应用数学和力学，第10卷第1期(1989年1月)
Applied Mathematics and Mechanics
应用数学和力学编委会编
重 庆 出 版 社 出 版

矩形板在横向压力和面内压缩共同作用下的后屈曲*

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摘 要

本文从Kármán板大挠度方程出发，先用Galerkin法将横向载荷作用转化为初挠度影响，然后以挠度为摄动参数，采用直接摄动法研究简支矩形板在横向载荷与面内压缩共同作用下的后屈曲性态。

本文讨论了两种面内边界条件，同时计及板初始几何缺陷的影响。计算结果与实验结果的比较表明二者是一致的。

一、引 言

在工程结构（如桥梁结构，船舶结构，海洋平台结构等）中，薄板构件往往受到横向压力和面内压缩同时作用。若横向压力很大，则为大挠度问题；若横向压力较小，随着面内压缩的增加板将发生屈曲。弄清矩形板在横向压力与面内压缩共同作用下的后屈曲性态具有重要的实际意义。

对于矩形板在面内压缩作用下的后屈曲性态已作过广泛的研究，相比之下，当有横向压力作用时，矩形板在面内压缩作用下的屈曲和后屈曲则研究得甚少，仅有少数文献可供资用[1~5]。因此，有必要作进一步的分析和研究。

本文讨论计及横向压力时矩形板在面内压缩作用下的后屈曲性态。在不变的，较小的横向压力作用下，矩形板产生一初始挠度，由于该初挠度的存在直接影响矩形板的屈曲与后屈曲性态，其表现类似有初始几何缺陷的板。

本文从Kármán板大挠度方程出发，先用Galerkin法将横向压力作用转化为初挠度影响，然后以挠度为摄动参数，采用直接摄动法研究矩形板在横向压力与面内压缩共同作用下的后屈曲性态。

本文讨论了两种面内边界条件，一种为纵向边缘可移简支，一种为纵向边缘不可移简支。

本文同时计及板的初始几何缺陷，其形式取作和矩形板小挠度解的形式一致。

* 钱伟长推荐.

二、基 本 方 程

假定四边简支矩形板的长为a，宽为b，厚度为t，受到对边均布压力p，并有线性分布横向载荷 $q=q_0-q_1(x/a-1/2)$，若 $q_1=0$，即为均布横向压力。取坐标系如图1所示。并以W和ϕ表示挠度和应力函数，以W_0^*表示初始几何缺陷。那么，计及横向压力q时，Kármán 板大挠度方程可表为如下形式：

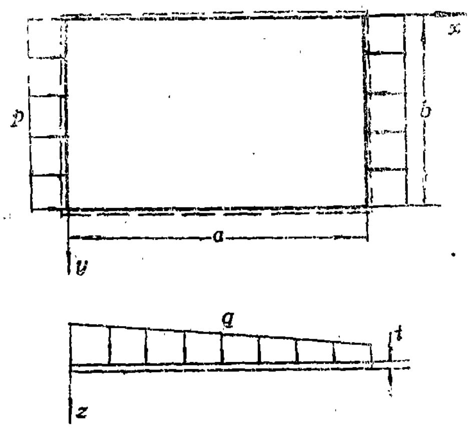


图1 矩形板受横向压力与面内压缩共同作用

$$D\nabla^4 W=t\Big[\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 W}{\partial x^2}-2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 W}{\partial x\partial y}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 W}{\partial y^2}$$

$$+\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 W_0^*}{\partial x^2}-2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 W_0^*}{\partial x\partial y}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 W_0^*}{\partial y^2}\Big]+q \tag{2.1}$$

$$\nabla^4\phi=E\Big[\Big(\frac{\partial^2 W}{\partial x\partial y}\Big)^2-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}+2\frac{\partial^2 W}{\partial x\partial y}\frac{\partial^2 W_0^*}{\partial x\partial y} \tag{2.2}$$

$$-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W_0^*}{\partial y^2}-\frac{\partial^2 W}{\partial y^2}\frac{\partial^2 W_0^*}{\partial x^2}\Big] \tag{2.3}$$

其中 $$\nabla^4=\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}$$

$D=\dfrac{Et^3}{12(1-\nu^2)}$为弯曲刚度，$E$和$\nu$分别为弹性模数和Poisson比。

板中的内力

$$N_x=t\frac{\partial^2\phi}{\partial y^2},\quad N_{xy}=-t\frac{\partial^2\phi}{\partial x\partial y},\quad N_y=t\frac{\partial^2\phi}{\partial x^2} \tag{2.4}$$

面内位移U,V与W,W_0^*及ϕ的关系为

$$\left.\begin{aligned}&\frac{\partial U}{\partial x}+\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2+\frac{\partial W}{\partial x}\frac{\partial W_0^*}{\partial x}=\frac{1}{E}\left(\frac{\partial^2\phi}{\partial y^2}-\nu\frac{\partial^2\phi}{\partial x^2}\right)\\&\frac{\partial V}{\partial y}+\frac{1}{2}\left(\frac{\partial W}{\partial y}\right)^2+\frac{\partial W}{\partial y}\frac{\partial W_0^*}{\partial y}=\frac{1}{E}\left(\frac{\partial^2\phi}{\partial x^2}-\nu\frac{\partial^2\phi}{\partial y^2}\right)\\&\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}+\frac{\partial W}{\partial x}\frac{\partial W}{\partial y}+\frac{\partial W}{\partial x}\frac{\partial W_0^*}{\partial y}+\frac{\partial W}{\partial y}\frac{\partial W_0^*}{\partial x}\\&\quad=-\frac{2(1+\nu)}{E}\frac{\partial^2\phi}{\partial x\partial y}\end{aligned}\right\}\tag{2.5}$$

假定边界支承为四边简支的，那么边界条件为

$$x=0,\ a;\quad W=W_{,xx}=0,\quad N_{xy}=0 \tag{2.6a}$$

$$\int_0^b N_x dy+P=0 \tag{2.6b}$$

$$y=0,\ b;\ W=W_{,yy}=0,\ N_{xy}=0 \tag{2.7a}$$

$$\int_0^a N_y dx=0\quad\text{（纵向边缘可移简支）} \tag{2.7b}$$

$$V=\text{常数}\quad\text{（纵向边缘不可移简支）} \tag{2.7c}$$

单位端部缩短

$$\begin{aligned}\frac{\Delta_x}{a}&=-\frac{1}{ab}\int_0^b\int_0^a\frac{\partial U}{\partial x}dxdy\\&=-\frac{1}{ab}\int_0^b\int_0^a\left[\frac{1}{E}\left(\frac{\partial^2\phi}{\partial y^2}-\nu\frac{\partial^2\phi}{\partial x^2}\right)-\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2-\frac{\partial W}{\partial x}\frac{\partial W_0^*}{\partial x}\right]dxay\end{aligned}\tag{2.8}$$

$$\begin{aligned}\frac{\Delta_y}{b}&=-\frac{1}{ab}\int_0^a\int_0^b\frac{\partial V}{\partial y}dydx\\&=-\frac{1}{ab}\int_0^a\int_0^b\left[\frac{1}{E}\left(\frac{\partial^2\phi}{\partial x^2}-\nu\frac{\partial\phi^2}{\partial y^2}\right)-\frac{1}{2}\left(\frac{\partial W}{\partial y}\right)^2-\frac{\partial W}{\partial y}\frac{\partial W_0^*}{\partial y}\right]dydx\end{aligned}\tag{2.9}$$

引进

$$\begin{aligned}&\bar{x}=\frac{\pi}{a}x,\quad \bar{y}=\frac{\pi}{b}y,\quad \beta=\frac{a}{b},\\&\overline{W}=\frac{W}{t}\sqrt{12(1-\nu^2)},\quad \overline{W}_0^*=\frac{W_0^*}{t}\sqrt{12(1-\nu^2)},\\&\bar{\phi}=\frac{\phi t}{D},\quad \lambda_x=\frac{\sigma_x}{\sigma_{cl}},\quad K_q=\frac{qa^4\sqrt{12(1-\nu^2)}}{\pi^4Dt},\\&\sigma_{cl}=\frac{4\pi^2D}{b^2t},\quad \delta_x=\frac{12(1-\nu^2)b^2}{4\pi^2t^2}\frac{\Delta_x}{a},\quad \delta_y=\frac{12(1-\nu^2)b^2}{4\pi^2t^2}\frac{\Delta_y}{b}\end{aligned}\tag{2.10}$$

那么，方程(2.1)(2.2)可化为如下无量纲形式（略去字母上的“—”号）

$$\begin{aligned}\nabla^4W=\beta^2\Big[&\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2W}{\partial x^2}-2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2W}{\partial x\partial y}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2W}{\partial y^2}\\&+\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2W_0^*}{\partial x^2}-2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2W_0^*}{\partial x\partial y}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2W_0^*}{\partial y^2}\Big]+K_q\end{aligned}\tag{2.11}$$

$$\nabla^4\phi=\beta^2\left[\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}+2\frac{\partial^2 W}{\partial x\partial y}\frac{\partial^2 W_0^*}{\partial x\partial y}-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W_0^*}{\partial y^2}-\frac{\partial^2 W}{\partial y^2}\frac{\partial^2 W_0^*}{\partial x^2}\right] \tag{2.12}$$

其中
$$\nabla^4=\frac{\partial^4}{\partial x^4}+2\beta^2\frac{\partial^4}{\partial x^2\partial y^2}+\beta^4\frac{\partial^4}{\partial y^4} \tag{2.13}$$

边界条件化为

$$x=0,\ \pi;\quad W=W_{,xx}=0,\quad \phi_{,xy}=0 \tag{2.14a}$$

$$\frac{1}{\pi}\int_0^\pi \beta^2\frac{\partial^2\phi}{\partial y^2}dy+4\lambda_x\beta^2=0 \tag{2.14b}$$

$$y=0,\ \pi;\quad W=W_{,yy}=0,\quad \phi_{,xy}=0 \tag{2.15a}$$

$$\frac{1}{\pi}\int_0^\pi \frac{\partial^2\phi}{\partial x^2}dx=0\quad \text{（可移简支）} \tag{2.15b}$$

$$\delta_y=0\qquad \text{（不可移简支）} \tag{2.15c}$$

单位端部缩短化为

$$\delta_x=-\frac{1}{4\pi^2\beta^2}\int_0^\pi\int_0^\pi\left[\left(\beta^2\frac{\partial^2\phi}{\partial y^2}-\nu\frac{\partial^2\phi}{\partial x^2}\right)-\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2-\frac{\partial W}{\partial x}\frac{\partial W_0^*}{\partial x}\right]dxdy \tag{2.16}$$

$$\delta_y=-\frac{1}{4\pi^2\beta^2}\int_0^\pi\int_0^\pi\left[\left(\frac{\partial^2\phi}{\partial x^2}-\nu\beta^2\frac{\partial^2\phi}{\partial y^2}\right)-\frac{1}{2}\beta^2\left(\frac{\partial W}{\partial y}\right)^2-\beta^2\frac{\partial W}{\partial y}\frac{\partial W_0^*}{\partial y}\right]dydx \tag{2.17}$$

方程(2.11)至(2.17)可以用来研究简支矩形板在横向压力和面内压缩共同作用下的曲屈问题。

三、分析方法与渐近解

设

$$W=W_T=W_L+W_1^*,\quad \phi=\phi_T=\phi_L+\phi_1^* \tag{3.1}$$

其中W_1^*为横向载荷K_q引起的初始挠度，W_L为面内压缩引起的附加挠度；ϕ_1^*，ϕ_L为和W_1^*，W_L相对应的应力函数。那么，W_1^*，ϕ_1^*满足方程

$$\nabla^4 W_1^*=K_q,\quad \nabla^4\phi_1^*=0 \tag{3.2}$$

设方程(3.2)的解为

$$W_1^*=\varepsilon\left[\sum_{m=1}A_m^*\sin mx\,\sin ny\right] \tag{3.3}$$

$$\phi_1^*=0$$

并取
$$K_q=\varepsilon k_1=\varepsilon\left[k_{10}-k_{11}\left(\frac{x}{\pi}-\frac{1}{2}\right)\right] \tag{3.4}$$

将式(3.3)、(3.4)代入(3.2)，利用Galerkin法我们容易得到

$$A_m^*=\begin{cases}\dfrac{16k_{10}}{mn(m^2+n^2\beta^2)^2\pi^2} & (m=\text{奇数},\ n=1)\\[2ex] \dfrac{8k_{11}}{mn(m^2+n^2\beta^2)^2\pi^2} & (m=\text{偶数},\ n=1)\end{cases} \tag{3.5}$$

而W_L,ϕ_L满足方程

$$\nabla^4 W_L=\beta^2\Big[\frac{\partial^2\phi_L}{\partial y^2}\frac{\partial^2 W_L}{\partial x^2}-2\frac{\partial^2\phi_L}{\partial x\partial y}\frac{\partial^2 W_L}{\partial x\partial y}+\frac{\partial^2\phi_L}{\partial x^2}\frac{\partial^2 W_L}{\partial y^2}+\frac{\partial^2\phi_L}{\partial y^2}\frac{\partial^2 W^*}{\partial x^2}-2\frac{\partial^2\phi_L}{\partial x\partial y}\frac{\partial^2 W^*}{\partial x\partial y}+\frac{\partial^2\phi_L}{\partial x^2}\frac{\partial^2 W^*}{\partial y^2}\Big] \tag{3.6}$$

$$\nabla^4\phi_L=\beta^2\Big[\Big(\frac{\partial^2 W_L}{\partial x\partial y}\Big)^2-\frac{\partial^2 W_L}{\partial x^2}\frac{\partial^2 W_L}{\partial y^2}+2\frac{\partial^2 W_L}{\partial x\partial y}\frac{\partial^2 W^*}{\partial x\partial y}-\frac{\partial^2 W_L}{\partial x^2}\frac{\partial^2 W^*}{\partial y^2}-\frac{\partial^2 W_L}{\partial y^2}\frac{\partial^2 W^*}{\partial x^2}\Big] \tag{3.7}$$

其中 $$W^*=W_0^*+W_1^* \tag{3.8}$$

设方程(3.6)、(3.7)的解为如下渐近展开式

$$W_L(x,y,\varepsilon)=\sum_{n=1}\varepsilon^n W_n(x,y),\quad \phi_L(x,y,\varepsilon)=\sum_{n=0}\varepsilon^n\phi_n(x,y) \tag{3.9}$$

并取初始几何缺陷（无量纲形式）

$$W_0^*=\varepsilon A_0^*\sin mx\,\sin ny \tag{3.10}$$

由于在初挠度W_1^*中仅$\varepsilon[A_m^*\sin mx\sin ny]$项对屈曲和后屈曲产生显著影响，因此，可将$W^*$表为

$$W^*=\varepsilon\mu A_{11}^{(1)}\sin mx\,\sin ny \tag{3.11}$$

其中 $$\mu=\frac{A_0^*+A_m^*}{A_{11}^{(1)}} \tag{3.12}$$

比照参考文献 [6]，我们发现方程的形式和解的形式都相同，因此，我们可以直接调用文[6]的结果，即对于

1. **纵向边缘可移简支**

$$\lambda_x=\frac{1}{4\beta^2}\Big\{\frac{(m^2+n^2\beta^2)^2}{(1+\mu)m^2}+\frac{1}{16}\frac{m^4+n^4\beta^4}{m^2}(1+2\mu)W_{Lm}^2+\frac{1}{256}\Big(2(1+\mu)^2(1+2\mu)^2\Big[\frac{m^4(m^4+n^4\beta^4)}{g_{13}}+\frac{n^4\beta^4(m^4+n^4\beta^4)}{g_{31}}\Big]-(1+\mu)(1+2\mu)[2(1+\mu)^2+(1+2\mu)]\Big[\frac{m^8}{g_{13}}+\frac{n^8\beta^8}{g_{31}}\Big]\Big)W_{Lm}^4+\cdots\Big\} \tag{3.13}$$

$$\delta_x=\lambda_x+\frac{1}{32}\frac{m^2}{\beta^2}(1+2\mu)W_{Lm}^2+\frac{1}{256}\frac{m^2}{\beta^2}(1+\mu)^2(1+2\mu)^2\Big[\frac{m^4}{g_{13}}+\frac{n^4\beta^4}{g_{31}}\Big]W_{Lm}^4+\cdots \tag{3.14}$$

其中 $$\mu=\frac{(A_0^*+A_m^*)\varepsilon}{A_{11}^*\varepsilon} \tag{3.15}$$

而 $$A_{11}^{(1)}\varepsilon=W_{Lm}+\frac{1}{16}(1+\mu)^2(1+2\mu)m^2\Big[\frac{m^4}{g_{13}}+\frac{n^4\beta^4}{g_{31}}\Big]W_{Lm}^3+\cdots \tag{3.16}$$

及

$$\left.\begin{aligned}g_{13}&=m^2[(m^2+9n^2\beta^2)^2(1+\mu)-(m^2+n^2\beta^2)^2]\\ g_{31}&=m^2[(9m^2+n^2\beta^2)^2(1+\mu)-9(m^2+n^2\beta^2)^2]\end{aligned}\right\} \tag{3.17}$$

2. 纵向边缘不可移简支

$$\lambda_x=\frac{1}{4\beta^2}\left\{\frac{(m^2+n^2\beta^2)^2}{(1+\mu)(m^2+\nu n^2\beta^2)}+\frac{1}{16}\frac{m^4+3n^4\beta^4}{m^2+\nu n^2\beta^2}(1+2\mu)W_{Lm}^2\right.$$
$$+\frac{1}{256}\left(2(1+\mu)^2(1+2\mu)^2\left[\frac{m^4(m^4+3n^4\beta^4)}{g_{13}}+\frac{n^4\beta^4(m^4+3n^4\beta^4)}{g_{31}}\right]\right.$$
$$\left.\left.-(1+\mu)(1+2\mu)[2(1+\mu)^2+(1+2\mu)]\left[\frac{m^8}{g_{13}}+\frac{n^8\beta^8}{g_{31}}\right]\right)W_{Lm}^4+\cdots\right\} \tag{3.18}$$

$$\delta_x=(1-\nu^2)\lambda_x+\frac{1}{32}\frac{m^2+\nu n^2\beta^2}{\beta^2}(1+2\mu)W_{Lm}^2+\frac{1}{256}\frac{(m^2+\nu n^2\beta^2)^2}{\beta^2}$$
$$\cdot(1+\mu)^2(1+2\mu)^2\left[\frac{m^4}{g_{13}}+\frac{n^4\beta^4}{g_{31}}\right]W_{Lm}^4+\cdots \tag{3.19}$$

其中
$$\mu=\frac{(A_0^*+A_m^*)\varepsilon}{A_{11}^{(1)}\varepsilon} \tag{3.20}$$

而

$$A_{11}^{(1)}\varepsilon=W_{Lm}+\frac{1}{16}(1+\mu)^2(1+2\mu)\left[\frac{m^4(m^2+\nu n^2\beta^2)}{g_{13}}+\frac{n^4\beta^4(m^2+\nu n^2\beta^2)}{g_{13}}\right]W_{Lm}^3+\cdots \tag{3.21}$$

及

$$\left.\begin{aligned}g_{13}&=(m^2+9n^2\beta^2)^2(m^2+\nu n^2\beta^2)(1+\mu)-(m^2+n^2\beta^2)^2(m^2+9\nu n^2\beta^2)\\g_{31}&=(9m^2+n^2\beta^2)^2(m^2+\nu n^2\beta^2)(1+\mu)-(m^2+n^2\beta^2)^2(9m^2+\nu n^2\beta^2)\end{aligned}\right\} \tag{3.22}$$

需要指出，在式(3.15)，(3.20)中，当$W_{Lm}=0$时，若$A_0^*+A_m^*=0$，则$\mu=0$，此时λ_x对应线性临界值；若$A_0^*+A_m^*\neq0$，则$\mu=\infty$，此时$\lambda_x=0$。

由式(3.13)至(3.22)可以看出，当有横向载荷作用时，即便$W_0^*=0$，矩形板的载荷-挠度曲线或载荷-端部缩短曲线类似有初始几何缺陷的板。

当横向压力$q=0$时，则$W_1^*=0$，由式(3.1)此时$W=W_T=W_L$。重新回到文[6]的结果。

四、结果和讨论

根据式(3.13)至(3.22)，我们分别计算了对应两种面内边界条件，长宽比$\beta=2,3$的三块简支矩形板在横向载荷和面内单向压缩共同作用下的后屈曲平衡路径。板的几何参数，弹性常数及横向载荷在表1中给出。计算结果如图2，图3所示。图示表明

(1) 挠度-载荷曲线，挠度-端部缩短曲线类似有初始几何缺陷的板。

(2) 由于横向载荷的存在，板产生一初始挠度，至使挠度-载荷曲线起始点偏离坐标原点。其偏离程度与横向压力大小及屈曲模态有关。

(3) 纵向边缘可移简支与纵向边缘不可移简支相比其曲线较为平缓，因此，对应纵向边缘不可移简支矩形板具有较高的屈后强度。

图4为本文结果与文[3,4]实验结果的比较，可以看出，实验结果与理论曲线间有相当好的符合。其中3号板的实验结果与初始几何缺陷$W_0^*/t=0.05$的理论曲线相当符合，而1号板的实验结果更接近初始几何缺陷$W_0^*/t=0.1$的理论曲线.

表1

板No.	a (mm)	b (mm)	t (mm)	E (10^4kg/mm²)	ν	q_0 (kg/cm²)	q_1 (kg/cm²)
1	1000	500	6	2.08	0.33	0.1	0.1
2	1200	400	6	2.09	0.33	0.3	0.0
3	1200	400	6	2.09	0.33	0.5	0.0

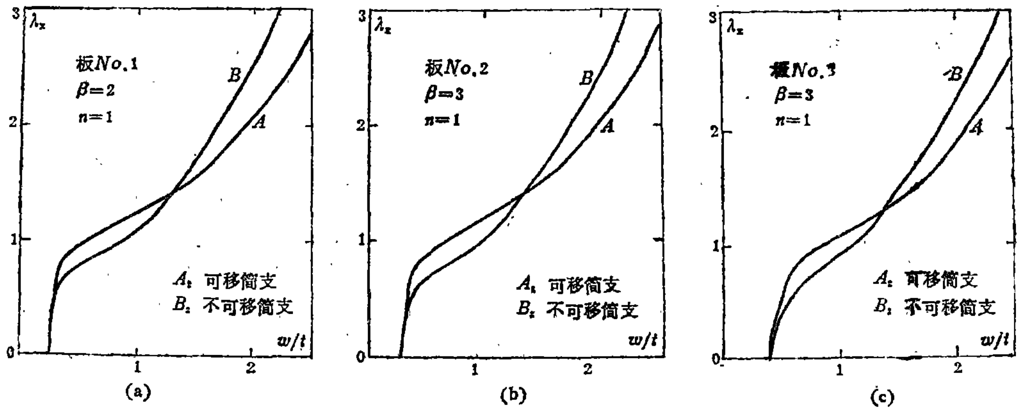


图2 后屈曲载荷-挠度曲线

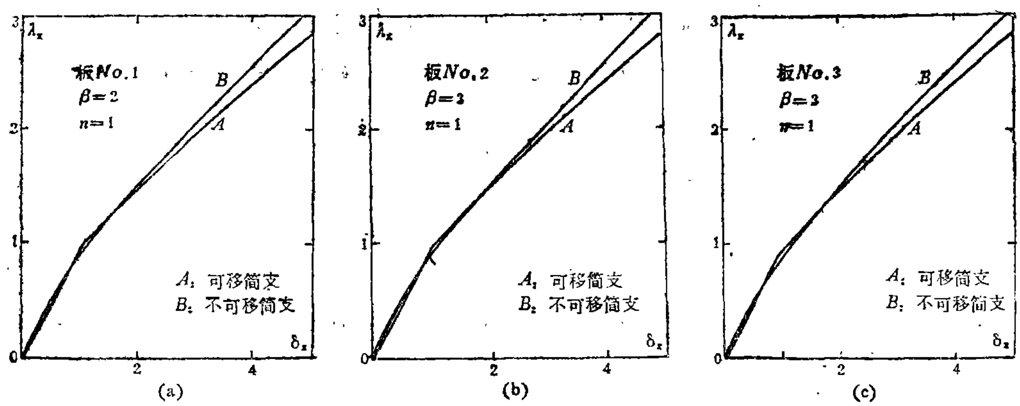
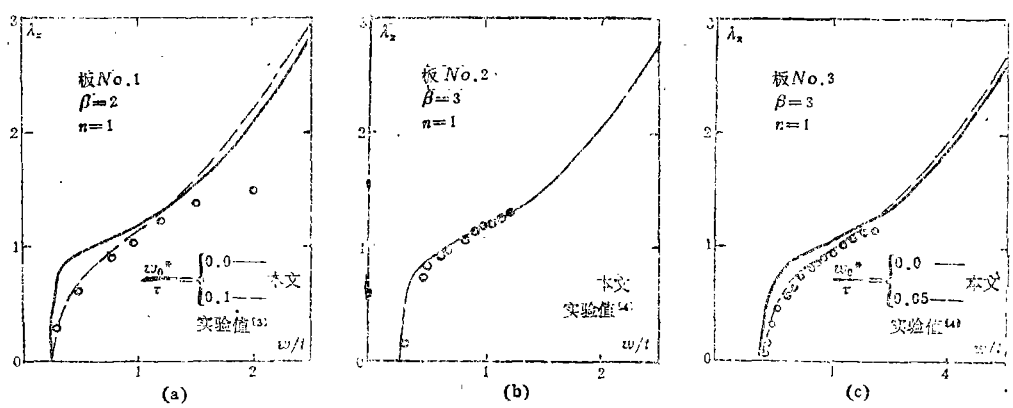


图3 后屈曲载荷-端部缩短曲线



—— 本文 ○ 实验值

图4 后曲屈载荷-挠度曲线，理论与实验结果比较

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Postbuckling of Rectangular Plates under Uniaxial Compression Combined with Lateral Pressure

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Abstract

Based on the nonlinear large deflection equations of von Kármán plates, the lateral pressure is first converted into an initial deflection by Galerkin method, the postbuckling behavior of simply supported rectangular plates under uniaxial compression combined with lateral pressure is then studied applying perturbation method by taking deflection as perturbation parameter.

Two types of in-plane boundary conditions and the effects of initial geometric imperfection are also considered. It is found that the theoretical results are in good accordance with experiments.