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单向压缩简支矩形板后屈曲摄动分析*

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摘　要

本文从 Kármán 板大挠度方程出发，以挠度为摄动参数，采用直接摄动法，研究了简支矩形板在单向压缩作用下的后屈曲性态。

本文讨论了两种面内边界条件，同时考虑了初始挠度的影响。计算结果与实验结果的比较表明二者是一致的。

本文所用的方法，没有见到有人发表过。作者认为，对于矩形板后屈曲分析，本文是比较简明的。

一、引　言

简支矩形板在单向压缩下屈曲是弹性稳定理论中的经典问题，又是一个重要问题。弄清矩形板的后屈曲性态，以及初挠度对屈后性态的影响，对充分认识和利用平板的后屈曲超载性能具有十分重要的意义。

一般认为，矩形板具有稳定的后屈曲平衡路径。Stein (1959)[1]曾发现，随着加载过程的延续，矩形板在后屈曲阶段将出现纵向波形的跳跃。这种现象后来被称为二次屈曲，并引起了许多学者的注意和研究[3],[4],[6],[7],[8],[10]。由于各人所用方法不同，所得结果存在明显的差异。目前，对矩形板二次屈曲的机理和性态还缺乏统一的认识。因此，有必要对矩形板的后屈曲性态作进一步的分析讨论。

在矩形板屈曲和后屈曲分析中广泛采用 Galerkin 法和能量法，摄动技术仅被用于非线性代数方程[2],[5],[9]。

需要特别指出，文[1]曾以载荷为摄动参数，采用连续摄动法求得四边简支矩形板后屈曲平衡路径的二级渐近表达式。但是很多情况说明，采用载荷为摄动参数往往使载荷挠度关系曲线在挠度较大时并不收敛于真实解。

一方面为了避免能量法、Galerkin 法带来的繁复，另一方面为了减小以载荷作摄动参数造成的偏差，本文将以挠度为摄动参数，采用直接摄动法，将 Kármán 板大挠度方程化为一系列线性方程求解，在求得大挠度渐近解的基础上，利用边界条件直接导得后屈曲平衡路径的四级渐近表达式。渐近解中含有初挠度的影响。初挠度的形式取作和矩形板小挠度解的形式一致。

* 钱伟长推荐。

本文讨论了两种面内边界条件，一种为纵向边缘可移简支，一种为纵向边缘不可移简支。

二、基 本 方 程

假定四边简支矩形板的长为a，宽为b，厚度为t，受到对边均布压力。取坐标系如图1所示。并以W^*和W分别表示初始的和附加的挠度，以ϕ表示应力函数，那么 Kármán 板大挠度方程可表为如下形式：

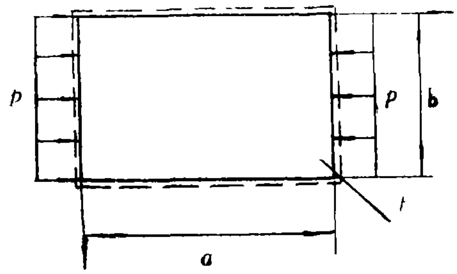


图1 单向压缩简支矩形板

$$D\nabla^4 W=t\left[\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 W}{\partial x^2}-2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 W}{\partial x\partial y}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 W^*}{\partial x^2}-2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 W^*}{\partial x\partial y}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 W^*}{\partial y^2}\right] \tag{2.1}$$

$$\nabla^4\phi=E\left[\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}+2\frac{\partial^2 W}{\partial x\partial y}\frac{\partial^2 W^*}{\partial x\partial y}-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W^*}{\partial y^2}-\frac{\partial^2 W}{\partial y^2}\frac{\partial^2 W^*}{\partial x^2}\right] \tag{2.2}$$

其中

$$\nabla^4=\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4} \tag{2.3}$$

$D=Et^3/12(1-\nu^2)$为弯曲刚度，E和ν分别为弹性模数和 Poisson 比。板中的内力

$$N_x=t\frac{\partial^2\phi}{\partial y^2},\quad N_{xy}=-t\frac{\partial^2\phi}{\partial x\partial y},\quad N_y=t\frac{\partial^2\phi}{\partial x^2} \tag{2.4}$$

面内位移U，V与W，W^*及ϕ的关系为

$$\left.\begin{aligned}&\frac{\partial U}{\partial x}+\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2+\frac{\partial W}{\partial x}\frac{\partial W^*}{\partial x}=\frac{1}{E}\left(\frac{\partial^2\phi}{\partial y^2}-\nu\frac{\partial^2\phi}{\partial x^2}\right)\\&\frac{\partial V}{\partial y}+\frac{1}{2}\left(\frac{\partial W}{\partial y}\right)^2+\frac{\partial W}{\partial y}\frac{\partial W^*}{\partial y}=\frac{1}{E}\left(\frac{\partial^2\phi}{\partial x^2}-\nu\frac{\partial^2\phi}{\partial y^2}\right)\\&\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}+\frac{\partial W}{\partial x}\frac{\partial W}{\partial y}+\frac{\partial W}{\partial x}\frac{\partial W^*}{\partial y}+\frac{\partial W}{\partial y}\frac{\partial W^*}{\partial x}=-\frac{2(1+\nu)}{E}\frac{\partial^2\phi}{\partial x\partial y}\end{aligned}\right\} \tag{2.5}$$

假定边界支承为四边简支的，那么边界条件为

$$x=0,a;\ W=W_{,xx}=0,\ N_{xy}=0 \tag{2.6a}$$

$$\int_0^b N_x dy+p=0 \tag{2.6b}$$

$$y=0,b;\ W=W_{,yy}=0,\ N_{xy}=0 \tag{2.7a}$$

$$\int_0^a N_y dx=0 \quad \text{(可移简支)} \tag{2.7b}$$

$$V=\text{const} \quad \text{(不可移简支)} \tag{2.7c}$$

单位轴向缩短

$$\frac{\Delta_x}{a}=-\frac{1}{ab}\int_0^b\int_0^a\frac{\partial U}{\partial x}dxdy$$

$$=-\frac{1}{ab}\int_0^b\int_0^a\left[\frac{1}{E}\left(\frac{\partial^2\phi}{\partial y^2}-\nu\frac{\partial^2\phi}{\partial x^2}\right)-\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2-\frac{\partial W}{\partial x}\frac{\partial W^*}{\partial x}\right]dxdy \tag{2.8}$$

$$\frac{\Delta_y}{b}=-\frac{1}{ab}\int_0^a\int_0^b\frac{\partial V}{\partial y}dydx$$

$$=-\frac{1}{ab}\int_0^a\int_0^b\left[\frac{1}{E}\left(\frac{\partial^2\phi}{\partial x^2}-\nu\frac{\partial^2\phi}{\partial y^2}\right)-\frac{1}{2}\left(\frac{\partial W}{\partial y}\right)^2-\frac{\partial W}{\partial y}\frac{\partial W^*}{\partial y}\right]dydx \tag{2.9}$$

引进

$$\bar{x}=\frac{\pi}{a}x,\ \bar{y}=\frac{\pi}{b}y,\bar{W}=\frac{W}{t}\sqrt{12(1-\nu^2)},\ \bar{W}^*=\frac{W^*}{t}\sqrt{12(1-\nu^2)},\ \beta=\frac{a}{b},$$

$$\bar{\phi}=\frac{\phi t}{D},\ \lambda_x=\frac{\sigma}{\sigma_{cl}},\ \sigma_{cl}=\frac{4\pi^2 D}{b^2 t},\ \delta_x=\frac{\Delta_x/a}{\sigma_{cl}/E}=\frac{12(1-\nu^2)b^2}{4\pi^2 t^2}\frac{\Delta_x}{a},$$

$$\delta_y=\frac{\Delta_y/b}{\sigma_{cl}/E}=\frac{12(1-\nu^2)b^2}{4\pi^2t^2}\frac{\Delta_y}{b} \tag{2.10}$$

那么，Kármán 方程可化为如下无量纲形式(略去字母上的“—”号)

$$\nabla^4 W=\beta^2\left[\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 W}{\partial x^2}-2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 W}{\partial x\partial y}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 W^*}{\partial x^2}\right.$$

$$\left.-2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 W^*}{\partial x\partial y}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 W^*}{\partial y^2}\right] \tag{2.11}$$

$$\nabla^4\phi=\beta^2\left[\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}+2\frac{\partial^2 W}{\partial x\partial y}\frac{\partial^2 W^*}{\partial x\partial y}-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W^*}{\partial y^2}-\frac{\partial^2 W}{\partial y^2}\frac{\partial^2 W^*}{\partial x^2}\right] \tag{2.12}$$

其中

$$\nabla^4=\frac{\partial^4}{\partial x^4}+2\beta^2\frac{\partial^4}{\partial x^2\partial y^2}+\beta^4\frac{\partial^4}{\partial y^4} \tag{2.13}$$

边界条件化为

$$x=0,\ \pi;\ W=W_{,xx}=0,\ \frac{\partial^2\phi}{\partial x\partial y}=0 \tag{2.14a}$$

$$\frac{1}{\pi}\int_0^\pi\beta^2\frac{\partial^2\phi}{\partial y^2}dy+4\lambda_x\beta^2=0 \tag{2.14b}$$

$$y=0,\ \pi;\ W=W_{,yy}=0,\ \frac{\partial^2\phi}{\partial x\partial y}=0 \tag{2.15a}$$

$$\frac{1}{\pi}\int_0^{\pi}\frac{\partial^2\phi}{\partial x^2}dx=0 \qquad (可移简支) \tag{2.15b}$$

$$\delta_y=b \qquad (不可移简支) \tag{2.15c}$$

单位轴向缩短化为

$$\delta_x=-\frac{1}{4\pi^2\beta^2}\int_0^{\pi}\int_0^{\pi}\left[\left(\beta^2\frac{\partial^2\phi}{\partial y^2}-\nu\frac{\partial^2\phi}{\partial x^2}\right)-\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2-\frac{\partial W}{\partial x}\frac{\partial W^*}{\partial x}\right]dxdy \tag{2.16}$$

$$\delta_y=-\frac{1}{4\pi^2\beta^2}\int_0^{\pi}\int_0^{\pi}\left[\left(\frac{\partial^2\phi}{\partial x^2}-\nu\beta^2\frac{\partial^2\phi}{\partial y^2}\right)-\frac{1}{2}\beta^2\left(\frac{\partial W}{\partial y}\right)^2-\beta^2\frac{\partial W}{\partial y}\frac{\partial W^*}{\partial y}\right]dydx \tag{2.17}$$

方程(2.11)至(2.17)为简支矩形板后屈曲问题控制方程。一般说来，要想求得其精确解是困难的。本文将以直接摄动法来构造其渐近解。

三、大挠度渐近解

设方程(2.11)、(2.12)的解为如下渐近展开式：

$$W(x,y,\varepsilon)=\sum_{n=1}\varepsilon^n W_n(x,y),\quad \phi(x,y,\varepsilon)=\sum_{n=0}\varepsilon^n\phi_n(x,y) \tag{3.1}$$

并假定板的初始挠度(或初始缺陷)与小挠度解的形式相同，即

$$W^*=\varepsilon A_{11}^{*}\sin mx\sin ny=\varepsilon\mu A_{11}^{(1)}\sin mx\sin ny \tag{3.2}$$

其中 $\mu=A_{11}^{*}/A_{11}^{(1)}$ 为缺陷参数。

将式(3.1)、(3.2)代入方程(2.11)、(2.12)得各级摄动方程，逐级求解便可获得足够精确的解。

$O(1)$：

$$\nabla^4\phi_0=0 \tag{3.3}$$

计及边界条件(2.14a)、(2.15a)，取满足方程(3.3)的解为

$$\phi_0=-B_{00}^{(0)}\frac{y^2}{2}b_{00}^{(0)}\frac{x^2}{2} \tag{3.4}$$

那么

$O(\varepsilon)$：

$$\left.\begin{aligned}&\nabla^4W_1+\beta^2B_{00}^{(0)}\frac{\partial^2W_1}{\partial x^2}+b_{00}^{(0)}\beta^2\frac{\partial^2W_1}{\partial y^2}=(\beta^2B_{00}^{(0)}m^2\\&\qquad+b_{00}^{(0)}n^2\beta^2)\,\mu A_{11}^{(1)}\sin mx\sin ny\\&\nabla^4\phi_1=0\end{aligned}\right\} \tag{3.5}$$

计及边界条件，取满足方程(3.5)的解为

$$W_1=A_{11}^{(1)}\sin mx\sin ny,\quad \phi_1=0 \tag{3.6}$$

将式(3.6)代入方程(3.5)得

$$\beta^2B_{00}^{(0)}m^2+b_{00}^{(0)}n^2\beta^2=\frac{(m^2+n^2\beta^2)^2}{(1+\mu)} \tag{3.7}$$

那么

$O(\varepsilon^2)$：

$$\left.\begin{aligned}&\nabla^4W_2+\beta^2B_{00}^{(0)}\frac{\partial^2W_2}{\partial x^2}+b_{00}^{(0)}\beta^2\frac{\partial^2W_2}{\partial y^2}=0\\&\nabla^4\phi_2=\frac{1}{2}A_{11}^{(1)}A_{11}^{(1)}m^2n^2\beta^2(1+2\mu)(\cos 2mx+\cos 2ny)\end{aligned}\right\} \tag{3.8}$$

计及边界条件，取满足方程(3.8)的解为

$$W_2=0,\quad \phi_2=-B_{00}^{(2)}\frac{y^2}{2}-b_{00}^{(2)}\frac{x^2}{2}+B_{20}^{(2)}\cos 2mx+B_{02}^{(2)}\cos 2ny \tag{3.9}$$

将式(3.9)代入方程(3.8)得

$$B_{20}^{(2)}=\frac{1}{32}\frac{n^2\beta^2}{m^2}(1+2\mu)A_{11}^{(1)}A_{11}^{(1)},\quad B_{02}^{(2)}=\frac{1}{32}\frac{m^2}{n^2\beta^2}(1+2\mu)A_{11}^{(1)}A_{11}^{(1)} \tag{3.10}$$

那么

$$\left.\begin{aligned}O(\varepsilon^3):\quad &\nabla^4W_3+\beta^2B_{00}^{(0)}\frac{\partial^2W_3}{\partial x^2}+b_{00}^{(0)}\beta^2\frac{\partial^2W_3}{\partial y^2}=[\beta^2B_{00}^{(2)}m^2-b_{00}^{(2)}n^2\beta^2-2m^2n^2\beta^2(B_{20}^{(2)}\\&+B_{02}^{(2)})](1+\mu)A_{11}^{(1)}\sin mx\sin ny+2m^2n^2\beta^2(1+\mu)A_{11}^{(1)}B_{02}^{(2)}\sin mx\sin 3ny\\&+2m^2n^2\beta^2(1+\mu)A_{11}^{(1)}B_{20}^{(2)}\sin 3mx\sin ny\\&\nabla^4\phi_3=0\end{aligned}\right\} \tag{3.11}$$

计及边界条件，取满足方程(3.11)的解为

$$W_3=A_{13}^{(3)}\sin mx\sin 3ny+A_{31}^{(3)}\sin 3mx\sin ny,\quad \phi_3=0 \tag{3.12}$$

将式(3.12)代入方程(3.11)得

$$\beta^2B_{00}^{(2)}m^2+b_{00}^{(2)}n^2\beta^2=(m^4+n^4\beta^4)(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}/16 \tag{3.13}$$

及

$$\left.\begin{aligned}A_{13}^{(3)}&=\frac{1}{16}\frac{m^4}{(m^2+9n^2\beta^2)^2-(\beta^2B_{00}^{(0)}m^2+b_{00}^{(0)}9n^2\beta^2)}(1+\mu)(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}\\A_{31}^{(3)}&=\frac{1}{16}\frac{n^4\beta^4}{(9m^2+n^2\beta^2)^2-(\beta^2B_{00}^{(0)}9m^2+b_{00}^{(0)}n^2\beta^2)}(1+\mu)(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}\end{aligned}\right\} \tag{3.14}$$

那么

$$\begin{aligned}O(\varepsilon^4):\quad &\nabla^4W_4+\beta^2B_{00}^{(0)}\frac{\partial^2W_4}{\partial x^2}+b_{00}^{(0)}\beta^2\frac{\partial^2W_4}{\partial y^2}=0\\&\nabla^4\phi_4=A_{11}^{(1)}A_{13}^{(3)}m^2n^2\beta^2(1+\mu)(-\cos 2ny+4\cos 4ny+4\cos 2mx\cos 2ny\\&\quad-\cos 2mx\cos 4ny)+A_{11}^{(1)}A_{31}^{(3)}m^2n^2\beta^2(1+\mu)(-\cos 2mx+4\cos 4mx\\&\quad+4\cos 2mx\cos 2ny-\cos 4mx\cos 2ny)\end{aligned} \tag{3.15}$$

计及边界条件，取满足方程(3.15)的解为

$$W_4=0$$

$$\left.\begin{aligned}\phi_4=&-B_{00}^{(4)}\frac{y^2}{2}-b_{00}^{(4)}\frac{x^2}{2}+B_{20}^{(4)}\cos 2mx+B_{02}^{(4)}\cos 2ny+B_{22}^{(4)}\cos 2mx\cos 2ny\\&+B_{40}^{(4)}\cos 4mx+B_{04}^{(4)}\cos 4ny+B_{24}^{(4)}\cos 2mx\cos 4ny+B_{42}^{(4)}\cos 4mx\cos 2ny\end{aligned}\right\} \tag{3.16}$$

将式(3.16)代入方程(3,15)可以求得一系列系数表达式，不过我们感兴趣的仅 $B_{20}^{(4)}$ 和 $B_{02}^{(4)}$，

即

$$\left.\begin{aligned}B_{20}^{(4)}=&-\frac{1}{256}\frac{n^2\beta^2}{m^2}\frac{n^4\beta^4}{(9m^2+n^2\beta^2)^2-(\beta^2B_{00}^{(0)}9m^2+b_{00}^{(0)}n^2\beta^2)}(1+\mu)^2(1\\&+2\mu)A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}\\B_{02}^{(4)}=&-\frac{1}{256}\frac{m^2}{n^2\beta^2}\frac{m^4}{(m^2+9n^2\beta^2)^2-(\beta^2B_{00}^{(0)}m^2+b_{00}^{(0)}9n^2\beta^2)}(1+\mu)^2(1\\&+2\mu)A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}\end{aligned}\right\}\tag{3.17}$$

进一步，由五级摄动方程我可以导得

$$\begin{aligned}\beta^2B_{00}^{(4)}m^2+b_{00}^{(4)}n^2\beta^2=&-\frac{1}{256}(1+2\mu)[2(1+2\mu)^2+(1+\\&+2\mu)]\left[\frac{m^8}{(m^2+9n^2\beta^2)^2-(\beta^2B_{00}^{(0)}m^2+b_{00}^{(0)}9n^2\beta^2)}\right.\\&\left.+\frac{n^8\beta^8}{(9m^2+n^2\beta^2)^2-(\beta^2B_{00}^{(0)}9m^2+b_{00}^{(0)}n^2\beta^2)}\right]A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}\end{aligned}\tag{3.18}$$

至此，我们已经可以写出大挠度渐近解

$$W=\varepsilon[A_{11}^{(1)}\sin mx\sin ny]+\varepsilon^3[A_{13}^{(3)}\sin mx\sin 3ny+A_{31}^{(3)}\sin 3mx\sin ny]+O(\varepsilon^5)\tag{3.19}$$

$$\begin{aligned}\phi=&-B_{00}^{(0)}\frac{y^2}{2}-b_{00}^{(0)}\frac{x^2}{2}+\varepsilon^2\left[-B_{00}^{(2)}\frac{y^2}{2}-b_{00}^{(2)}\frac{x^2}{2}+B_{20}^{(2)}\cos 2mx+B_{02}^{(2)}\cos 2ny\right]\\&+\varepsilon^4\left[-B_{00}^{(4)}\frac{y^2}{2}-b_{00}^{(4)}\frac{x^2}{2}+B_{20}^{(4)}\cos 2mx+B_{02}^{(4)}\cos 2ny+B_{22}^{(4)}\cos 2mx\cos 2ny\right.\\&\left.+B_{40}^{(4)}\cos 4mx+B_{04}^{(4)}\cos 4ny+B_{24}^{(4)}\cos 2mx\cos 4ny+B_{42}^{(4)}\cos 4mx\cos 2ny\right]+O(\varepsilon^6)\end{aligned}\tag{3.20}$$

其中除系数 $B_{00}^{(i)}$，$b_{00}^{(i)}$ $(i=0,2,4,\cdots)$倘未确定外，其它系数皆可表为 $A_{11}^{(1)}$的形式。

四、后屈曲平衡路径

现在考虑两种不同的面内边界条件。

1. 纵向边缘可移简支

由边界条件(2.15b)，我们有

$$b_{00}^{(i)}=0\qquad(i=0,2,4,\cdots)\tag{4.1}$$

那么，由式(3.7)、(3.13)、(3.18)有

$$\beta^2B_{00}^{(0)}=\frac{(m^2+n^2\beta^2)^2}{(1+\mu)m^2}\tag{4.2}$$

$$\beta^2B_{00}^{(2)}=\frac{1}{16}\frac{m^4+n^4\beta^4}{m^2}(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}\tag{4.3}$$

$$\begin{aligned}\beta^2B_{00}^{(4)}=&-\frac{1}{256}(1+\mu)(1+2\mu)[2(1+\mu)^2+(1+2\mu)]n^2\beta^2\\&\cdot\left[\frac{m^2}{n^2\beta^2}\frac{m^4}{(m^2+9n^2\beta^2)^2(1+\mu)-(m^2+n^2\beta^2)^2}\right.\end{aligned}$$

$$+\frac{n^2\beta^2}{m^2}\frac{n^4\beta^4}{(9m^2+n^2\beta^2)^2(1+\mu)-9(m^2+n^2\beta^2)^2}\Big]A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)} \tag{4.4}$$

将式(3.20)代入边界条件(2.14b)得

$$4\lambda_x\beta^2=\beta B_{00}^{(0)}+\varepsilon^2\beta^2B_{00}^{(2)}+\varepsilon^4\beta^2B_{00}^{(4)}+\cdots \tag{4.5}$$

将式(4.2)、(4.3)、(4.4)代入(4.5)得

$$\lambda_x=\frac{1}{4\beta^2}\Big\{\frac{(m^2n^2\beta^2)^2}{(1+\mu)m^2}+\frac{1}{16}\frac{m^4+n^4\beta^4}{m^2}(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}\varepsilon^2$$

$$-\frac{1}{256}(1+\mu)(1+2\mu)[2(1+\mu)^2+(1+2\mu)]n^2\beta^2$$

$$\cdot\Big[\frac{m^2}{n^2\beta^2}\frac{m^4}{(m^2+9n^2\beta^2)^2(1+\mu)-(m^2+n^2\beta^2)^2}$$

$$+\frac{n^2\beta^2}{m^2}\frac{n^4\beta^4}{(9m^2+n^2\beta^2)^2(1+\mu)-9(m^2+n^2\beta^2)^2}\Big]A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}\varepsilon^4+\cdots\Big\} \tag{4.6}$$

将式(3.19)、(3.20)及(3.2)代入(2.16)得

$$\delta_x=\lambda_x+\frac{1}{32}\frac{m^2}{\beta^2}(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}\varepsilon^2+\cdots \tag{4.7}$$

其中摄动参数 $A_{11}^{(1)}\varepsilon$具有明显的物理意义。因为由式(3.19)

当 $x=\pi/2m$，$y=\pi/2n$ 时，最大无量纲挠度

$$w_m=A_{11}^{(1)}\varepsilon-(A_{13}^{(3)}+A_{31}^{(3)})\varepsilon^3+O(\varepsilon^5) \tag{4.8}$$

反之

$$A_{11}^{(1)}\varepsilon=w_m+\frac{1}{16}(1+\mu)^2(1+2\mu)\Big[\frac{m^4}{(m^2+9n^2\beta^2)^2(1+\mu)-(m^2+n^2\beta^2)^2}$$

$$+\frac{n^4\beta^4}{(9m^2+n^2\beta^2)^2(1+\mu)-9(m^2+n^2\beta^2)^2}\Big]w_m^3+O(w_m^5) \tag{4.9}$$

将式(4.9)代入(4.6)、(4.7)我们得到以最大无量纲挠度为摄动参数的表达式：

$$\lambda_x=\frac{1}{4\beta^2}\Big\{\frac{(m^2+n^2\beta^2)^2}{(1+\mu)m^2}+\frac{1}{16}\frac{m^4+n^4\beta^4}{m^2}(1+2\mu)w_m^2+\frac{1}{256}n^2\beta^2\Big\langle(1+\mu)(1+2\mu)[2(1$$

$$+\mu)(1+2\mu)-2(1+\mu)^2-(1+2\mu)]\Big[\frac{m^2}{n^2\beta^2}\frac{m^4}{(m^2+9n^2\beta^2)^2(1+\mu)-(m^2+n^2\beta^2)^2}$$

$$+\frac{n^2\beta^2}{m^2}\frac{n^4\beta^4}{(9m^2+n^2\beta^2)^2(1+\mu)-9(m^2+n^2\beta^2)^2}\Big]$$

$$+2(1+\mu)^2(1+2\mu)^2\Big[\frac{m^2n^2\beta^2}{(m^2+9n^2\beta^2)^2(1+\mu)-(m^2+n^2\beta^2)^2}$$

$$+\frac{m^2n^2\beta^2}{(9m^2+n^2\beta^2)^2(1+\mu)-9(m^2+n^2\beta^2)^2}\Big]\Big\rangle w_m^4+\cdots\Big\} \tag{4.10}$$

$$\delta_x=\lambda_x+\frac{1}{32}\frac{m^2}{\beta^2}(1+2\mu)w_m^2+\frac{1}{256}\frac{m^2}{\beta^2}(1+\mu)^2(1+2\mu)^2$$

$$\cdot\Big[\frac{m^4}{(m^2+9n^2\beta^2)^2(1+\mu)-(m^2+n^2\beta^2)^2}+\frac{n^4\beta^4}{(9m^2+n^2\beta^2)^2(1+\mu)-9(m^2+n^2\beta^2)^2}\Big]$$

$$\cdot w_m^4+\cdots \tag{4.11}$$

2. 纵向边缘不可移简支

由边界条件(2.15c)我们有$\delta_y=0$

将式(3.19)、(3.20)及(3.2)代入(2.17)我们导得

$$-(b_{00}^{(0)}+\varepsilon^2 b_{00}^{(2)}+\varepsilon^4 b_{00}^{(4)}+\cdots)+\nu(\beta^2 B_{00}^{(0)}+\varepsilon^2\beta^2 B_{00}^{(2)}+\varepsilon^4\beta^4 B_{00}^{(4)}+\cdots)$$
$$-\frac{1}{8}(1+2\mu)n^2\beta^2 A_{11}^{(1)}A_{11}^{(1)}\varepsilon^2+\cdots=0 \tag{4.12}$$

令$\varepsilon\to 0$，有

$$b_{00}^{(0)}=\nu\beta^2 B_{00}^{(0)} \tag{4.13}$$

代入式(3.7)得

$$\beta^2 B_{00}^{(0)}=\frac{(m^2+n^2\beta^2)^2}{(1+\mu)(m^2+\nu n^2\beta^2)} \tag{4.14}$$

将(3.7)、(3.13)、(3.18)三式相加并计及式(4.13)、(4.14)我们有

$$m^2(\beta^2 B_{00}^{(0)}+\varepsilon^2\beta^2 B_{00}^{(2)}+\varepsilon^4\beta^2 B_{00}^{(4)}+\cdots)+n^2\beta^2(b_{00}^{(0)}+\varepsilon^2 b_{00}^{(2)}+\varepsilon^4 b_{00}^{(4)}+\cdots)$$
$$=\frac{(m^2+n^2\beta^2)^2}{1+\mu}+\frac{1}{16}(m^4+n^4\beta^4)(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}\varepsilon^2-\frac{1}{256}(1+\mu)(1+2\mu)[2(1$$
$$+\mu)^2+(1+2\mu)]\cdot\Big[\frac{m^8(m^2+\nu n^2\beta^2)}{(m^2+9n^2\beta^2)^2(m^2+\nu n^2\beta^2)(1+\mu)-(m^2+n^2\beta^2)^2(m^2+9\nu n^2\beta^2)}$$
$$+\frac{n^8\beta^8(m^2+\nu n^2\beta^2)}{(9m^2+n^2\beta^2)^2(m^2+\nu n^2\beta^2)(1+\mu)-(m^2+n^2\beta^2)^2(9m^2+\nu n^2\beta^2)}\Big]A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}\varepsilon^4$$
$$+\cdots \tag{4.15}$$

对于纵边不可移简支，式(4.5)仍然成立。

由式(4.5)、(4.12)、(4.15)，我们导得

$$\lambda_x=\frac{1}{4\beta^2}\Big\{\frac{(m^2+n^2\beta^2)^2}{(1+\mu)(m^2+\nu n^2\beta^2)}+\frac{1}{16}\frac{m^4+3n^4\beta^4}{m^2+\nu n^2\beta^2}(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}\varepsilon^2$$
$$-\frac{1}{256}(1+\mu)(1+2\mu)[2(1+\mu)^2$$
$$+(1+2\mu)]\Big[\frac{m^8}{(m^2+9n^2\beta^2)^2(m^2+\nu n^2\beta^2)(1+\mu)-(m^2+n^2\beta^2)^2(m^2+9\nu n^2\beta^2)}$$
$$+\frac{n^8\beta^8}{(9m^2+n^2\beta^2)^2(m^2+\nu n^2\beta^2)(1+\mu)-(m^2+n^2\beta^2)^2(9m^2+\nu n^2\beta^2)}\Big]A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}\varepsilon^4$$
$$+\cdots\Big\} \tag{4.16}$$

将式(3.19)、(3.20)及(3.2)代入(2.16)并计及(4.12)，我们有

$$\delta_x=(1-\nu^2)\lambda_x+\frac{1}{32}\frac{m^2+\nu n^2\beta^2}{\beta^2}(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}\varepsilon^2+\cdots \tag{4.17}$$

对于纵边不可移简支，式(4.8)同样有效，因此

$$A_{11}^{(1)}\varepsilon=w_m+\frac{1}{16}(1+\mu)^2(1+2\mu)$$

$$\cdot\left[\frac{m^4(m^2+\nu n^2\beta^2)}{(m^2+9n^2\beta^2)^2(m^2+\nu n^2\beta^2)(1+\mu)-(m^2+n^2\beta^2)^2(m^2+9\nu n^2\beta^2)}\right.$$

$$\left.+\frac{n^4\beta^4(m^2+\nu n^2\beta^2)}{(9m^2+n^2\beta^2)^2(m^2+\nu n^2\beta^2)(1+\mu)-(m^2+n^2\beta^2)^2(9m^2+\nu n^2\beta^2)}\right]w_m^3+O(w_m^5) \tag{4.18}$$

将式(4.18)代入(4.16)、(4.17)我们得到以最大无量纲挠度为摄动参数的表达式：

$$\lambda_x=\frac{1}{4\beta^2}\left\{\frac{(m^2+n^2\beta^2)^2}{(1+\mu)(m^2+\nu n^2\beta^2)}+\frac{1}{16}\frac{m^4+3n^4\beta^4}{m^2+\nu n^2\beta^2}(1+2\mu)w_m^2\right.$$

$$+\frac{1}{256}\left\langle 2(1+\mu)^2(1+2\mu)^2\right.$$

$$\cdot\left[\frac{m^4(m^4+3n^4\beta^4)}{(m^2+9n^2\beta^2)^2(m^2+\nu n^2\beta^2)(1+\mu)-(m^2+n^2\beta^2)^2(m^2+9\nu n^2\beta^2)}\right.$$

$$\left.+\frac{n^4\beta^4(m^4+3n^4\beta^4)}{(9m^2+n^2\beta^2)^2(m^2+\nu n^2\beta^2)(1+\mu)-(m^2+n^2\beta^2)^2(9m^2+\nu n^2\beta^2)}\right]$$

$$-(1+\mu)(1+2\mu)[2(1+\mu)^2$$

$$+(1+2\mu)]\left[\frac{m^8}{(m^2+9n^2\beta^2)^2(m^2+\nu n^2\beta^2)(1+\mu)-(m^2+n^2\beta^2)^2(m^2+9\nu n^2\beta^2)}\right.$$

$$\left.\left.\left.+\frac{n^8\beta^8}{(9m^2+n^2\beta^2)^2(m^2+\nu n^2\beta^2)(1+\mu)-(m^2+n^2\beta^2)^2(9m^2+\nu n^2\beta^2)}\right]\right\rangle w_m^4+\cdots\right\} \tag{4.19}$$

$$\delta_x=(1-\nu^2)\lambda_x+\frac{1}{32}\frac{m^2+\nu n^2\beta^2}{\beta^2}(1+2\mu)w_m^2+\frac{1}{256}\frac{(m^2+\nu n^2\beta^2)^2}{\beta^2}(1+\mu)^2(1+2\mu)^2$$

$$\cdot\left[\frac{m^4}{(m^2+9n^2\beta^2)^2(m^2+\nu n^2\beta^2)(1+\mu)-(m^2+n^2\beta^2)^2(m^2+9\nu n^2\beta^2)}\right.$$

$$\left.+\frac{n^4\beta^4}{(9m^2+n^2\beta^2)^2(m^2+\nu n^2\beta^2)(1+\mu)-(m^2+n^2\beta^2)^2(9m^2+\nu n^2\beta^2)}\right]w_m^4+\cdots \tag{4.20}$$

在式(4.10)、(4.19)中，当 $w_m=0$，即为线性临界值．可见纵边不可移简支和可移简支的线性临界力，后屈曲平衡路径都是不同的．

五、结 果 和 讨 论

根据渐近分析导出的公式，我们分别计算了两种面内边界条件，对应不同长宽比（$\beta=1,2,4$）矩形板的后屈曲平衡路径．其中Poisson比取 $\nu=0.3$，初挠度取 $W^*/t=0.0,0.1$．计算结果如图2、图3所示．可以看出：

1．对应不同纵向半波数m的$\lambda\sim\delta$曲线是相交的，且随着矩形板长宽比β的增加，其交点愈来愈接近初始屈曲点．

2．对应纵向半波数 m 值较高的曲线能量水平较低，因此实际结构将由低波数跳到高波数，在较低的能量水平上继续保持稳定的平衡．

3．由于初挠度的影响将使波形转变提前发生．这一点和实验中观察到的现象是一致的[10]．

以纵向半波数跳跃为特征的多次屈曲反映了矩形板后屈曲的复杂过程．图 4 为长宽比

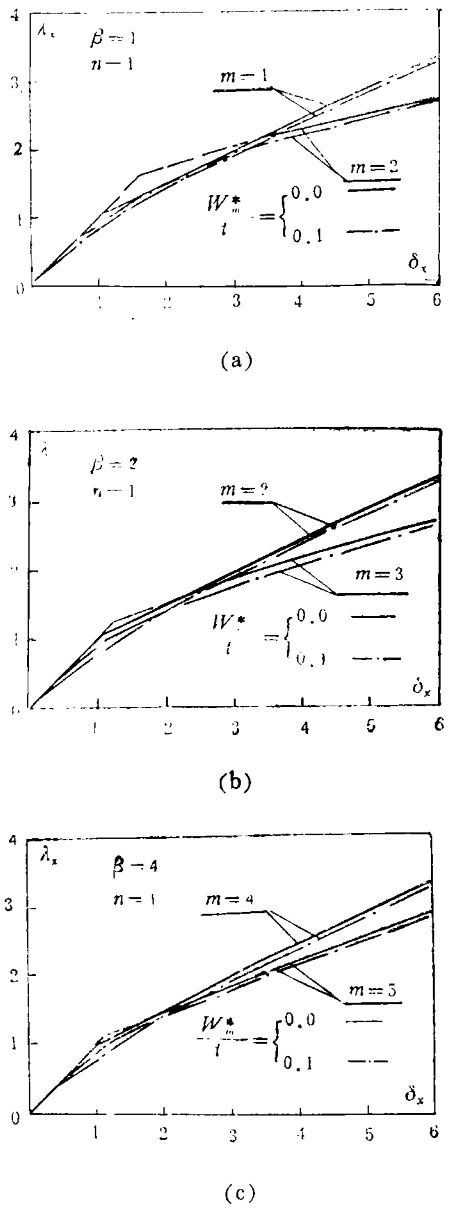


图2 纵边可移简支板载荷-纵向缩短曲线

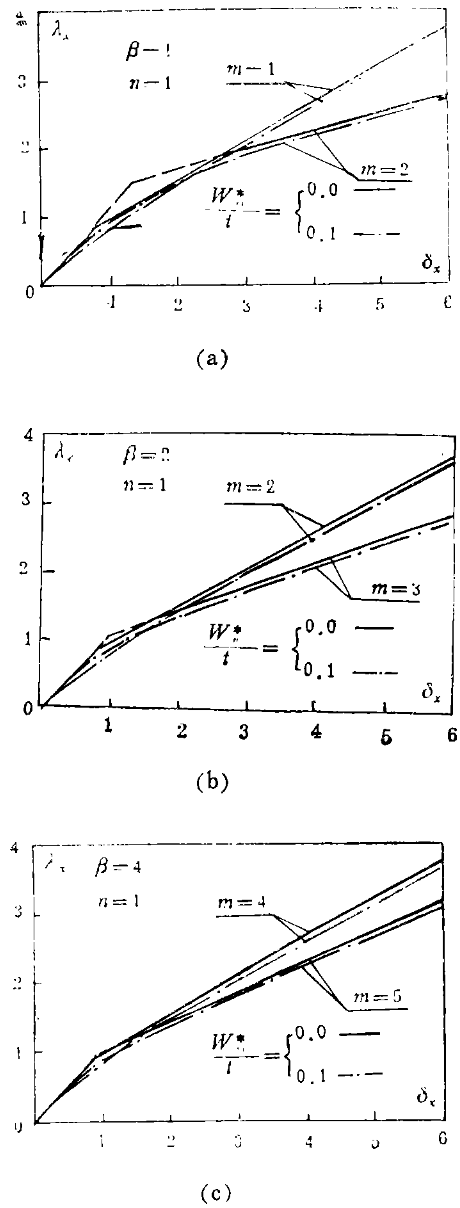


图3 纵边不可移简支板载荷-纵向缩短曲线

$\beta=4$的矩形板后屈曲阶段纵向半波数m从$5\to6\to7\to8$的$\lambda\sim\delta$曲线。图中双点划线为文[1]实验曲线。实验表明，当纵向半波数$m>6$时，矩形板开始出现塑性变形，因此，当m值较大时实验曲线要比理论曲线低得多，而在$m<7$的范围内，实验曲线与理论曲线相当接近。

图5为本文计算结果与文[1]摄动解及实验结果的比较。Stein 曾指出实验数据有相当大的离散。事实上，大部分实验数据落在本文计算结果$m=4$和$m=5$的两条线之间，且当纵向缩短较大时，实验数据接近$m=5$的理论曲线。

图 6 为本文渐近解与实验结果的比较。试件的几何尺寸，特性参数以及纵边边界支承条件详见参考文献[10]。图示表明，在理论和实验间得到了非常合理的符合，

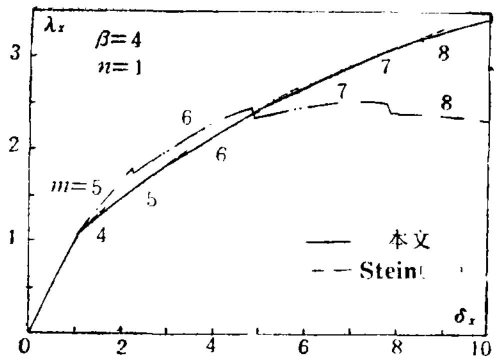


图4 载荷-纵向缩短理论与实验比较

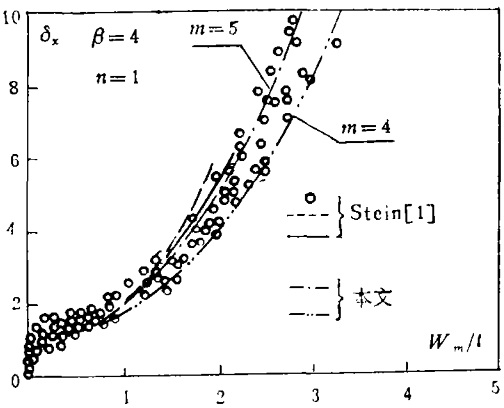
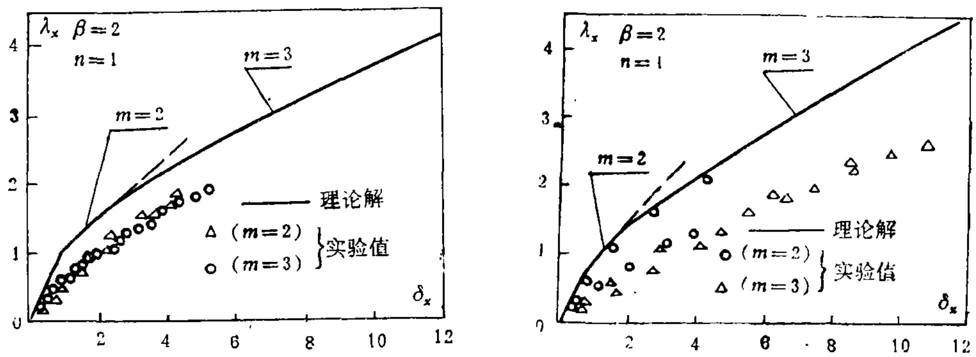


图5 纵向缩短-挠度理论与实验比较



(a) 纵边可移简支　　(b) 纵边不可移简支

图6 理论与实验结果比较

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Perturbation Analyses for the Postbuckling of Simply Supported Rectangular Plates under Uniaxial Compression

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Abstract

In this paper, applying perturbation method to von Kármán nonlinear large deflection equations of plates by taking deflection as perturbation parameter, the postbuckling behavior of simply supported rectangular plates under uniaxial compression is investigated. Two types of in-plane boundary conditions are now considered and the effects of initial imperfections are also studied. It is found that the theoretical results are in good agreement with experiments.

The method suggested in this paper which has not been found in previous papers is rather simple and easy for the postbuckling analysis of rectangular plates.