon_1^iB_{2,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\varepsilon_2^{m+\alpha_6}v_{nm}^{(3)}\Big)+$$

$$\varepsilon_1^{-2}\Big(\sum_{i=0}^{2}\varepsilon_1^iB_{2,i}\Big)\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\varepsilon_2^{m+\alpha_8}v_{nm}^{(4)}\Big)\Bigg]-$$

$$\Gamma\Bigg[\Bigg(\frac{\partial^2}{\partial y^2}\Big(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n}\varepsilon_2^{m}\varphi_{nm}\Big)\Bigg)(\mathrm{L}_0)+$$

$$\left(\frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_1}\varepsilon_2^{m+\beta_2}\psi_{nm}^{(1)}\right)+\frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_3}\varepsilon_2^{m+\beta_4}\psi_{nm}^{(2)}\right)+\right.$$

$$\varepsilon_1^{-2}\left(\sum_{i=0}^{2}\varepsilon_1^{i}B_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_5}\varepsilon_2^{m+\beta_6}\psi_{nm}^{(3)}\right)+$$

$$\left.\varepsilon_1^{-2}\left(\sum_{i=0}^{2}\varepsilon_1^{i}B_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_7}\varepsilon_2^{m+\beta_8}\psi_{nm}^{(4)}\right)\right)(L_1)+$$

$$\left.\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n}\varepsilon_2^{m}\varphi_{nm}\right)\right)(K_0)+$$

$$\left(\varepsilon_1^{-2}\left(\sum_{i=0}^{2}\varepsilon_1^{i}A_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_1}\varepsilon_2^{m+\beta_2}\psi_{nm}^{(1)}\right)+\right.$$

$$\varepsilon_1^{-2}\left(\sum_{i=0}^{2}\varepsilon_1^{i}A_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_3}\varepsilon_2^{m+\beta_4}\psi_{nm}^{(2)}\right)+$$

$$\left.\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_5}\varepsilon_2^{m+\beta_6}\psi_{nm}^{(3)}\right)+\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_7}\varepsilon_2^{m+\beta_8}\psi_{nm}^{(4)}\right)\right)(K_1)-$$

$$2\left(\frac{\partial^2}{\partial x\partial y}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n}\varepsilon_2^{m}\varphi_{nm}\right)\right)(J_0)-$$

$$2\left(\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^{i}A_{1,i}\right)\frac{\partial}{\partial y}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_1}\varepsilon_2^{m+\beta_2}\psi_{nm}^{(1)}\right)+\right.$$

$$\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^{i}A_{1,i}\right)\frac{\partial}{\partial y}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_3}\varepsilon_2^{m+\beta_4}\psi_{nm}^{(2)}\right)+$$

$$\frac{\partial}{\partial x}\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^{i}B_{1,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_5}\varepsilon_2^{m+\beta_6}\psi_{nm}^{(3)}\right)+$$

$$\left.\left.\left.\frac{\partial}{\partial x}\varepsilon_1^{-1}\left(\sum_{i=0}^{2}\varepsilon_1^{i}B_{1,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\varepsilon_2^{m+\alpha_8}\psi_{nm}^{(4)}\right)\right)(J_1)\right]\right\}=q(x,y), \qquad (4)$$

$$\left\{\frac{\partial^4}{\partial x^4}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n}\varepsilon_2^{m}\varphi_{nm}\right)+\delta_1\frac{\partial^4}{\partial x^2\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n}\varepsilon_2^{m}\varphi_{nm}\right)+\right.$$

$$\delta_2\left[\frac{\partial^4}{\partial y^4}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n}\varepsilon_2^{m}\varphi_{nm}\right)-\left(\frac{\partial^2}{\partial x\partial y}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n}\varepsilon_2^{m}w_{nm}\right)\right)^2+\right.$$

$$\left.\left.\left(\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n}\varepsilon_2^{m}w_{nm}\right)\right)\left(\frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n}\varepsilon_2^{m}w_{nm}\right)\right)\right]\right\}+$$

$$\left\{\left[\varepsilon_1^{-4}\left(\sum_{i=0}^{4}\varepsilon_1^{i}A_{4,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_1}\varepsilon_2^{m+\beta_2}\psi_{nm}^{(1)}\right)+\right.\right.$$

$$\varepsilon_1^{-4}\left(\sum_{i=0}^{4}\varepsilon_1^{i}A_{4,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_3}\varepsilon_2^{m+\beta_4}\psi_{nm}^{(2)}\right)+$$

$$\left.\frac{\partial^4}{\partial x^4}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_5}\varepsilon_2^{m+\beta_6}\psi_{nm}^{(3)}\right)+\frac{\partial^4}{\partial x^4}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_7}\varepsilon_2^{m+\beta_8}\psi_{nm}^{(4)}\right)\right]+$$

$$\delta_1\left(\varepsilon_1^{-2}\left(\sum_{i=0}^{2}\varepsilon_1^{i}A_{2,i}\right)\frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_1}\varepsilon_2^{m+\beta_2}\psi_{nm}^{(1)}\right)+\right.$$

$$\varepsilon_1^{-2}\left(\sum_{i=0}^{2}\varepsilon_1^{i}A_{2,i}\right)\frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_3}\varepsilon_2^{m+\beta_4}\psi_{nm}^{(2)}\right)+$$

$$\frac{\partial^2}{\partial x^2}\varepsilon_1^{-2}\left[\left(\sum_{i=0}^{2}\varepsilon_1^{i}B_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_5}\varepsilon_2^{m+\beta_6}\psi_{nm}^{(3)}\right)+\right.$$

$$\left(\sum_{i=0}^{2}\varepsilon_1^{i}B_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_7}\varepsilon_2^{m+\beta_8}\psi_{nm}^{(4)}\right)\Bigg]+$$

$$\delta_2\Bigg[\frac{\partial^4}{\partial y^4}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_1}\varepsilon_2^{m+\beta_2}\psi_{nm}^{(1)}\right)+\frac{\partial^4}{\partial y^4}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_3}\varepsilon_2^{m+\beta_4}\psi_{nm}^{(2)}\right)+$$

$$\varepsilon_1^{-2}\left(\sum_{i=0}^{4}\varepsilon_1^{i}B_{4,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_5}\varepsilon_2^{m+\beta_6}\psi_{nm}^{(3)}\right)+$$

$$\varepsilon_1^{-2}\left(\sum_{i=0}^{4}\varepsilon_1^{i}B_{4,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\beta_7}\varepsilon_2^{m+\beta_8}\psi_{nm}^{(4)}\right)-$$

$$\left[\frac{\partial^2}{\partial x\partial y}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n}\varepsilon_2^{m}w_{nm}\right)\right](\mathrm{J}_0)-$$

$$\Bigg\{\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^{i}A_{1,i}\right)\frac{\partial}{\partial y}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\varepsilon_2^{m+\alpha_2}v_{nm}^{(1)}\right)+$$

$$\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^{i}A_{1,i}\right)\frac{\partial}{\partial y}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\varepsilon_2^{m+\alpha_4}v_{nm}^{(2)}\right)+$$

$$\frac{\partial}{\partial x}\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^{i}B_{1,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\varepsilon_2^{m+\alpha_6}v_{nm}^{(3)}\right)+$$

$$\frac{\partial}{\partial x}\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^{i}B_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\varepsilon_2^{m+\alpha_8}v_{nm}^{(4)}\right)\Bigg\}(\mathrm{J}_1)+$$

$$\left[\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n}\varepsilon_2^{m}w_{nm}\right)\right](\mathrm{K}_0)+$$

$$\Bigg[\Bigg(\varepsilon_1^{-2}\left(\sum_{i=0}^{2}\varepsilon_1^{i}A_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\varepsilon_2^{m+\alpha_2}v_{nm}^{(1)}\right)+$$

$$\varepsilon_1^{-2}\left(\sum_{i=0}^{2}\varepsilon_1^{i}A_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\varepsilon_2^{m+\alpha_4}v_{nm}^{(2)}\right)+$$

$$\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\varepsilon_2^{m+\alpha_6}v_{nm}^{(3)}\right)+\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\varepsilon_2^{m+\alpha_8}v_{nm}^{(4)}\right)\Bigg)(\mathrm{K}_1)\Bigg]\Bigg\}=0, \tag{5}$$

其中 $\Gamma = 1+\varepsilon\gamma(1-0.5)$,

$$\mathrm{L}_0=\varepsilon_1^{-2}\left(\sum_{i=0}^{2}\varepsilon_1^{i}A_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\varepsilon_2^{m+\alpha_2}v_{nm}^{(1)}\right)+$$

$$\varepsilon_1^{-2}\left(\sum_{i=0}^{2}\varepsilon_1^{i}A_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\varepsilon_2^{m+\alpha_4}v_{nm}^{(2)}\right)+$$

$$\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\varepsilon_2^{m+\alpha_6}v_{nm}^{(3)}\right)+\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\varepsilon_2^{m+\alpha_8}v_{nm}^{(4)}\right),$$

$$\mathrm{L}_1=\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n}\varepsilon_2^{m}w_{nm}\right)+\mathrm{L}_0,$$

$$\mathrm{K}_0=\frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\varepsilon_2^{m+\alpha_2}v_{nm}^{(1)}\right)+\frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\varepsilon_2^{m+\alpha_4}v_{nm}^{(2)}\right)+$$

$$\varepsilon_1^{-2}\left(\sum_{i=0}^{2}\varepsilon_1^{i}B_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\varepsilon_2^{m+\alpha_6}v_{nm}^{(3)}\right)+$$

$$\varepsilon_1^{-2}\left(\sum_{i=0}^{2}\varepsilon_1^{i}B_{2,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\varepsilon_2^{m+\alpha_8}v_{nm}^{(4)}\right),$$

$$\mathrm{K}_1=\frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n}\varepsilon_2^{m}w_{nm}\right)+\mathrm{K}_0,$$

$$
\begin{aligned}
J_0 = {} & \varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^{i}A_{1,i}\right)\frac{\partial}{\partial y}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_1}\varepsilon_2^{m+\alpha_2}v_{nm}^{(1)}\right)+ \\
& \varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^{i}A_{1,i}\right)\frac{\partial}{\partial y}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_3}\varepsilon_2^{m+\alpha_4}v_{nm}^{(2)}\right)+ \\
& \frac{\partial}{\partial x}\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^{i}B_{1,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_5}\varepsilon_2^{m+\alpha_6}v_{nm}^{(3)}\right)+ \\
& \frac{\partial}{\partial x}\varepsilon_1^{-1}\left(\sum_{i=0}^{1}\varepsilon_1^{i}B_{1,i}\right)\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n+\alpha_7}\varepsilon_2^{m+\alpha_8}v_{nm}^{(4)}\right),
\end{aligned}
$$

$$
J_1 = \frac{\partial^2}{\partial x\,\partial y}\left(\sum_{n=0}^{N}\sum_{m=0}^{M}\varepsilon_1^{n}\varepsilon_2^{m}w_{nm}\right)+J_0.
$$

同样,将(2),(3)以及展开的微分算子代入边界条件(1),可得到边界层型函数表示的边界条件.

取 $\alpha_1=\alpha_3=\alpha_5=\alpha_7=2,\ \beta_1=\beta_3=\beta_5=\beta_7=4,\ \alpha_2=\alpha_4=\alpha_6=\alpha_8=\beta_2=\beta_4=\beta_6=\beta_8=0$[6]. 将它们代入(4)、(5)及诸边界条件,然后比较第一个大括号中 $\varepsilon_1\varepsilon_2$ 同次幂的系数可得挠度函数和应力函数的递推方程和递推边界条件:

$$
\Gamma\left[\frac{\partial^2\varphi_{00}}{\partial y^2}\frac{\partial^2 w_{00}}{\partial x^2}+\frac{\partial^2\varphi_{00}}{\partial x^2}\frac{\partial^2 w_{00}}{\partial y^2}-\frac{\partial^2\varphi_{00}}{\partial x\partial y}\frac{\partial^2 w_{00}}{\partial x\partial y}\right]=-q(x,y), \tag{6}
$$

$$
\frac{\partial^4\varphi_{00}}{\partial x^4}+\delta_1\frac{\partial^4\varphi_{00}}{\partial x^2\partial y^2}+\delta_2\left[\frac{\partial^4\varphi_{00}}{\partial x^4}-\frac{\partial^2 w_{00}}{\partial x\partial y}\frac{\partial^2 w_{00}}{\partial x\partial y}+\frac{\partial^2\varphi_{00}}{\partial x^2}\frac{\partial^2 w_{00}}{\partial y^2}\right]=0, \tag{7}
$$

$$
\left\{
\begin{aligned}
& w_{00}\big|_{x=0}=\frac{\partial w_{00}}{\partial x}\Big|_{x=0}=w_{00}\big|_{x=1}=\frac{\partial w_{00}}{\partial x}\Big|_{x=1}=0, \\
& w_{00}\big|_{y=0}=\frac{\partial w_{00}}{\partial y}\Big|_{y=0}=w_{00}\big|_{y=b/a}=\frac{\partial w_{00}}{\partial y}\Big|_{y=b/a}=0, \\
& w_{00}\big|_{\substack{x=0\\y=0}}=\frac{\partial w_{00}}{\partial x}\Big|_{\substack{x=0\\y=0}}=\frac{\partial w_{00}}{\partial y}\Big|_{\substack{x=0\\y=0}}=0, \\
& w_{00}\big|_{\substack{x=1\\y=b/a}}=\frac{\partial w_{00}}{\partial x}\Big|_{\substack{x=1\\y=b/a}}=\frac{\partial w_{00}}{\partial y}\Big|_{\substack{x=1\\y=b/a}}=0, \\
& \left[\frac{\partial^2\varphi_{00}}{\partial x^2}-\mu_{21}\frac{\partial^2\varphi_{00}}{\partial y^2}\right]\Big|_{x=0}=0,\ \left[\frac{\partial^2\varphi_{00}}{\partial x^2}-\mu_{21}\frac{\partial^2\varphi_{00}}{\partial y^2}\right]\Big|_{x=1}=0, \\
& \left[\frac{\partial^2\varphi_{00}}{\partial y^2}-\mu_{12}\frac{\partial^2\varphi_{00}}{\partial x^2}\right]\Big|_{y=0}=0,\ \left[\frac{\partial^2\varphi_{00}}{\partial y^2}-\mu_{12}\frac{\partial^2\varphi_{00}}{\partial x^2}\right]\Big|_{y=b/a}=0.
\end{aligned}
\right. \tag{8}
$$

…… …… ……

由(4)和(5)式的第二个大括号,分别比较 $v_{nm}^{(1)}$ 和 $\phi_{nm}^{(1)}$, $v_{nm}^{(2)}$ 和 $\phi_{nm}^{(2)}$, $v_{nm}^{(3)}$ 和 $\phi_{nm}^{(3)}$, $v_{nm}^{(4)}$ 和 $\phi_{nm}^{(4)}$ 的 $\varepsilon_1\varepsilon_2$ 同次幂的系数,我们获得边界层型函数的递推方程:

$$
\left\{
\begin{aligned}
& \Gamma^2 A_{4,0}v_{00}^{(1)}-\frac{\partial^2\varphi_{00}}{\partial y^2}A_{2,0}v_{00}^{(1)}=0, \\
& \Gamma^2 A_{4,0}v_{00}^{(2)}-\frac{\partial^2\varphi_{00}}{\partial y^2}A_{2,0}v_{00}^{(2)}=0, \\
& \Gamma^2\delta_2 B_{4,0}v_{00}^{(3)}-\frac{\partial^2\varphi_{00}}{\partial x^2}B_{2,0}v_{00}^{(3)}=0, \\
& \Gamma^2\delta_2 B_{4,0}v_{00}^{(4)}-\frac{\partial^2\varphi_{00}}{\partial x^2}B_{2,0}v_{00}^{(4)}=0,
\end{aligned}
\right. \tag{9}
$$

$$\begin{cases} A_{4,0}\phi_{00}^{(1)} + \delta_2 A_{2,0} v_{00}^{(1)}\left\{\dfrac{\partial^2 w_{00}}{\partial y^2} + \dfrac{1}{2}B_{2,0}v_{00}^{(3)} + \dfrac{1}{2}B_{2,0}v_{00}^{(4)}\right\} = 0, \\ A_{4,0}\phi_{00}^{(2)} + \delta_2 A_{2,0} v_{00}^{(2)}\left\{\dfrac{\partial^2 w_{00}}{\partial y^2} + \dfrac{1}{2}B_{2,0}v_{00}^{(3)} + \dfrac{1}{2}B_{2,0}v_{00}^{(4)}\right\} = 0, \\ B_{4,0}\phi_{00}^{(3)} + B_{2,0}v_{00}^{(3)}\left\{\dfrac{\partial^2 w_{00}}{\partial x^2} + \dfrac{1}{2}A_{2,0}v_{00}^{(1)} + \dfrac{1}{2}A_{2,0}v_{00}^{(2)}\right\} = 0, \\ B_{4,0}\phi_{00}^{(4)} + B_{2,0}v_{00}^{(4)}\left\{\dfrac{\partial^2 w_{00}}{\partial x^2} + \dfrac{1}{2}A_{2,0}v_{00}^{(1)} + \dfrac{1}{2}A_{2,0}v_{00}^{(2)}\right\} = 0. \end{cases} \tag{10}$$

…… …… ……

2 挠度函数 $W(x, y)$ 和应力函数 $\Phi(x, y)$ 的渐近解

若 $\delta_2 = E_2/E_1 < 1$, 将其视为小参数, 由正则摄动法求解 w_{00} 和 φ_{00}, 令

$$w_{00} = \sum_{i=0}^{P}\delta_2^i w_{00i}(x, y), \tag{11}$$

$$\varphi_{00} = \sum_{i=0}^{P}\delta_2^i \varphi_{00i}(x, y). \tag{12}$$

将(11)、(12)代入(6),(7),比较 δ_2 的同次幂的系数, 得 w_{000} 和 φ_{000} 的递推方程和递推边界条件:

$$\Gamma\left[\frac{\partial^2\varphi_{000}}{\partial y^2}\frac{\partial^2 w_{000}}{\partial x^2} + \frac{\partial^2\varphi_{000}}{\partial x^2}\frac{\partial^2 w_{000}}{\partial y^2} - \frac{\partial^2\varphi_{000}}{\partial x\partial y}\frac{\partial^2 w_{000}}{\partial x\partial y}\right] = -q(x, y), \tag{13}$$

$$\frac{\partial^4\varphi_{000}}{\partial x^4} + \delta_1\frac{\partial^4\varphi_{000}}{\partial x^2\partial y^2} = 0, \tag{14}$$

$$\begin{cases} w_{000}\big|_{x=0} = \dfrac{\partial w_{000}}{\partial x}\Big|_{x=0} = w_{000}\big|_{x=1} = \dfrac{\partial w_{000}}{\partial x}\Big|_{x=1} = 0, \\ w_{000}\big|_{y=0} = \dfrac{\partial w_{000}}{\partial y}\Big|_{y=0} = 0, \\ w_{000}\big|_{y=b/a} = \dfrac{\partial w_{000}}{\partial y}\Big|_{y=b/a} = 0, \\ w_{000}\big|_{\substack{x=0\\y=0}} = \dfrac{\partial w_{000}}{\partial x}\Big|_{\substack{x=0\\y=0}} = \dfrac{\partial w_{000}}{\partial y}\Big|_{\substack{x=0\\y=0}} = 0, \\ w_{000}\big|_{\substack{x=1\\y=b/a}} = \dfrac{\partial w_{000}}{\partial x}\Big|_{\substack{x=1\\y=b/a}} == 0, \dfrac{\partial w_{000}}{\partial y}\Big|_{\substack{x=1\\y=b/a}} = 0, \\ \left[\dfrac{\partial^2\varphi_{000}}{\partial x^2} - \mu_{21}\dfrac{\partial^2\varphi_{000}}{\partial y^2}\right]\Big|_{x=0} = \left[\dfrac{\partial^2\varphi_{000}}{\partial x^2} - \mu_{21}\dfrac{\partial^2\varphi_{000}}{\partial y^2}\right]\Big|_{x=1} = 0, \\ \left[\dfrac{\partial^2\varphi_{000}}{\partial y^2} - \mu_{12}\dfrac{\partial^2\varphi_{000}}{\partial x^2}\right]\Big|_{y=0} = \left[\dfrac{\partial^2\varphi_{000}}{\partial y^2} - \mu_{12}\dfrac{\partial^2\varphi_{000}}{\partial x^2}\right]\Big|_{y=b/a} = 0. \end{cases} \tag{15a}$$

…… …… ……

$$\sum_{i=0}^{n}\frac{\partial^2\varphi_{00(n-i)}}{\partial y^2}\frac{\partial^2 w_{00i}}{\partial x^2} + \sum_{i=0}^{n}\frac{\partial^2\varphi_{00(n-i)}}{\partial x^2}\frac{\partial^2 w_{00i}}{\partial y^2} -$$

$$2\sum_{i=0}^{n}\frac{\partial^2\varphi_{00(n-i)}}{\partial x\partial y}\frac{\partial^2 w_{00i}}{\partial x\partial y} = 0, \tag{15b}$$

$$\frac{\partial^4 \varphi_{00n}}{\partial x^4} + \delta_1 \frac{\partial^4 \varphi_{00n}}{\partial x^2 \partial y^2} + \frac{\partial^4 \varphi_{00(n-1)}}{\partial x^4} - \sum_{i=0}^{n} \frac{\partial^2 w_{00(n-i-1)}}{\partial x \partial y} \frac{\partial^2 w_{00i}}{\partial x \partial y} + \sum_{i=0}^{n} \frac{\partial^2 w_{00(n-i-1)}}{\partial x^2} \frac{\partial^2 w_{00i}}{\partial y^2} = 0 \cdot \tag{15c}$$

$$\begin{cases} w_{00n}\,|_{x=0} = \left.\dfrac{\partial w_{00n}}{\partial x}\right|_{x=0} = w_{00n}\,|_{x=1} = \left.\dfrac{\partial w_{00n}}{\partial x}\right|_{x=1} = 0, \\ w_{00n}\,|_{y=0} = \left.\dfrac{\partial w_{00n}}{\partial y}\right|_{y=0} = 0, \\ w_{00n}\,|_{y=b/a} = \left.\dfrac{\partial w_{00n}}{\partial y}\right|_{y=b/a} = 0, \\ w_{00n}\,|_{\substack{x=0\\y=0}} = \left.\dfrac{\partial w_{00n}}{\partial x}\right|_{\substack{x=0\\y=0}} = \left.\dfrac{\partial w_{00n}}{\partial y}\right|_{\substack{x=0\\y=0}} = 0, \\ w_{00n}\,|_{\substack{x=1\\y=b/a}} = \left.\dfrac{\partial w_{00n}}{\partial x}\right|_{\substack{x=1\\y=b/a}} = \left.\dfrac{\partial w_{00n}}{\partial y}\right|_{\substack{x=1\\y=b/a}} = 0, \\ \left.\left[\dfrac{\partial^2 \varphi_{00n}}{\partial x^2} - \mu_{21} \dfrac{\partial^2 \varphi_{00n}}{\partial y^2}\right]\right|_{x=0} = \left.\left[\dfrac{\partial^2 \varphi_{00n}}{\partial x^2} - \mu_{21} \dfrac{\partial^2 \varphi_{00n}}{\partial y^2}\right]\right|_{x=1} = 0, \\ \left.\left[\dfrac{\partial^2 \varphi_{00n}}{\partial y^2} - \mu_{12} \dfrac{\partial^2 \varphi_{00n}}{\partial x^2}\right]\right|_{y=0} = \left.\left[\dfrac{\partial^2 \varphi_{00n}}{\partial y^2} - \mu_{12} \dfrac{\partial^2 \varphi_{00n}}{\partial x^2}\right]\right|_{y=b/a} = 0 \cdot \end{cases} \tag{15d}$$

利用分离变量法, 可以求得(14) 式满足边界条件(15) 的解为:

$$\varphi_{000} = \left[\frac{\delta_1 + \mu_{21}}{\mu_{21}}\left(\left(1 - \exp\left[n\pi\sqrt{\delta_1}\,\frac{a}{b}\right]\right)c_1 + \left(1 - \exp\left[-n\pi\sqrt{\delta_1}\,\frac{a}{b}\right]\right)c_2\right)x - \frac{\delta_1 + \mu_{21}}{\mu_{21}}c' + c_1\exp\left[n\pi\sqrt{\delta_1}\,\frac{ax}{b}\right] + c_2\exp\left[-n\pi\sqrt{\beta_1}\,\frac{ax}{b}\right]\right]\sin\frac{n\pi a}{b}y \cdot \tag{16}$$

其中 c_1, c_2 是待定系数, $c' = c_1 + c_2$• 将 φ_{000} 代入(13) 式, 得:

$$\Gamma\left[\frac{n^2\pi^2 a^2}{b^2}X(x)\sin\frac{n\pi a}{b}y\,\frac{\partial^2 w_{000}}{\partial x^2} - X''(x)\sin\frac{n\pi a}{b}y\,\frac{\partial^2 w_{000}}{\partial y^2}\,\frac{n\pi a}{b}X'(x)\cos\frac{n\pi a}{b}y\,\frac{\partial^2 w_{000}}{\partial x \partial y}\right] = q(x, y), \tag{17}$$

其中 $X(x)$ 是 φ_{000} 中 x 的变量部分•

在(17) 中 $|\gamma| < 1$, 可视为小参数, 令

$$w_{000} = \sum_{i=0}^{L} \gamma^i w_{000i}, \tag{18}$$

将(18) 代入(17), 比较 γ 的同次幂的系数得:

$$\frac{n^2\pi^2 a^2}{b^2}X(x)\sin\frac{n\pi}{b}y\,\frac{\partial^2 w_{0000}}{\partial x^2} - X''(x)\sin\frac{n\pi}{b}y\,\frac{\partial^2 w_{0000}}{\partial y^2} + \frac{n\pi a}{b}X'(x)\cos\frac{n\pi a}{b}y\,\frac{\partial^2 w_{0000}}{\partial x \partial y} = q(x, y), \tag{19}$$

…… …… ……

$$\frac{n^2\pi^2 a^2}{b^2}X(x)\sin\frac{n\pi}{b}y\,\frac{\partial^2 w_{000i}}{\partial x^2} - X''(x)\sin\frac{n\pi}{b}y\,\frac{\partial^2 w_{000i}}{\partial y^2} + \frac{n\pi a}{b}X'(x)\cos\frac{n\pi a}{b}y\,\frac{\partial^2 w_{000i}}{\partial x \partial y} +$$

$$\varepsilon\left(x-\frac{1}{2}\right)\left[\frac{n^2\pi^2a^2}{b^2}X(x)\sin\frac{n\pi}{b}y\frac{\partial^2 w_{000(i-1)}}{\partial x^2}-X''(x)\sin\frac{n\pi}{b}y\frac{\partial^2 w_{000(i-1)}}{\partial y^2}\frac{n\pi a}{b}X'(x)\cos\frac{n\pi a}{b}y\frac{\partial^2 w_{000(i-1)}}{\partial x\partial y}\right]=0, \quad (20)$$

如果 $q(x,y)$ 给定，我们可以用傅立叶级数法求出 w_{0000}；然后利用(20)式可求出诸 w_{000n}，从而求得 w_{000}。 类似可求得了 w_{00} 和 φ_{00}。 将 w_{00} 和 φ_{00} 代入(4)和(5)的第一个大括号(取 $n=1$, $m=0$ 且带有负下标的项取 0)，我们获得关于 w_{10} 和 φ_{10} 的偏微分方程。 解微分方程，并利用边界条件，可求得 w_{10} 和 φ_{10}。 这样可逐次求得 w_{nm}，φ_{nm} ($n=0,1,\cdots,N$; $m=0,1,\cdots,M$)。

非常显然，变厚度正交各向异性矩形板的非线性非对称弯曲问题的边界层型函数应与等厚度正交各向异性矩形板的非线性非对称弯曲问题的边界层型函数应相同[6]。

3 讨 论

在本文中，利用修正的"修正的两变量法"和"混合摄动法"，并在 $G<E_2<E_1$ 以及厚度变化缓慢($|\gamma|<1$)的假设下，引进了四个小参数，对四边固支的变厚度正交各向异性矩形板的非线性非对称弯曲问题进行了研究。 得到了挠度函数 $W(x,y)$ 和应力函数 $\Phi(x,y)$ 对 ε_1 为 N 阶及对 ε_2 为 M 阶的一致有效渐近解。 变厚度正交各向异性矩形板的非线性非对称弯曲问题的边界层型函数应与等厚度正交各向异性矩形板的非线性非对称弯曲问题的边界层型函数完全一致。 当 $\gamma=0$ 时，本文的所有结果与文献[6]给出的等厚度正交各向异性矩形板的非线性非对称弯曲问题的结果一致。

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Uniformly Valid Asymptotic Solutions of the Nonlinear Unsymmetrical Bending for Orthotropic Rectangular Thin Plate of Four Clamped Edges With Variable Thickness

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Abstract: By using "the method of modified two_variable", "the method of mixing perturbation" and introducing four small parameters, the problem of the non_linear unsymmetrical bending for orthotropic rectangular thin plate with linear variable thickness is studied. And the uniformly valid asymptotic solution of N th_order for eqsilon 1 and M th_order for epsilon 2 of the deflection functions and stress function are obtained.

Key words: orthotropic rectangular thin plate with variable thickness; four clampled edge; nonlinear unsymmetrical bending; method of modified two_variable; method of mixing perturbation; uniformly valid asymptotic solution