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具有复杂边界条件的杆的振动分析

唐驾时　李　骊　霍拳忠

（湖南大学）　（天津大学）

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摘　要

本文研究一端带有集中质量并支以弹簧另一端作支承运动的杆的纵向振动．由于这个问题的边界条件比较复杂，且要考虑阻尼，因此本文只求稳态周期解．首先分析线性系统；然后考虑材料非线性，用摄动法求具有非线性边界条件的非线性方程的近似解析解．

一端带有集中质量并支以弹簧另一端作支承运动的杆的纵向有阻尼振动问题，由于它的边界条件比较复杂，且要考虑阻尼，因此即使就是线性系统，求解也比较麻烦．我们考虑支承运动为多项谐波激励，且假设杆是细长均匀的．首先分析线性系统，求杆纵向振动的稳态周期解；然后考虑材料非线性，它表现为应力是应变的非线性函数，用摄动法求其近似解析解，具体作法是：将解进行摄动展开后用谐波平衡来分离变量，通过求解空间问题的常微分方程组来得到各阶摄动方程的解．

符号说明

$u(x,t)$：杆x截面的位移

$s(t)$：支承端位移

D_1：内阻尼系数

m：杆端集中质量

ρ：杆密度

k：弹簧刚度

D_2：外阻尼系数

l：杆长度

E：弹性模量

A：杆截面积

一、线性系统分析

考虑应力是应变的线性函数时，图1所示的一端带有集中质量并支以弹簧另一端作支承运动的杆的纵向振动方程为

$$\rho A\frac{\partial^2 u}{\partial t^2}=EA\frac{\partial^2 u}{\partial x^2}-D_1\frac{\partial^3 u}{\partial x^2\partial t}-D_2\frac{\partial u}{\partial t} \tag{1.1}$$

边界条件为

$$\left.\begin{aligned}&u(0,t)=s(t)\\&EA\frac{\partial u(l,t)}{\partial x}=-m\frac{\partial^2 u(l,t)}{\partial t^2}-ku(l,t)\end{aligned}\right\}\tag{1.2}$$

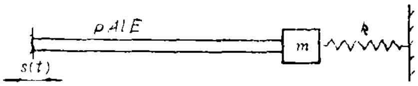

图 1 杆振动模型

式 (1.1) 可以简化为

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}-\mu_1\frac{\partial^3 u}{\partial x^2\partial t}-\mu_2\frac{\partial u}{\partial t}\tag{1.3}$$

其中

$$a^2=\frac{E}{\rho},\quad \mu_1=\frac{D_1}{\rho A},\quad \mu_2=\frac{D_2}{\rho A}\tag{1.4}$$

考虑支承运动为周期函数，按正弦级数展开，它的前 n 项为

$$s(t)=s_0+\sum_{j=1}^{n}s_j\sin j\omega t\tag{1.5}$$

设式 (1.3) 的周期解为

$$u(x,t)=f(x)+\sum_{j=1}^{n}[g_j(x)\cos j\omega t+h_j(x)\sin j\omega t]\tag{1.6}$$

将式(1.6)代入式(1.3)，利用谐波平衡，得

$$f''(x)=0\tag{1.7}$$

$$\left.\begin{aligned}&a^2g_j''(x)-\mu_1 j\omega h_j''(x)+j^2\omega^2 g_j(x)-\mu_2 j\omega h_j(x)=0\\&a^2h_j''(x)+\mu_1 j\omega g_j''(x)+j^2\omega^2 h_j(x)+\mu_2 j\omega g_j(x)=0\end{aligned}\right\}\tag{1.8}$$

式 (1.7) 的解为

$$f(x)=F_0+F_1x\tag{1.9}$$

式 (1.8) 为常微分方程组，设其解为

$$\left.\begin{aligned}&g_j(x)=A_j\exp(p_jx)\\&h_j(x)=B_j\exp(p_jx)\end{aligned}\right\}\tag{1.10}$$

将式(1.10)代入式(1.8)，并令

$$\alpha_j=\frac{\mu_1 j\omega}{a^2},\quad \beta_j=\frac{\mu_2 j\omega}{a^2},\quad \gamma_j=\frac{j\omega}{a}\tag{1.11}$$

则有

$$p_j^4+\frac{2(\gamma_j^2+\alpha_j\beta_j)}{1+\alpha_j^2}p_j^2+\frac{\beta_j^2+\gamma_j^4}{1+\alpha_j^2}=0$$

上方程有四个复根

$$\left.\begin{aligned}&p_{1j}=\lambda_{1j}+i\lambda_{2j},\qquad p_{2j}=-\lambda_{1j}-i\lambda_{2j}\\&p_{3j}=-\lambda_{1j}+i\lambda_{2j},\quad p_{4j}=\lambda_{1j}-i\lambda_{2j}\end{aligned}\right\}\tag{1.12}$$

其中

$$\left.\begin{aligned}&\lambda_{1j}=\sqrt{\frac{\sqrt{(\gamma_j^4+\beta_j^2)(1+\alpha_j^2)}-(\gamma_j^2+\alpha_j\beta_j)}{2(1+\alpha_j^2)}}\\&\lambda_{2j}=\sqrt{\frac{\sqrt{(\gamma_j^4+\beta_j^2)(1+\alpha_j^2)}+(\gamma_j^2+\alpha_j\beta_j)}{2(1+\alpha_j^2)}}\end{aligned}\right\}\tag{1.13}$$

且式(1.10)中的常数A_j，B_j有如下关系

$$\left.\begin{array}{ll} B_{1j}=iA_{1j}, & B_{2j}=iA_{2j} \\ B_{3j}=-iA_{3j}, & B_{4j}=-iA_{4j} \end{array}\right\} \tag{1.14}$$

将式(1.12)，(1.14)代入式(1.10)，就得到四个复数形式的线性无关的解，将它们的线性组合与式(1.9)一起代入式(1.6)，得

$$\begin{aligned} u(x,t)=F_0+F_1x+\sum_{j=1}^{n}\{&[\exp[\lambda_{1j}x](C_{1j}\cos\lambda_{2j}x+C_{2j}\sin\lambda_{2j}x) \\ &+\exp[-\lambda_{1j}x](C_{3j}\cos\lambda_{2j}x+C_{4j}\sin\lambda_{2j}x)]\cos j\omega t \\ &+\exp[\lambda_{1j}x](C_{2j}\cos\lambda_{2j}x-C_{1j}\sin\lambda_{2j}x) \\ &-\exp[-\lambda_{1j}x](C_{4j}\cos\lambda_{2j}x-C_{3j}\sin\lambda_{2j}x)]\sin j\omega t\} \end{aligned} \tag{1.15}$$

由边界条件式(1.2)可求得式(1.15)中的常数为

$$\left.\begin{array}{ll} F_0=s_0, & F_1=-\dfrac{ks_0}{EA+kl} \\ C_{1j}=\dfrac{b_{1j}d_{1j}-b_{2j}d_{2j}}{d_{1j}^2+d_{2j}^2}, & C_{2j}=\dfrac{b_{2j}d_{1j}+b_{1j}d_{2j}}{d_{1j}^2+d_{2j}^2} \\ C_{3j}=-C_{1j}, & C_{4j}=-s_j+C_{2j} \end{array}\right\} \tag{1.16}$$

其中

$$\left.\begin{aligned} d_{1j}&=\lambda_{1j}\cos\lambda_{2j}l\,\mathrm{ch}\lambda_{1j}l-\lambda_{2j}\sin\lambda_{2j}l\,\mathrm{sh}\lambda_{1j}l+\frac{k-mj^2\omega^2}{EA}\cos\lambda_{2j}l\,\mathrm{sh}\lambda_{1j}l \\ d_{2j}&=\lambda_{1j}\sin\lambda_{2j}l\,\mathrm{sh}\lambda_{1j}l+\lambda_{2j}\cos\lambda_{2j}l\,\mathrm{ch}\lambda_{1j}l+\frac{k-mj^2\omega^2}{EA}\sin\lambda_{2j}l\,\mathrm{ch}\lambda_{1j}l \\ b_{1j}&=\frac{1}{2}\exp[-\lambda_{1j}l]s_j\left(-\lambda_{1j}\sin\lambda_{2j}l+\lambda_{2j}\cos\lambda_{2j}l+\frac{k-mj^2\omega^2}{EA}\sin\lambda_{2j}l\right) \\ b_{2j}&=\frac{1}{2}\exp[-\lambda_{1j}l]s_j\left(\lambda_{1j}\cos\lambda_{2j}l-\lambda_{2j}\sin\lambda_{2j}l-\frac{k-mj^2\omega^2}{EA}\cos\lambda_{2j}l\right) \end{aligned}\right\} \tag{1.17}$$

将式(1.16)代入式(1.15)，则方程(1.1)的解为

$$\left.\begin{aligned} u(x,t)=s_0-\frac{ks_0}{EA+kl}x+\sum_{j=1}^{n}[&(2C_{1j}\cos\lambda_{2j}x\,\mathrm{sh}\lambda_{1j}x+2C_{2j}\sin\lambda_{2j}x\,\mathrm{ch}\lambda_{1j}x \\ &-s_j\exp[-\lambda_{1j}x]\sin\lambda_{2j}x)\cos j\omega t+(-2C_{1j}\sin\lambda_{2j}x\,\mathrm{ch}\lambda_{1j}x \\ &+2C_{2j}\cos\lambda_{2j}x\,\mathrm{sh}\lambda_{1j}x+s_j\exp[-\lambda_{1j}x]\cos\lambda_{2j}x)\sin j\omega t] \end{aligned}\right\} \tag{1.18}$$

二、非线性系统分析

如果考虑应力是应变的非线性函数，一般来说，$\sigma(\partial u/\partial x)$的泰勒级数展开式为

$$\sigma\left(\frac{\partial u}{\partial x}\right)=E\frac{\partial u}{\partial x}\left[1+E_1\frac{\partial u}{\partial x}+E_2\left(\frac{\partial u}{\partial x}\right)^2+\cdots\right]$$

其中E_1，E_2为无量纲量。略去上式中立方以上的高阶项，则

$$\sigma\left(\frac{\partial u}{\partial x}\right)=E\frac{\partial u}{\partial x}\left(1+E_1\frac{\partial u}{\partial x}\right) \tag{2.1}$$

此时，杆的运动微分方程为

$$\rho A\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial x}\left[A\sigma\left(\frac{\partial u}{\partial x}\right)-D_1\frac{\partial^2 u}{\partial x\partial t}\right]-D_2\frac{\partial u}{\partial t}$$

考虑到式(2.1)，上式变为

$$\rho A\frac{\partial^2 u}{\partial t^2}=EA\left(\frac{\partial^2 u}{\partial x^2}+2E_1\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\right)-D_1\frac{\partial^3 u}{\partial x^2\partial t}-D_2\frac{\partial u}{\partial t} \tag{2.2}$$

边界条件

$$\left.\begin{aligned}&u(0,t)=s(t)\\&EA\frac{\partial u(l,t)}{\partial x}\left[1+E_1\frac{\partial u(l,t)}{\partial x}\right]=-m\frac{\partial^2 u(l,t)}{\partial t^2}-ku(l,t)\end{aligned}\right\} \tag{2.3}$$

我们考虑弱非线性情况，用摄动法来解方程(2.2)。首先将式(2.2)，(2.3)无量纲化，设

$$\left.\begin{aligned}&u^*=\frac{1}{L}u, && x^*=\frac{1}{L}x\\&t^*=\frac{1}{L}\sqrt{\frac{E}{\rho}}\,t, && \omega^*=L\sqrt{\frac{\rho}{E}}\,\omega\\&l^*=\frac{1}{L}l, && k^*=\frac{1}{EA}Lk\\&m^*=\frac{1}{LA\rho}m, && s_j^*=\frac{1}{L}s_j\quad(j=0,1,2,\cdots n)\\&\mu_1^*=\frac{1}{\sqrt{E\rho}\,A}D_1 && \mu_2^*=\frac{1}{\sqrt{E\rho}\,A}LD_2\end{aligned}\right\} \tag{2.4}$$

将式(2.4)代入式(2.2)，(2.3)，不妨去掉"*"号，并让$L=l$，则式(2.2)，(2.3)无量纲化后的形式为

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}=-2E_1\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+\mu_1\frac{\partial^3 u}{\partial x^2\partial t}+\mu_2\frac{\partial u}{\partial t} \tag{2.5}$$

边界条件

$$\left.\begin{aligned}&u(0,t)=s_0+\sum_{j=1}^{n}s_j\sin j\omega t\\&\frac{\partial u(1,t)}{\partial x}\left[1+E_1\frac{\partial u(1,t)}{\partial x}\right]=-m\frac{\partial^2 u(1,t)}{\partial t^2}-ku(1,t)\end{aligned}\right\} \tag{2.6}$$

设式(2.5)的解为

$$u(x,t)=u_0(x,t)+\varepsilon u_1(x,t)+\varepsilon^2 u_2(x,t)+\cdots \tag{2.7}$$

因为$E_1\partial u/\partial x$，μ_1，μ_2，ε可以视为同一量级，故在式(2.5)的右端添上小参数ε，然后将式(2.7)代入式(2.5)得

$$\frac{\partial^2 u_0}{\partial x^2}+\varepsilon\frac{\partial^2 u_1}{\partial x^2}+\cdots-\frac{\partial^2 u_0}{\partial t^2}-\varepsilon\frac{\partial^2 u_1}{\partial t^2}-\cdots=-2\varepsilon E_1\left(\frac{\partial u_0}{\partial x}+\varepsilon\frac{\partial u_1}{\partial x}+\cdots\right)$$

$$\cdot\left(\frac{\partial^2 u_0}{\partial x^2}+\varepsilon\frac{\partial^2 u_1}{\partial x^2}+\cdots\right)+\varepsilon\mu_1\frac{\partial^3 u_0}{\partial x^2\partial t}+\varepsilon^2\mu_1\frac{\partial^3 u_1}{\partial x^2\partial t}+\cdots$$

$$+\varepsilon\mu_2\frac{\partial u_0}{\partial t}+\varepsilon^2\mu_2\frac{\partial u_1}{\partial t}+\cdots$$

比较上式两边 ε 的同次幂系数得

$$\frac{\partial^2 u_0}{\partial x^2}-\frac{\partial^2 u_0}{\partial t^2}=0 \tag{2.8}$$

$$\frac{\partial^2 u_1}{ux^2}-\frac{\partial^2 u_1}{\partial t^2}=-2E_1\frac{\partial u_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2}+\mu_1\frac{\partial^3 u_0}{\partial x^2\partial t}+\mu_2\frac{\partial u_0}{\partial t} \tag{2.9}$$

边界条件相应为

$$\left.\begin{aligned}&u_0(0,t)=s_0+\sum_{j=1}^{n}s_j\sin j\omega t\\&\frac{\partial u_0(1,t)}{\partial x}=-m\frac{\partial^2 u_0(1,t)}{\partial t^2}-ku_0(1,t)\end{aligned}\right\} \tag{2.10}$$

$$\left.\begin{aligned}&u_1(0,t)=0\\&\frac{\partial u_1(1,t)}{\partial x}+E_1\left[\frac{\partial u_0(1,t)}{\partial x}\right]^2=-m\frac{\partial^2 u_1(1,t)}{\partial t^2}-ku_1(1,t)\end{aligned}\right\} \tag{2.11}$$

设方程(2.8)的周期解为

$$u_0(x,t)=a(x)+\sum_{j=1}^{n}[b_j(x)\cos j\omega t+C_j(x)\sin j\omega t] \tag{2.12}$$

将式(2.12) 代入式 (2.8)，通过谐波平衡可得一组常微分方程．解这组方程，利用边界条件式(2.10)，得

$$u_0(x,t)=s_0+\sum_{j=1}^{n}(s_j\cos j\omega x+C_j\sin j\omega x)\sin j\omega t \tag{2.13}$$

其中

$$C_j=\frac{j\omega\sin j\omega+(mj^2\omega^2-k)\cos j\omega}{j\omega\cos j\omega-(mj^2\omega^2-k)\sin j\omega}s_j \tag{2.14}$$

这里，我们假设

$$j\omega\cos j\omega-(mj^2\omega^2-k)\sin j\omega\neq 0$$

式(2.13)实际上就是不计阻尼时线性方程的解．

将式(2.13)代入式(2.9)的右端

$$\frac{\partial^2 u_1}{\partial x^2}-\frac{\partial^2 u_1}{\partial t^2}=E_1\sum_{j=1}^{n}j^3\omega^3(\alpha_j\cos 2j\omega x+\beta_j\sin 2j\omega x)+\sum_{j=1}^{n}(-\mu_1 j^3\omega^3$$

$$+\mu_2 j\omega)(s_j\cos j\omega x+C_j\sin j\omega x)\cos j\omega t-E_1\sum_{j=1}^{n}j^3\omega^3(\alpha_j\cos 2j\omega x$$

$$+\beta_j\sin 2j\omega x)\cos 2j\omega t+\frac{1}{2}E_1\sum_{j=1}^{n}\sum_{\substack{i=1\\i\neq j}}^{n}ji^2\omega^3[\gamma_{ji}\cos(j+i)\omega x$$

$$+\delta_{ji}\cos(j-i)\omega x+\zeta_{ji}\sin(j+i)\omega x+\eta_{ji}\sin(j-i)\omega x]$$
$$\cdot[-\cos(j+i)\omega t+\cos(j-i)\omega t] \tag{2.15}$$

其中

$$\left.\begin{aligned} &\alpha_j=s_jC_j, && \beta_j=\frac{1}{2}(C_j^2-s_j^2)\\ &\gamma_{ji}=C_js_i+s_jC_i, && \delta_{ji}=C_js_i-s_jC_i\\ &\zeta_{ji}=-s_js_i+C_jC_i, && \eta_{ji}=-s_js_i-C_jC_i. \end{aligned}\right\} \tag{2.16}$$

因此方程(2.15)的周期解可设为

$$u_1(x,t)=g(x)+\sum_{j=1}^{n}[(-\mu_1j^3\omega^3+\mu_2j\omega)d_j(x)\cos j\omega t$$
$$-E_1j^3\omega^3h_j(x)\cos 2j\omega t]+\sum_{j=1}^{n}\sum_{\substack{i=1\\i\neq j}}^{n}\frac{1}{2}E_1ji^2\omega^3[-m_{ji}(x)$$
$$\cdot\cos(j+i)\omega t+n_{ji}(x)\cos(j-i)\omega t] \tag{2.17}$$

将式(2.17)代入式(2.9)，同样地利用谐波平衡，可得一组常微分方程

$$\left.\begin{aligned} &g''(x)=E_1\sum_{j=1}^{n}j^3\omega^3(\alpha_j\cos 2j\omega x+\beta_j\sin 2j\omega x)\\ &d_j''(x)+(j\omega)^2d_j(x)=s_j\cos j\omega x+C_j\sin j\omega x\\ &h_j''(x)+(2j\omega)^2h_j(x)=\alpha_j\cos 2j\omega x+\beta_j\sin 2j\omega x\\ &m_{ji}''(x)+(j+i)^2\omega^2m_{ji}(x)=\gamma_{ji}\cos(j+i)\omega x+\delta_{ji}\cos(j-i)\omega x\\ &\qquad+\zeta_{ji}\sin(j+i)\omega x+\eta_{ji}\sin(j-i)\omega x\\ &n_{ji}''(x)+(j-i)^2\omega^2n_{ji}(x)=\gamma_{ji}\cos(j+i)\omega x+\delta_{ji}\cos(j-i)\omega x\\ &\qquad+\zeta_{ji}\sin(j+i)\omega x+\eta_{ji}\sin(j-i)\omega x \end{aligned}\right\} \tag{2.18}$$

对这些方程逐一求解得

$$\left.\begin{aligned} &g(x)=G_0+G_1x-\frac{1}{4}E_1\sum_{j=1}^{n}j\omega(\alpha_j\cos 2j\omega x+\beta_j\sin 2j\omega x)\\ &d_j(x)=D_{1j}\cos j\omega x+D_{2j}\sin j\omega x+\frac{1}{2j\omega}x(-C_j\cos j\omega x+s_j\sin j\omega x)\\ &h_j(x)=H_{1j}\cos 2j\omega x+H_{2j}\sin 2j\omega x+\frac{1}{4j\omega}x(-\beta_j\cos 2j\omega x+\alpha_j\sin 2j\omega x)\\ &m_{ji}(x)=M_{1ji}\cos(j+i)\omega x+M_{2ji}\sin(j+i)\omega x+\frac{1}{2(j+i)\omega}x[-\zeta_{ji}\cos(j+i)\omega x\\ &\qquad+\gamma_{ji}\sin(j+i)\omega x]+\frac{1}{4ji\omega^2}[\delta_{ji}\cos(j-i)\omega x+\eta_{ji}\sin(j-i)\omega x]\\ &n_{ji}(x)=N_{1ji}\cos(j-i)\omega x+N_{2ji}\sin(j-i)\omega x+\frac{1}{2(j-i)\omega}x[-\zeta_{ji}\cos(j-i)\omega x\\ &\qquad+\gamma_{ji}\sin(j-i)\omega x]-\frac{1}{4ji\omega^2}[\delta_{ji}\cos(j+i)\omega x+\eta_{ji}\sin(j+i)\omega x] \end{aligned}\right\} \tag{2.19}$$

因此式(2.17)成为

$$u_1=G_0+G_1x=\sum_{j=1}^{n}\Big\{-\frac{1}{4}E_1j\omega(\alpha_j\cos2j\omega x+\beta_j\sin2j\omega x)+(-\mu_1j^3\omega^3$$

$$+\mu_2j\omega)\Big[D_{1j}\cos j\omega x+D_{2j}\sin j\omega x+\frac{1}{2j\omega}x(-C_1\cos j\omega x$$

$$+s_j\sin j\omega x)\Big]\cos j\omega t-E_1j^3\omega^3\Big[H_{1j}\cos2j\omega x+H_{2j}\sin2j\omega x$$

$$+\frac{1}{4j\omega}x(-\beta_j\cos2j\omega x+\alpha_j\sin2j\omega x)\Big]\cos2j\omega t\Big\}$$

$$-\frac{1}{2}E_1\sum_{j=1}^{n}\sum_{\substack{i=1\\i\neq j}}^{n}ji^2\omega^3\Big\{M_{1ji}\cos(j+i)\omega x+M_{2ji}\sin(j+i)\omega x$$

$$+\frac{1}{2(j+i)\omega}x[-\zeta_{ji}\cos(j+i)\omega x+\gamma_{ji}\sin(j+i)\omega x]$$

$$+\frac{1}{4ji\omega^2}[\delta_{ji}\cos(j-i)\omega x+\eta_{ji}\sin(j-i)\omega x]\Big\}\cos(j+i)\omega t$$

$$+\frac{1}{2}E_1\sum_{j=1}^{n}\sum_{\substack{i=1\\i\neq j}}^{n}ji^2\omega^3\Big\{N_{1ji}\cos(j-i)\omega x+N_{2ji}\sin(j-i)\omega x$$

$$+\frac{1}{2(j-i)\omega}x[-\zeta_{ji}\cos(j-i)\omega x+\gamma_{ji}\sin(j-i)\omega x]$$

$$-\frac{1}{4ji\omega^2}[\delta_{ji}\cos(j+i)\omega x+\eta_{ji}\sin(j+i)\omega x]\Big\}\cos(j-i)\omega t \tag{2.20}$$

由式(2.11)的第一个边界条件得

$$\left.\begin{aligned}&D_{1j}=0,\quad G_0=\frac{1}{4}E_1\sum_{j=1}^{n}j\omega\alpha_j, &&H_{1j}=0,\\&N_{1ji}=\frac{1}{4ji\omega^2}\delta_{ji}, &&M_{1ji}=-\frac{1}{4ji\omega^2}\delta_{ji}\end{aligned}\right\} \tag{2.21}$$

由式(2.11)的第二个边界条件得

$$G_1+\sum_{j=1}^{n}\Big\{-\frac{1}{2}E_1j^2\omega^2(-\alpha_j\sin2j\omega+\beta_j\cos2j\omega)+(-\mu_1j^3\omega^3+\mu_2j\omega)\Big[j\omega D_{2j}\cos j\omega$$

$$+\frac{1}{2j\omega}(-C_1\cos j\omega+s_j\sin j\omega)+\frac{1}{2}(-C_j\sin j\omega+s_j\cos j\omega)\Big]\cos j\omega t$$

$$-E_1\Big[2j^4\omega^4H_{2j}\cos2j\omega+\frac{1}{4}j^2\omega^2(-\beta_j\cos2j\omega+\alpha_j\sin2j\omega)$$

$$+\frac{1}{2}j^3\omega^3(\beta_j\sin2j\omega+\alpha_j\cos2j\omega)\Big]\cos2j\omega t\Big\}-\frac{1}{2}E_1\sum_{j=1}^{n}\sum_{\substack{i=1\\i\neq j}}^{n}ji^2\omega^3$$

$$\cdot\Big\{(j+i)\omega[-M_{1ji}\sin(j+i)\omega+M_{2ji}\cos(j+i)\omega]+\frac{1}{2(j+i)\omega}[-\zeta_{ji}\cos(j+i)\omega$$

$$+\gamma_{ji}\sin(j+i)\omega]+\frac{1}{2}[\zeta_{ji}\sin(j+i)\omega+\gamma_{ji}\cos(j+i)\omega]$$

$$+\frac{j-i}{4ji\omega}[-\delta_{ji}\sin(j-i)\omega+\eta_{ji}\cos(j-i)\omega]\Big\}\cos(j+i)\omega t$$

$$+\frac{1}{2}E_1\sum_{j=1}^{n}\sum_{\substack{i=1\\i\neq j}}^{n}ji^2\omega^3\Big\{(j-i)\omega[-N_{1ji}\sin(j-i)\omega+N_{2ji}\cos(j-i)\omega]$$

$$+\frac{1}{2(j-i)\omega}[-\zeta_{ji}\cos(j-i)\omega+\gamma_{ji}\sin(j-i)\omega]+\frac{1}{2}[\zeta_{ji}\sin(j-i)\omega$$

$$+\gamma_{ji}\cos(j-i)\omega]-\frac{j+i}{4ji\omega}[-\delta_{ji}\sin(j+i)\omega+\eta_{ji}\cos(j+i)\omega]\Big\}\cos(j-i)\omega t$$

$$+\sum_{j=1}^{n}\Big[\frac{1}{2}E_1j^2\omega^2\Big(\frac{s_j^2+C_j^2}{2}+\beta_j\cos2j\omega-\alpha_j\sin2j\omega\Big)-\frac{1}{2}E_1j^2\omega^2\Big(\frac{s_j^2+C_j^2}{2}$$

$$+\beta_j\cos2j\omega+\alpha_j\sin2j\omega\Big)\cos2j\omega t\Big]+\frac{1}{4}E_1\sum_{j=1}^{n}\sum_{\substack{i=1\\i\neq j}}^{n}ji\omega^2[\zeta_{ji}\cos(j+i)\omega$$

$$-\eta_{ji}\cos(j-i)\omega-\gamma_{ji}\sin(j+i)\omega+\delta_{ji}\sin(j-i)\omega][-\cos(j+i)\omega t+\cos(j-i)\omega t]$$

$$=-K(G_0+G_1)+\sum_{j=1}^{n}\Big\{\frac{1}{4}KE_1j\omega(\alpha_j\cos2j\omega+\beta_j\sin2j\omega)$$

$$+(-\mu_1j^3\omega^3+\mu_2j\omega)(mj^2\omega^2-K)\Big[D_{2j}\sin j\omega+\frac{1}{2j\omega}(-C_j\cos j\omega$$

$$+s_j\sin j\omega)\Big]\cos j\omega t-E_1j^3\omega^3(4mj^2\omega^2-K)\Big[H_{2j}\sin2j\omega$$

$$+\frac{1}{4j\omega}(-\beta_j\cos2j\omega+\alpha_j\sin2j\omega)\Big]\Big\}\cos2j\omega t-\frac{1}{2}E_1\sum_{j=1}^{n}\sum_{\substack{i=1\\i\neq j}}^{n}ji^2\omega^3[m(j+i)^2\omega^2$$

$$-K]\Big\{M_{1ji}\cos(j+i)\omega+M_{2ji}\sin(j+i)\omega+\frac{1}{2(j+i)\omega}[-\zeta_{ji}\cos(j+i)\omega$$

$$+\gamma_{ji}\sin(j+i)\omega]+\frac{1}{4ji\omega^2}[\delta_{ji}\cos(j-i)\omega+\eta_{ji}\sin(j-i)\omega]\Big\}\cos(j+i)\omega t$$

$$+\frac{1}{2}E_1\sum_{j=1}^{n}\sum_{\substack{i=1\\i\neq j}}^{n}ji^2\omega^3[m(j-i)^2\omega^2-K]\Big\{N_{1ji}\cos(j-i)\omega+N_{2ji}\sin(j-i)\omega$$

$$+\frac{1}{2(j-i)\omega}[-\zeta_{ji}\cos(j-i)\omega+\gamma_{ji}\sin(j-i)\omega]-\frac{1}{4ji\omega^2}[\delta_{ji}\cos(j+i)\omega$$

$$+\eta_{ji}\sin(j+i)\omega]\Big\}\cos(j-i)\omega t$$

令同次谐波的系数相等，可确定上式的常数为

$$G_1=-\frac{1}{4(1+K)}E_1\sum_{j=1}^{n}[j^2\omega^2(s_j^2+C_j^2)-Kj\omega(\alpha_j\cos2j\omega+\beta_j\sin2j\omega-\alpha_j)]$$

$$\left.\begin{aligned}
D_{2j}&=\frac{[-(mj^2\omega^2-K)C_j-j\omega s_j+C_j]\cos j\omega+[(mj^2\omega^2-K)s_j-j\omega C_j-s_j]\sin j\omega}{2j\omega[j\omega\cos j\omega-(mj^2\omega^2-K)\sin j\omega]}\\
H_{2j}&=[-(s_j^2+C_j^2)+[-\beta_j(4mj^2\omega^2-K)-\beta_j-2j\omega\alpha_j]\cos 2j\omega+[\alpha_j(4mj^2\omega^2-K)\\
&\quad+\alpha_j-2j\omega\beta_j]\sin 2j\omega]\cdot[4j\omega[2j\omega\cos 2j\omega-(4mj^2\omega^2-K)\sin 2j\omega]]^{-1}\\
M_{2ji}&=\frac{1}{(j+i)\omega\cos(j+i)\omega-[m(j+i)^2\omega^2-K]\sin(j+i)\omega}\Big\{M_{1ji}[(m(j+i)^2\omega^2-K)\\
&\quad\cdot\cos(j+i)\omega+(j+i)\omega\sin(j+i)\omega]+\Big[-\frac{m(j+i)^2\omega^2-K}{2(j+i)\omega}\zeta_{ji}-\frac{1}{2}\gamma_{ji}\\
&\quad-\frac{1}{2j\omega}\zeta_{ji}+\frac{1}{2(j+i)\omega}\zeta_{ji}\Big]\cos(j+i)\omega+\Big[\frac{m(j+i)^2\omega^2-K}{2(j+i)\omega}\gamma_{ji}-\frac{1}{2}\zeta_{ji}\\
&\quad+\frac{1}{2j\omega}\gamma_{ji}-\frac{1}{2(j+i)\omega}\gamma_{ji}\Big]\sin(j+i)\omega+\Big[\frac{m(j+i)^2\omega^2-K}{4ji\omega^2}\delta_{ji}\\
&\quad+\frac{1}{2j\omega}\eta_{ji}-\frac{j-i}{4ji\omega}\eta_{ji}\Big]\cos(j-i)\omega+\Big[\frac{m(j+i)^2\omega^2-K}{4ji\omega^2}\eta_{ji}\\
&\quad-\frac{1}{2j\omega}\delta_{ji}+\frac{j-i}{4ji\omega}\delta_{ji}\Big]\sin(j-i)\omega\Big\}\\
N_{2ji}&=\frac{1}{(j-i)\omega\cos(j-i)\omega-[m(j-i)^2\omega^2-K]\sin(j-i)\omega}\Big\{N_{1ji}[(m(j-i)^2\omega^2\\
&\quad-K)\cos(j-i)\omega+(j-i)\omega\sin(j-i)\omega]+\Big[-\frac{m(j-i)^2\omega^2-K}{2(j-i)\omega}\zeta_{ji}\\
&\quad-\frac{1}{2}\gamma_{ji}+\frac{1}{2j\omega}\eta_{ji}+\frac{1}{2(j-i)\omega}\zeta_{ji}\Big]\cos(j-i)\omega+\Big[\frac{m(j-i)^2\omega^2-K}{2(j-i)\omega}\gamma_{ji}\\
&\quad-\frac{1}{2}\zeta_{ji}-\frac{1}{2j\omega}\delta_{ji}-\frac{1}{2(j-i)\omega}\gamma_{ji}\Big]\sin(j-i)\omega+\Big[-\frac{m(j-i)^2\omega^2-K}{4ji\omega^2}\delta_{ji}\\
&\quad-\frac{1}{2j\omega}\zeta_{ji}+\frac{j+i}{4ji\omega}\eta_{ji}\Big]\cos(j+i)\omega+\Big[-\frac{m(j-i)^2\omega^2-K}{4ji\omega^2}\eta_{ji}\\
&\quad+\frac{1}{2j\omega}\gamma_{ji}-\frac{j+i}{4ji\omega}\delta_{ji}\Big]\sin(j+i)\omega\Big\}
\end{aligned}\right\}\tag{2.22}$$

这里，我们假设

$$2j\omega\cos 2j\omega-(4mj^2\omega^2-K)\sin 2j\omega\neq 0$$
$$(j+i)\omega\cos(j+i)\omega-[m(j+i)^2\omega^2-K]\sin(j+i)\omega\neq 0$$
$$(j-i)\omega\cos(j-i)\omega-[m(j-i)^2\omega^2-K]\sin(j-i)\omega\neq 0$$

因此方程 (2.5) 的近似解析解为

$$\begin{aligned}
u(x,t)&=u_0(x,t)+\varepsilon u_1(x,t)\\
&=Y_0(x)+\sum_{j=1}^{n}[Y_{1j}(x)\cos j\omega t+Y_{2j}(x)\sin j\omega t+Z_j(x)\cos 2j\omega t]\\
&\quad+\sum_{j=1}^{n}\sum_{\substack{i=1\\ i\neq j}}^{n}[W_{1ji}(x)\cos(j+i)\omega t+W_{2ji}(x)\cos(j-i)\omega t]
\end{aligned}\tag{2.23}$$

其中

$$
\left.\begin{aligned}
&Y_0(x)=s_0+G_1x-\frac{1}{4}E_1\sum_{j=1}^{n}j\omega(\alpha_j\cos2j\omega x+\beta_j\sin2j\omega x-\alpha_j)\\
&Y_{1j}(x)=(-\mu_1j^3\omega^3+\mu_2j\omega)\left[D_{2j}\sin j\omega x+\frac{1}{2j\omega}x(-C_j\cos j\omega x+s_j\sin j\omega x)\right]\\
&Y_{2j}(x)=s_j\cos j\omega x+C_j\sin j\omega x\\
&Z_j(x)=-E_1j^3\omega^3\left[H_{2j}\sin2j\omega x+\frac{1}{4j\omega}x(-\beta_j\cos2j\omega x+\alpha_j\sin2j\omega x)\right]\\
&W_{1ji}(x)=-\frac{1}{2}E_1ji^2\omega^3\Big\{M_{1ji}\cos(j+i)\omega x+M_{2ji}\sin(j+i)\omega x\\
&\qquad+\frac{1}{2(j+i)\omega}x[-\zeta_{ji}\cos(j+i)\omega x+\gamma_{ji}\sin(j+i)\omega x]\\
&\qquad+\frac{1}{4ji\omega^2}[\delta_{ji}\cos(j-i)\omega x+\eta_{ji}\sin(j-i)\omega x]\Big\}\\
&W_{2ji}(x)=\frac{1}{2}E_1ji^2\omega^3\Big\{N_{1ji}\cos(j-i)\omega x+N_{2ji}\sin(j-i)\omega x\\
&\qquad+\frac{1}{2(j-i)\omega}x[-\zeta_{ji}\cos(j-i)\omega x+\gamma_{ji}\sin(j-i)\omega x]\\
&\qquad-\frac{1}{4ji\omega^2}[\delta_{ji}\cos(j+i)\omega x+\eta_{ji}\sin(j+i)\omega x]\Big\}
\end{aligned}\right\}\qquad(2.24)
$$

杆（或柱）的纵向强迫振动在工程上是常见的，例如内燃机气门推杆的振动，建筑物基础桩的振动等等，都可以归结为本文提出的这类力学问题。

我们把上面推导的结果，应用到研究内燃机气门推杆的振动问题上，获得了气门的运动规律，其计算结果与实验结果接近。

对于一端作支承运动，另一端或自由，或固定，或用弹簧支承，或带有集中质量的这类杆的纵向振动问题，都可以用上述方法来求其稳态周期解。

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Vibration Analysis of a Rod with Complex Boundary Conditions

Tang Jia-shi

(Hunan University, Changsha)

Li Li Huo Quan-zhong

(Tianjin University, Tianjin)

Abstract

This paper deals with the forced longitudinal vibration of a rod carrying a concentrated mass and supported by a spring at one end. The vibration of the rod is excited by the motion of the supporting point at the other end. Since the boundary conditions of the problem are complex and it is necessary to consider the damping, we determine only the steady state periodic solution. First the linear system is anaysed; then the material nonlinearity is considered and the approximate analytic solution of nonlinear partial differential equation with nonlinear boundary conditions is obtained by the perturbation method.