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弹性基上锥壳一般弯曲问题的精确解

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摘　要

本文应用 Donnell 的简化假定，从弹性基上锥壳位移型微分方程组出发，通过引入一个位移函数$U(s,\theta)$（在极限情况下就退化成V.S. Vlasov对于圆柱壳所引的位移函数[5]），将基本微分方程组化成为一个八阶可解偏微分方程．这个方程的一般解用级数形式给出．对于在实际中有广泛应用价值的 Winkler 弹性基上锥壳的轴对称弯曲问题，本文给出了详细的数值结果，并求出了边缘荷载作用下的影响系数，这对计算弹性基上锥壳组合结构有着重要的意义．

一、引　言

弹性基上锥壳的计算问题，在容器设计和建筑工程中经常遇到．但锥壳在任意荷载下的一般弯曲问题，就作者所知，目前的研究并不多，N. J. Hoff[1]，P. Seld[2]研究了在任意荷载下锥壳的一般弯曲问题，但前者的结果只有当半锥顶角α不大于30°的小锥度条件下才是正确的，后者虽然通过保留由Hoff[1]省略了的项而得到了适应于任意锥度的基本方程，但没有给出[2]中方程(17)的一般解，并认为求解是非常困难的．A. Д. Коваленко[3]应用复数变换给出了锥壳的复数方程，用广义超几何函数表示了方程的解．我们认为锥壳的复数方程虽然有许多优点，但在位移计算，弹性连接条件的处理方面比较困难，相比之下采用位移解法似更有优点．特别是锥壳的特征问题，弹性基上锥壳的分析问题，则更宜于采用壳体的位移解法．本文作者之一曾对锥壳的位移解法系统地进行了研究[4]，同样以超几何函数给出一般解．

在[4]的基础上，本文系统地研究了弹性基上锥壳的一般弯曲问题．就作者所知，这个问题至今没有精确解．文中通过引入一个位移函数$U(s,\theta)$，将弹性基上锥壳用位移表示的基本微分方程组化成为一个八阶变系数线性偏微分方程，方程的一般解用级数形式给出．作为本文的特殊情况，弹性基上圆柱壳和圆薄板的一般弯曲问题的基本方程，可直接由本文的一般结果导出．

对于在工程实际中有着广泛应用价值的弹性基（Winkler模式）上锥在多种边界条件下的轴对称弯曲问题，本文给出了详细的数值结果；并给出了在边缘力作用下的影响系数，这对于计算弹性基上的锥壳组合结构有着重要的意义．

二、弹性基上锥壳的基本方程及其通解

当略去u，v方向上的地基抗力，弹性基上锥壳一般弯曲问题用位移表示的基本微分方程组可以写成以下形式：

$$\left.\begin{aligned} L^{11}(u)+L^{12}(v)+L^{13}(w)+q_1/K=0 \\ L^{21}(u)+L^{22}(v)+L^{23}(w)+q_2/K=0 \\ L^{31}(u)+L^{32}(v)+L^{33}(w)=q_n/D \end{aligned}\right\} \tag{2.1}$$

其中算子

$$L^{11}=\frac{\partial^2}{\partial s^2}+\frac{1}{s}\frac{\partial}{\partial s}-\frac{1}{s^2}+\frac{1-\mu}{2}\frac{\sec^2\varphi}{s^2}\frac{\partial^2}{\partial\theta^2},$$

$$L^{12}=\sec\varphi\left(\frac{1+\mu}{2}\frac{1}{s}\frac{\partial^2}{\partial s\partial\theta}-\frac{3-\mu}{2}\frac{1}{s^2}\frac{\partial}{\partial\theta}\right),$$

$$L^{13}=\mathrm{tg}\,\varphi\left(\mu\frac{1}{s}\frac{\partial}{\partial s}-\frac{1}{s^2}\right),\ L^{21}=\sec\varphi\left(\frac{1+\mu}{2}\frac{1}{s}\frac{\partial^2}{\partial s\partial\theta}+\frac{3-\mu}{2}\frac{1}{s^2}\frac{\partial}{\partial\theta}\right),$$

$$L^{22}=\frac{1-\mu}{2}\left(\frac{\partial^2}{\partial s^2}+\frac{1}{s}\frac{\partial}{\partial s}-\frac{1}{s^2}\right)+\frac{\sec^2\varphi}{s^2}\frac{\partial^2}{\partial\theta^2},$$

$$L^{23}=\frac{1}{s^2}\sec^2\varphi\sin\varphi\frac{\partial}{\partial\theta},\ L^{31}=\frac{K}{D}\mathrm{tg}\varphi\left(\mu\frac{1}{s}\frac{\partial}{\partial s}+\frac{1}{s^2}\right),$$

$$L^{32}=\frac{K}{D}\mathrm{tg}\varphi\sec\varphi\frac{\partial}{\partial\theta}$$

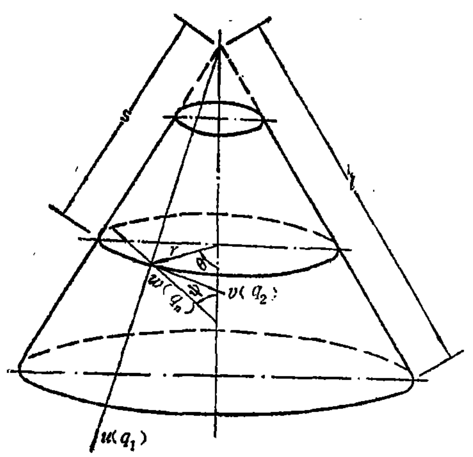
对于Winkler模式　　　　(2.2)

$$L^{33}=\nabla_s^2\nabla_s^2+\frac{K}{D}\mathrm{tg}^2\varphi\frac{1}{s^2}+\frac{k}{D}$$

对于双参数模式

$$L^{33}=\nabla_s^2\nabla_s^2+\frac{K}{D}\mathrm{tg}^2\varphi\frac{1}{s^2}+\frac{1}{D}(k-G_p\nabla_s^2)$$

其中 $\nabla_s^2=\frac{\partial^2}{\partial s^2}+\frac{1}{s}\frac{\partial}{\partial s}+\frac{\sec^2\varphi}{s^2}\frac{\partial^2}{\partial\theta^2}$，$K=Eh/(1-\mu^2)$，$D=Eh^3/12(1-\mu^2)$，$E$为壳体材料的弹性模量，$\mu$为壳体材料的 Poisson 比，$h$为锥壳的厚度，$G_p$为地基剪切模量，$k$为地基的 Winkler 模量，其余各量如图1所示。



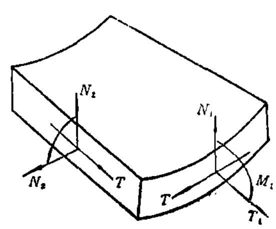


图 1

用位移分量表示的内力分量如下：

$$\left.\begin{aligned}
&T_1=\frac{K}{s}[(D_s+\mu)u+\mu\sec\varphi\partial_\theta v+\mu w\mathrm{tg}\varphi]\\
&T_2=\frac{K}{s}[(1+\mu D_s)u+\sec\varphi\partial_\theta v+w\mathrm{tg}\varphi]\\
&T=\frac{K}{s}[\sec\varphi\partial_\theta u+(D_s-1)v]\\
&M_1=-\frac{D}{s^2}[D_s(D_s-1)+\mu(\sec^2\varphi\partial_\theta^2+D_s)]w\\
&M_2=-\frac{D}{s^2}[\sec^2\varphi\partial_\theta^2+D_s+\mu D_s(D_s-1)]w\\
&H=(1-\mu)D\frac{\sec\varphi}{s^2}(1-D_s)\partial_\theta w\\
&N_1=-D\frac{\partial}{\partial s}(\nabla_s^2 w)\\
&N_2=-D\frac{\sec\varphi}{s}\frac{\partial}{\partial\theta}(\nabla_s^2 w)\\
&D_s=s\frac{\partial}{\partial s},\ \partial_\theta=\frac{\partial}{\partial\theta}
\end{aligned}\right\}\tag{2.3}$$

基本方程组(2.1)是变系数的线性微分方程组，不易求解，为此，令

$$s=\exp[t\cos\varphi]\tag{2.4}$$

其中t为新的自变量，在(2.4)的变换下，方程组(2.1)变成为：

$$\left.\begin{aligned}
&L_{11}(u)+L_{12}(v)+L_{13}(w)+r^2q_1/K=0\\
&L_{21}(u)+L_{22}(v)+L_{23}(w)+r^2q_2/K=0\\
&L_{31}(u)+L_{32}(v)+L_{33}(w)=r^4q_n/D
\end{aligned}\right\}\tag{2.5}$$

其中算子为

$$\left.\begin{aligned}
&L_{11}=\frac{\partial^2}{\partial t^2}+\frac{1-\mu}{2}\frac{\partial^2}{\partial\theta^2}-\cos^2\varphi,\ L_{12}=\frac{1+\mu}{2}\frac{\partial^2}{\partial t\partial\theta}-\frac{3-\mu}{2}\frac{\partial}{\partial\theta}\cos\varphi\\
&L_{13}=\sin\varphi\left(\mu\frac{\partial}{\partial t}-\cos\varphi\right),\ L_{21}=\frac{1+\mu}{2}\frac{\partial^2}{\partial t\partial\theta}+\frac{3-\mu}{2}\frac{\partial}{\partial\theta}\cos\varphi\\
&L_{22}=\frac{1-\mu}{2}\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial\theta^2}-\frac{1-\mu}{2}\cos^2\varphi,\ L_{23}=\sin\varphi\frac{\partial^2}{\partial\theta^2}\\
&L_{31}=\frac{K}{D}r^2\sin\varphi\left(\mu\frac{\partial}{\partial t}+\cos\varphi\right),\ L_{32}=\frac{K}{D}r^2\sin\varphi\frac{\partial}{\partial\theta}
\end{aligned}\right\}\tag{2.6}$$

对于Winkler模式

$$L_{33}=\left[\left(\frac{\partial}{\partial t}-2\cos\varphi\right)^2+\frac{\partial^2}{\partial\theta^2}\right]\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial\theta^2}\right)+\frac{K}{D}r^2\sin\varphi+r^4\frac{k}{D}$$

对于双参数模式

$$L_{33}=\left[\left(\frac{\partial}{\partial t}-2\cos\varphi\right)^2+\frac{\partial^2}{\partial\theta^2}\right]\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial\theta^2}\right)+\frac{K}{D}r^2\sin\varphi+r^4(k-G_p\nabla_s^2)/D$$

$$\nabla_t^2=\frac{1}{r^2}\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial\theta^2}\right)$$

式中，r为锥壳纬线圆的半径，注意到式(2.4)有：

$$r=s\cos\varphi=\cos\varphi\exp[t\cos\varphi] \tag{2.7}$$

(2.5)中的算子L_{ij} ($i=1,2;j=1,2,3$) 是常系数的线性偏微分算子，而因常系数偏微分算子对加法和乘法是可交换的，这样方程组(2.5)的通解可以用一个位移函数U (t,θ) 来表示，当位移分量u,v,w与位移函数$U(t,\theta)$有如下关系时，

$$\left.\begin{aligned}u&=-\sin\varphi L_u^t\ (U)-L_{u\varphi_1}^t\ (\psi_1)+L_{u\varphi_2}^t(\psi_2)\\ v&=-\sin\ \varphi\ L_v^t\ (U)+L_{v\varphi_1}^t\ (\psi_1)-L_{v\varphi_2}^t(\psi_2)\\ w&=L_w^t(U)\end{aligned}\right\} \tag{2.8}$$

则方程组(2.5)的前二式自动满足。这里，各微分算子分别为：

$$\left.\begin{aligned}L_u^t&=\mu\frac{\partial^3}{\partial t^3}-\frac{\partial^3}{\partial t\partial\theta^2}-\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial\theta^2}\right)\cos\varphi-\mu\frac{\partial}{\partial t}\cos^2\varphi+\cos^3\varphi\\ L_v^t&=(2+\mu)\frac{\partial^3}{\partial t^2\partial\theta}+\frac{\partial^3}{\partial\theta^3}+(1-\mu)\frac{\partial^2}{\partial t\partial\theta}\cos\varphi+\frac{\partial}{\partial\theta}\cos^2\varphi\\ L_w^t&=\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial\theta^2}\right)^2-2\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial\theta^2}\right)\cos^2\varphi+\cos^4\varphi\\ L_{u\varphi_1}^t&=L_{22},\ L_{u\varphi_2}^t=L_{12},\ L_{v\varphi_1}^t=L_{21},\ L_{v\varphi_2}^t=L_{11}\end{aligned}\right\} \tag{2.9}$$

$\psi_1(t,\theta)$，$\psi_2(t,\theta)$是分别与荷载q_1,q_2有关的两个函数。它们是方程

$$L_w^t(\psi_i)=-\frac{2}{1-\mu}\frac{q_ir^2}{K}\qquad(i=1,2) \tag{2.10}$$

的解。通常这里无需取方程(2.10)的一般解，只需是任意特解就是足够了。因为即使在这种情况下，也能使方程组(2.5)前二式自动满足。其解见[4]。

将(2.8)代入(2.5)的第三式中，即得位移函数$U(t,\theta)$应满足的方程：

1°对于Winkler模式

$$L_1^tL_2^t(U)+(1-\mu^2)\frac{K}{D}\sin^2\varphi r^2L_3^t(U)+r^4\frac{k}{D}L_2^t(U)=Q \tag{2.11}$$

2°对于双参数模式

$$L_1^tL_2^t\ (U)+(1-\mu^2)\frac{K}{D}\sin^2\varphi r^2L_3^t(U)+r^4\frac{1}{D}(k-G_p\nabla_t^2)L_2^t(U)=Q \tag{2.11$'$}$$

式中各微分算子

$$\left.\begin{aligned}L_1^t&=\left[\left(\frac{\partial}{\partial t}-2\cos\varphi\right)^2+\frac{\partial^2}{\partial\theta^2}\right]\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial\theta^2}\right)\\ L_2^t&=L_w^t,\ L_3^t=\frac{\partial^4}{\partial t^4}-\frac{\partial^2}{\partial t^2}\cos^2\varphi\end{aligned}\right\} \tag{2.12}$$

自由项

$$Q(t,\theta)=q_nr^4/D+\frac{1-\mu}{2}\frac{K}{D}r^2\sin\varphi\left\{\left[\mu\frac{\partial^3}{\partial t^3}-\frac{\partial^3}{\partial t\partial\theta^2}\right.\right.$$

$$\left.+\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial\theta^2}\right)\cos^2\varphi-\mu\frac{\partial}{\partial t}\cos^2\varphi-\cos^3\varphi\right]\psi_1$$

$$+[(2+\mu)\frac{\partial^3}{\partial t^2\partial\theta}+\frac{\partial^3}{\partial\theta^3}-(1-\mu)\frac{\partial^2}{\partial t\partial\theta}\cos\varphi+\frac{\partial}{\partial\theta}\cos^2\varphi]\psi_2\Big\} \tag{2.13}$$

(2.8)就是本文给出的用位移函数 $U(t,\ \theta)$ 表示的关于弹性基上锥壳一般弯曲问题的通解。(2.11)或(2.11)′为本文所得的基本方程。注意到(2.1)三个方程阶次的总和为8，而 $U(t,\theta)$ 满足一个八阶偏微分方程(2.11)或(2.11)′，恰好等于(2.1)的总阶次，从而这里给的通解(2.8)是恰当的。

由以上一般关系还可以导出两种极限情况下的结果。

1°$\varphi\to\pi/2$，则有$\cos\varphi\to0$，$\sin\varphi\to1$，$r\to a$，即对应于圆柱壳的情况，这时

$$u=-\mu\frac{\partial^3U}{\partial\xi^3}+\frac{\partial^3U}{\partial\xi\partial\theta^2}-\left(\frac{1-\mu}{2}\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\theta^2}\right)\psi_1+\frac{1+\mu}{2}\frac{\partial^2\psi_2}{\partial\xi\partial\theta}$$

$$v=-(2+\mu)\frac{\partial^3U}{\partial\xi^2\partial\theta}-\frac{\partial^3U}{\partial\xi^3}+\frac{1+\mu}{2}\frac{\partial^2\psi_1}{\partial\xi\partial\theta}-\left(\frac{\partial^2}{\partial\xi^2}+\frac{1-\mu}{2}\frac{\partial^2}{\partial\theta^2}\right)\psi_2 \tag{a}$$

$$w=\nabla_c^4U,\quad \nabla_c^4\psi_i=-\frac{2}{1-\mu}\frac{q_ia^2}{K}\qquad(i=1,2) \tag{b}$$

基本方程(2.11)及(2.11)′变成为:

（1） 对于Winkler模式

$$\nabla_c^8U+\frac{1-\mu^2}{\beta^2}\frac{\partial^4U}{\partial\xi^4}+a^4\frac{k}{D}\nabla_c^4U=Q^c \tag{c}$$

（2） 对于双参数模式

$$\nabla_c^8U+\frac{1-\mu^2}{\beta^2}\frac{\partial^4U}{\partial\xi^4}+a^4\frac{1}{D}(k-G_p\nabla_c^2)\nabla_c^4U=Q^c \tag{c$'$}$$

式中自由项

$$Q^c=a^4q_n/D+\frac{1-\mu}{2}\frac{K}{D}a^2\left\{\left[\mu\frac{\partial^3}{\partial\xi^3}-\frac{\partial^3}{\partial\xi\partial\theta^2}\right]\psi_1\right.$$

$$\left.+\left[(2+\mu)\frac{\partial^3}{\partial\xi^2\partial\theta}+\frac{\partial^3}{\partial\theta^3}\right]\psi_2\right\} \tag{d}$$

其中 $\xi=s/a$，$\beta^2=(h/a)^2/12$，$\nabla_c^2=\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\theta^2}$，$a$为圆柱壳的半径。如果不考虑地基影响，即 $k=0$，则(a)，(b)，(c)就是V.S. Vlasov 对于圆柱壳引入的位移函数和基本方程[5]。不过这里的特解求法与其不同。

2°.$\varphi\to0$，则有$\cos\varphi\to1$，$\sin\varphi\to0$，即对于圆薄板的情况，这时有

$$u=-\left[\frac{1-\mu}{2}(D_r^2-1)+\frac{\partial^2}{\partial\theta^2}\right]\psi_1+\left[\frac{1+\mu}{2}D_r-\frac{3-\mu}{2}\right]\frac{\partial\psi_2}{\partial\theta}$$

$$v=\left[\frac{1+\mu}{2}D_r+\frac{3-\mu}{2}\right]\frac{\partial\psi_1}{\partial\theta}-\left[D_r^2-1+\frac{1-\mu}{2}\frac{\partial^2}{\partial\theta^2}\right]\psi_2 \tag{e}$$

$$\left[\left(D_r^2+\frac{\partial^2}{\partial\theta^2}\right)^2-2\left(D_r^2-\frac{\partial^2}{\partial\theta^2}\right)+1\right]\psi_i=-\frac{2}{1-\mu}\frac{q_ir^2}{K}\qquad(i=1,2)$$

（1） 对于Winkler模式

$$\left[(D_r-2)^2+\frac{\partial^2}{\partial\theta^2}\right]\left(D_r^2+\frac{\partial^2}{\partial\theta^2}\right)w+r^4\frac{k}{D}w=r^4q_n/D \tag{f}$$

（2） 对于双参数模式

$$\left[(D_r-2)^2+\frac{\partial^2}{\partial\theta^2}\right]\left(D_r^2+\frac{\partial^2}{\partial\theta^2}\right)w-r^2\frac{G_p}{D}\left(D_r^2+\frac{\partial^2}{\partial\theta^2}\right)w$$

$$+r^4\frac{k}{D}w=r^4q_n/D \qquad \text{(f)}'$$

其中 $D_r=r\partial/\partial r$。可以验证(e)为圆薄板平面应力状态的通解和基本方程。式(f)或(f)′表示弹性基上薄板横向弯曲的基本方程。

以上导出结果的正确性说明了本文所得结果的正确性及一般性。

返回到初始自变量s。(2.8)，(2.10)及(2.11)，(2.11)′变为：

$$\begin{aligned} u&=-\sin\varphi\cos^3\varphi L_u^s(U)-\cos^2\varphi L_{u\varphi_1}^s(\psi_1)+\cos^2\varphi L_{u\varphi_2}^s(\psi_2)\\ v&=-\sin\varphi\cos^3\varphi L_v^s(U)+\cos^2\varphi L_{v\varphi_1}^s(\psi_1)-\cos^2\varphi L_{v\varphi_2}^s(\psi_2)\\ w&=\cos^4\varphi L_w^s(U)\end{aligned} \qquad (2.14)$$

$$L_w^s(\psi_i)=-\frac{2}{1-\mu}\frac{q_is^2\sec^2\varphi}{K} \qquad (i=1,2) \qquad (2.15)$$

1° 对于Winkler模式

$$L_1^sL_2^s(U)+(1-\mu^2)\frac{K}{D}s^2\mathrm{tg}^2\varphi L_3^s(U)+s^4\frac{k}{D}L_2^s(U)=Q \qquad (2.16)$$

2° 对于双参数模式

$$L_1^sL_2^s(U)+(1-\mu^2)\frac{k}{D}s^2\mathrm{tg}^2\varphi L_3^s(U)+\frac{s^4}{D}(k-G_p\nabla_s^2)L_2^s(U)=Q \qquad (2.16)'$$

这里已用到关系式$r=s\cos\varphi$。各微分算子为

$$\left.\begin{aligned} L_u^s&=(D_s-1)[(D_s-1)(\mu D_s-1)-\sec^2\varphi\partial_\theta^2]\\ L_v^s&=\sec\varphi[(2+\mu)D_s^2+\sec^2\varphi\partial_\theta^2+(1-\mu)D_s+1]\partial_\theta\\ L_w^s&=(D_s^2+\sec^2\varphi\partial_\theta^2)^2-2(D_s^2-\sec^2\varphi\partial_\theta^2)+1\\ L_{u\varphi_1}^s&=\frac{1-\mu}{2}(D_s^2-1)+\sec^2\varphi\partial_\theta^2\\ L_{u\varphi_2}^s&=\sec\varphi\left(\frac{1+\mu}{2}D_s-\frac{3-\mu}{2}\right)\partial_\theta\\ L_{v\varphi_1}^s&=\sec\varphi\left(\frac{1+\mu}{2}D_s+\frac{3-\mu}{2}\right)\partial_\theta\\ L_{v\varphi_2}^s&=D_s^2-1+\frac{1-\mu}{2}\sec^2\varphi\partial_\theta^2\\ L_1^s&=[(D_s-2)^2+\sec^2\varphi\partial_\theta^2](D_s^2+\sec^2\varphi\partial_\theta^2)\\ L_2^s&=L_w^s,\quad L_3^s=(D_s^2-1)D_s^2\end{aligned}\right\} \qquad (2.17)$$

自由项

$$\begin{aligned} Q(s,\theta)=&q_ns^4\sec^4\varphi/D+\frac{1-\mu}{2}\frac{K}{D}s^2\sec^3\varphi\sin\varphi\{[\mu(D_s^3-D_s)\\ &+D_s^2-1-(D_s+\sec^2\varphi)\partial_\theta^2]\psi_1+\sec\varphi[\partial_\theta^2\\ &+(2-\mu)D_s^2-(1-\mu)D_s+1]\partial_\theta\psi_2\}\end{aligned} \qquad (2.18)$$

将(2.14)代入(2.3)中，可以把内力分量用位移函数$U(s,\theta)$表示出来。这样弹性地基上锥壳的一般弯曲问题就化成在一定的边界条件下求解方程(2.16)或(2.16)′的数学问题。

三、基本方程（2·16）的精确解

以上对两种不同的地基模式，相应地导出它们的控制微分方程。以下只讨论Winkler地基上锥壳的一般弯曲问题的解。而对于双参数模式基上锥壳的一般弯曲问题将另文讨论。

将荷载展开成Fourier级数：

$$q_i(s,\ \theta)=q_{i0}+\sum_{n=1}^{\infty}[q_{inc}(s)\cos n\theta+q_{ins}(s)\sin n\theta] \qquad (i=1,2,n) \tag{3.1}$$

并取(2.16)的一般解为：

$$U(s,\theta)=U_0(s)+\sum_{n=1}^{\infty}[U_{nc}(s)\cos n\theta+U_{ns}(s)\sin n\theta] \tag{3.2}$$

将(3.1)，(3.2)代入(2.16)中，比较三角级数诸项的系数，同时引入算子$\delta_s(:)=s\cdot d(:)/ds$。即得两类问题的基本方程为：

1° $L_s(U_{nc})=Q_{nc} \quad (n=0,1,2,\cdots)$ (3.3)

2° $L_s(U_{ns})=Q_{ns} \quad (n=1,2,\cdots)$ (3.4)

其中

$$L_s=\prod_{i=1}^{8}(\delta_s-\alpha_i)+(1-\mu^2)\frac{K}{D}\mathrm{tg}^2\varphi s^2\prod_{i=1}^{4}(\delta_s-\beta_i)+\frac{k}{D}s^4\prod_{i=1}^{4}(\delta_s-\gamma_i) \tag{3.5}$$

各参数为($m=n\sec\varphi$)：

$$\left.\begin{aligned}&\alpha_1=m+2,\ \alpha_2=-m+2,\ \alpha_3=m+1,\ \alpha_4=-m+1\\&\alpha_5=m,\ \alpha_6=-m,\ \alpha_7=m-1,\ \alpha_8=-m-1\\&\beta_1=0,\ \beta_2=0,\ \beta_3=1,\ \beta_4=-1;\ \gamma_1=m-1,\ \gamma_2=-m-1\\&\gamma_3=m+1,\ \gamma_4=-(m+1)\end{aligned}\right\} \tag{3.6}$$

自由项

$$\begin{aligned}Q_{nc}=&s^4\sec^4\varphi q_{nnc}/D+\frac{1-\mu}{2}\frac{K}{D}s^2\sec^3\varphi\sin\varphi\{[\mu\delta_s^3\\&+\delta_s^2+(m^2-\mu)\delta_s+(m^2-1)]\psi_{1nc}+m[(2+\mu)\delta_s^2\\&-(1-\mu)\delta_s+1-m^2]\psi_{2ns}\}\end{aligned} \tag{3.7}$$

$$\begin{aligned}Q_{ns}=&s^4\sec^4\varphi q_{nns}/D+\frac{1-\mu}{2}\frac{K}{D}s^2\sec^3\varphi\sin\varphi\{[\mu\delta_s^3\\&+\delta_s^2+(m^2-\mu)\delta_s+(m^2-1)]\psi_{1ns}+m[(2+\mu)\delta_s^2\\&-(1-\mu)\delta_s+1-m^2]\psi_{2nc}\}\end{aligned} \tag{3.8}$$

由于(3.3)与(3.4)的结构相同，只要求解其中一个即可，如(3.3)。由线性微分方程的理论知，方程(3.3)的全解可写成：

$$U_{nc}=U_{nc}^H+U_{nc}^p \tag{3.9}$$

其中U_{nc}^H为齐次解，U_{nc}^p为特解。它们分别满足方程：

$$L_s(U^H_{nc})=0 \tag{3.10}$$

及

$$L_s(U^p_{nc})=Q_{nc} \tag{3.10'}$$

作变换，令$\zeta=(s/l)^2$，则方程(3.10)及(3.10)′变成为：

$$L_\zeta(U^H_{nc})=0 \tag{3.11}$$

及

$$L_\zeta(U^p_{nc})=Q_{nc} \tag{3.11'}$$

其中算子：

$$\delta_\zeta=\zeta\frac{d}{d\zeta}$$

$$L_\zeta=\prod_{i=1}^{8}(\delta_\zeta-\alpha_i/2)+(1-\mu^2)\frac{K}{D}\frac{l^2}{16}\mathrm{tg}^2\varphi\zeta\prod_{i=1}^{4}(\delta_\zeta-\beta_i/2)$$

$$+\frac{l^4k}{16D}\zeta^2\prod_{i=1}^{4}(\delta_\zeta-\gamma_i/2) \tag{3.12}$$

自由项

$$Q_{nc}(\zeta)=\frac{1}{16}\Big\{l^4\zeta^2\sec^4\varphi q_{nnc}/D+\frac{1-\mu}{2}\frac{K}{D}l^2\zeta\sec^3\varphi\sin\varphi\{[8\delta_\zeta^3+4\delta_\zeta^2+2(m^2-\mu)\delta_\zeta$$

$$+(m^2-1)]\psi_{1nc}+m[4(2+\mu)\delta_\zeta^2-2(1-\mu)\delta_\zeta+1-m^2]\psi_{2ns}\Big\} \tag{3.13}$$

设方程(3.11)的解为：

$$U^H_{nc}=\sum_{\nu=0}^{\infty}a_\nu\zeta^{\sigma+\nu}\qquad(a_0\neq0) \tag{3.14}$$

将(3.14)代入到(3.11)中，即得如递推关系：

$$a_\nu f_0(\sigma+\nu)+Aa_{\nu-1}f_1(\sigma+\nu-1)+Ba_{\nu-2}f_2(\sigma+\nu-2)=0 \tag{3.15}$$

其中

$$\left.\begin{aligned}
&f_0(\sigma+\nu)=\prod_{i=1}^{8}[(\sigma+\nu)-\alpha_i/2]\\
&f_1(\sigma+\nu-1)=\prod_{i=1}^{4}[(\sigma+\nu-1)-\beta_i/2]\\
&f_2(\sigma+\nu-2)=\prod_{i=1}^{4}[(\sigma+\nu-2)-\gamma_i/2]\\
&A=(1-\mu^2)\frac{K}{D}\frac{l^2}{16},\ B=l^4k/16D
\end{aligned}\right\} \tag{3.16}$$

其相应的指标方程为：

$$f_0(\sigma)=0 \tag{3.17}$$

由此得参数σ的八个根为：

$$\sigma_1=(m+2)/2,\ \sigma_2=(-m+2)/2,\ \sigma_3=(m+1)/2,\ \sigma_4=(-m+1)/2,$$

$$\sigma_5=m/2,\ \sigma_6=-m/2,\ \sigma_7=(m-1)/2,\ \sigma_8=(-m-1)/2 \tag{3.18}$$

1) 当$m\neq2$，或当$m/2$为大于2的整数时，则得齐次方程(3.11)的相应四个解为：

$$\left.\begin{aligned}&U_{nc}^{H(j)}=\sum_{\nu=0}^{\infty}a_\nu^j\zeta^{\sigma_j+\nu}\qquad(j=1,2,3,4)\\&a_\nu^jf_0(\sigma_j+\nu)+Aa_{\nu-1}^jf_1(\sigma_j+\nu-1)+Ba_{\nu-2}^jf_2(\sigma_j+\nu-2)=0\end{aligned}\right\}\tag{3.19}$$

由于σ_5与σ_1，σ_6与σ_2，σ_7与σ_3，σ_8与σ_4相差整数1．根据微分方程的解析理论[6]，相应于指标σ_5，σ_6，σ_7，σ_8的解析解中包含有对数项．其形式为：

$$U_{nc}^{H(j+4)}=U_{nc}^{H(j)}\ln\zeta+\sum_{\nu=0}^{\infty}(a_\nu^{j+4})'\zeta^{\sigma_{j+4}+\nu}$$

$$(a_\nu^{j+4})'f_0(\sigma_{j+4}+\nu)+A(a_{\nu-1}^{j+4})f_1(\sigma_{j+4}+\nu-1)+B(a_{\nu-2}^{j+4})'f_2(\sigma_{j+4}+\nu-2)$$
$$+a_{\nu-1}^j\frac{df_0(\sigma_{j+4}+\nu)}{d\sigma}+Aa_{\nu-2}^j\frac{df_1(\sigma_{j+4}+\nu-1)}{d\sigma}$$
$$+Ba_{\nu-3}^j\frac{df_2(\sigma_{j+4}+\nu-2)}{d\sigma}=0\qquad(j=1,2,3,4)\tag{3.20}$$

2） $m=2$时

如记 $\tau_1=2,\ \tau_2=0,\ \tau_3=-1/2,\ \tau_4=3/2,\ \tau_5=1/2,\ \tau_6=1,\ \tau_7=-1,\ \tau_8=-3/2$ (3.21)

有
$$U_{nc}^{H(j)}=\sum_{\nu=0}^{\infty}a_\nu^j\zeta^{\tau_j+\nu}\tag{3.22}$$

$$a_\nu^jf_0(\tau_j+\nu)+Aa_{\nu-1}^jf_1(\tau_j+\nu-1)+Ba_{\nu-2}^jf_2(\tau_j+\nu-2)=0$$
$$(j=1,2,3,4,5)$$

及
$$U_{nc}^{H(j+5)}=U_{nc}^{H(j)}\ln\zeta+\sum_{\nu=0}^{\infty}(a_\nu^{j+5})'\zeta^{\tau_{j+5}+\nu}\tag{3.23}$$

$$(a_\nu^{j+5})'f_0(\tau_{j+5}+\nu)+A(a_{\nu-1}^{j+5})'f_1(\tau_{j+5}+\nu-1)+B(a_{\nu-2}^{j+5})'f_2(\tau_{j+5}+\nu-2)$$
$$+a_{\nu-1}^j\frac{df_0(\tau_{j+5}+\nu)}{d\tau}+Aa_{\nu-2}^j\frac{df_1(\tau_{j+5}+\nu-1)}{d\tau}+Ba_{\nu-3}^j\frac{df_2(\tau_{j+5}+\nu-2)}{d\tau}=0$$
$$(j=1,2,3)$$

方程（3.11)′的特解要视自由项$Q_{nc}(\zeta)$的具体形式而定．对于本文讨论的分布荷载情况，自由项总可以用下式表示，即

$$Q_{nc}(\zeta)=\sum_{n'=0}^{N'}Q_{ncn'}\zeta^{\sigma_{n'}}\tag{3.24}$$

式中$Q_{ncn'}$为已知常数，N'为任意大整数．对于(3.24)的每一项$Q_{ncn'}\zeta^{\sigma_{n'}}$，方程(3.11)′变成：

$$L_\zeta(U_{ncn'}^p)=Q_{ncn'}\zeta^{\sigma_{n'}}\qquad(n'=\overline{1,N'})\tag{3.25}$$

方程(3.25)的解可取为：

$$U_{ncn'}^p=Q_{ncn'}\sum_{\nu=0}^{\infty}b_\nu^{n'}\zeta^{\sigma_{n'}+\nu}\tag{3.26}$$

其中

$$b_0^{n'}=1/g_0(\sigma_{n'})$$

$$b_\nu^n g_0(\sigma_{n'}+\nu)+Ab_{\nu-1}^{n'}\ g_1(\sigma_{n'}+\nu-1)+Bb_{\nu-2}^{n'}\ g_2(\sigma_{n'}+\nu-2)=0 \tag{3.27}$$

$$g_0(\sigma_{n'}+\nu)=\prod_{i=1}^{8}[(\sigma_{n'}+\nu)-\alpha_i/2]$$

$$g_1(\sigma_{n'}+\nu-1)=\prod_{i=1}^{4}[(\sigma_{n'}+\nu-1)-\beta_i/2]$$

$$g_2(\sigma_{n'}+\nu-2)=\prod_{i=1}^{4}[(\sigma_{n'}+\nu-2)-\gamma_i/2]$$

所以特解为:

$$U_{n\,c}^{p}=\sum_{n'=0}^{N'}U_{ncn'}^{p} \tag{3.28}$$

全解为:

$$U_{nc}=\sum_{\mu'=1}^{8}C_{\mu'}U_{nc}^{H(\mu')}+U_{nc}^{p} \tag{3.29}$$

其中 $c_{\mu'}(\mu'=\overline{1,8})$为待定常数，由边界条件确定。

对于$n=0(m=0)$，即对应于轴对称变形状态。当略去公共微分算子$\cos^2\varphi(\delta_s^2-1)$后，通解(2.14)及基本方程(2.15)，(2.16)变成:

$$u_0=-\sin\varphi\cos\varphi(\mu\delta_s-1)U_0+\frac{1-\mu}{2}\psi_1$$

$$w_0=\cos^2\varphi(\delta_s^2-1)U_0 \tag{3.30}$$

$$(\delta_s^2-1)\psi_1=-\frac{2}{1-\mu}\frac{q_i s^2}{K}\qquad(i=1,2) \tag{3.31}$$

$$(\delta_s-2)^2(\delta_s^2-1)\delta_s^2U_0+(1-\mu^2)\frac{K}{D}\mathrm{tg}^2\varphi s^2\delta_s^2U_0$$

$$+\frac{k}{D}s^4(\delta_s^2-1)U_0=Q(s) \tag{3.32}$$

自由项

$$Q(s)=q_n s^4\sec^2\varphi/D-\frac{1-\mu}{2}\frac{K}{D}s^2\mathrm{tg}\varphi\sec^2\varphi(\mu\delta_s+1)\psi_1 \tag{3.33}$$

方程(3.32)的齐次解为:

$\rho_1=1$，$\rho_2=0$，$\rho_3=1/2$，$\rho_4=1$，$\rho_5=0$，$\rho_6=-1/2$

$$U_0^{H(j)}=\sum_{\nu=0}^{8}c_\nu^j\xi^{\rho_j+\nu}\qquad(j=1,2,3) \tag{3.34}$$

$$c_\nu^j f_{00}(\rho_j+\nu)+Ac_{\nu-1}^j\ f_{11}(\rho_j+\nu-1)+Bc_{\nu-2}^j f_{22}(\rho_j+\nu-2)=0$$

$$U_0^{H(j+3)}=U_0^{H(j)}\ln\zeta+\sum_{\nu=0}^{\infty}(c_\nu^{j+3})'\zeta^{\rho_{j+3}+\nu}\qquad(j=1,2)\tag{3.35}$$

$$(c_\nu^{j+3})'f_{00}(\rho_{j+3}+\nu)+A(c_{\nu-1}^{j+3})'f_{11}(\rho_{j+3}+\nu-1)+B(c_{\nu-2}^{j+3})'f_{22}(\rho_{j+3}+\nu-2)+$$
$$+c_\nu^j\frac{df_{00}(\rho_{j+3}+\nu)}{d\rho}+A\,c_{\nu-1}^j\frac{df_{11}(\rho_{j+3}+\nu-1)}{d\rho}+B\,c_{\nu-2}^j\frac{df_{22}(\rho_{j+3}+\nu-2)}{d\rho}=0$$

$$U_0^{H(6)}=U_0^{H(3)}\ln\zeta+\sum_{\nu=0}^{\infty}(c_\nu^6)'\zeta^{\nu-1/2}\tag{3.36}$$

$$(c_\nu^6)'f_{00}(\nu-1/2)+A(c_{\nu-1}^6)'f_{11}(\nu-3/2)+B(c_{\nu-2}^6)'f_{22}(\nu-5/2)+$$
$$+c_{\nu-1}^3\frac{df_{00}(\nu-1/2)}{d\rho}+Ac_{\nu-2}^3\frac{df_{11}(\nu-3/2)}{d\rho}+B\,c_{\nu-3}^j\frac{df_{22}(\nu-5/2)}{d\rho}=0$$

其中

$$f_{00}(\rho+\nu)=(\rho+\nu)^2(\rho+\nu+1/2)(\rho+\nu-1)^2(\rho+\nu-1/2)\tag{3.37}$$
$$f_{11}(\rho+\nu-1)=(\rho+\nu-1)^2,\ f_{22}(\rho+\nu-2)=(\rho+\nu-5/2)(\rho+\nu-1/2)$$

方程(3.32)的特解可仿照一般问题特解的求法求出

以上所得的级数，按Cauchy判别法立即可知是收敛的。

四、数　值　结　果

应用本文所提出的精确计算方法，对实际工程中有广泛应用价值的弹性基上锥壳轴对称变形问题，进行了详细的数值计算。如图2所示的结构。

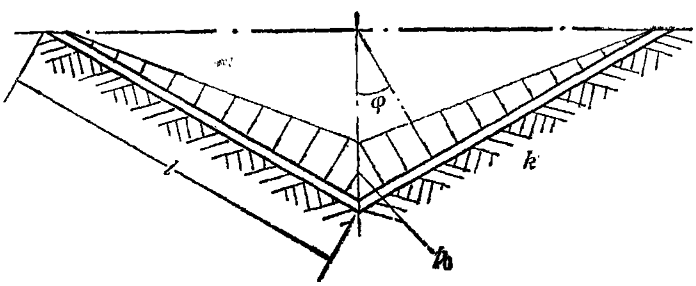


图　2

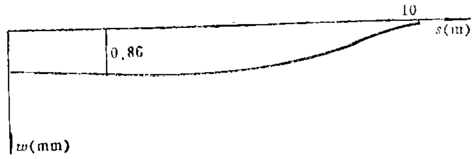
其中数据：$E=2.6\times10^9\text{kg/m}^2$，$\mu=0.167$，$l=10\text{m}$，
$h=0.3048\text{m}$，$\varphi=20^\circ$，$k=2.5\times10^6\text{kg/m}$，
$p_c=10^3\text{kg/m}^2$

1°周边固定

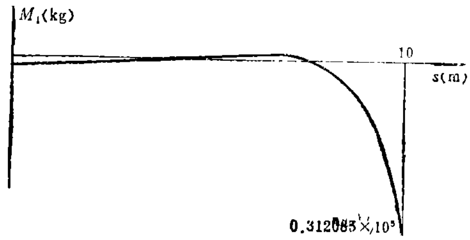
当周边固定时，相应的边界条件为：

$$u=w=\frac{dw}{ds}=0\tag{4.1}$$

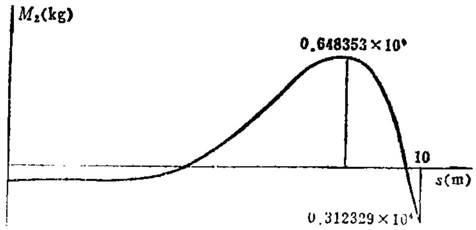
计算结果如图3所示。



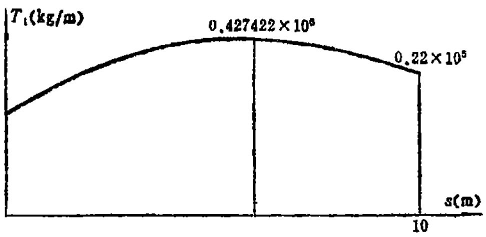
(a)



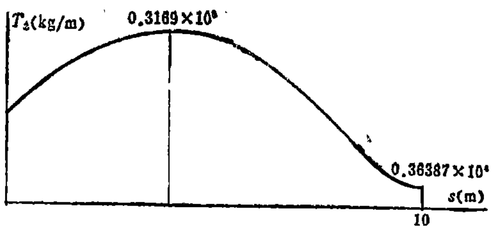
(b)



(c)

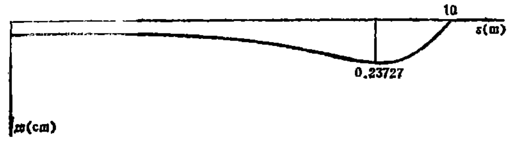


(d)

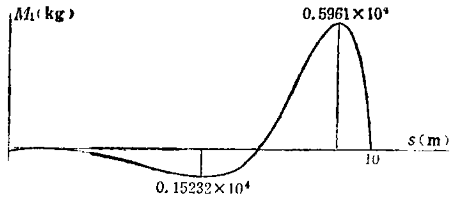


(e)

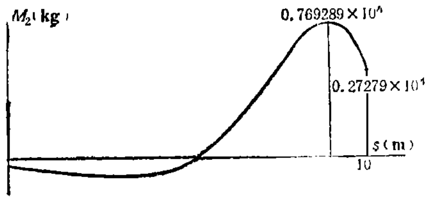
图 3



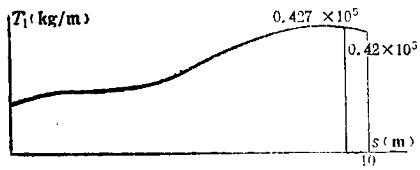
(a)



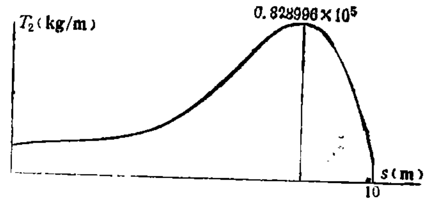
(b)



(c)



(d)



(e)

图 4

2° 周边简支

当周边简支时，相应的边界条件为:

$$u=w=M_1=0 \tag{4.2}$$

计算结果如图4所示。

3°边界上的影响系值

计算了单位边界力作用下的影响系数（如图5所示）。所得结果可用于弹性基上锥壳连结问题的计算，如表1所列。

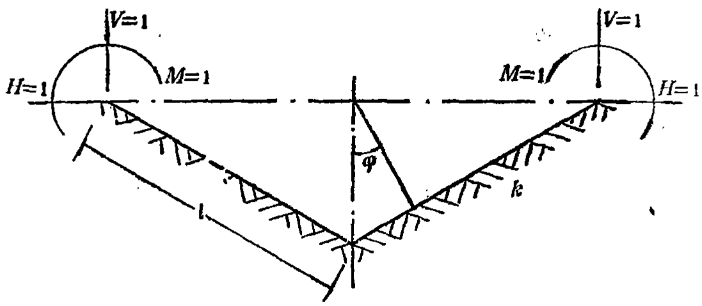


图 5

表 1

	H	U	M
U	0.7611809685E-8	−0.1301120033E-8	−0.6147245291E-8
W	0.2812641689E-7	0.1155594955E-6	−0.1026425259E-6
θ	−0.3164154379E-7	−0.7772973319E-7	0.1436657184E-7
T_1	0.9396926161	−0.3420200644	0.0000000000
T_2	−0.5112630694E-1	−0.3493415576E-1	0.24734454330E 1
M_1	0.0000000000	0.0000000000	1
M_2	0.9012809119E-1	0.8997059679E-1	0.4472711784E-1
N_1	0.3420200644	0.9396926161	0.0000000000

以上结果表明：边界效应是非常明显的，本文第一次给出弹性基上锥壳弯曲问题的精确解。所得结果不仅对于弹性基上锥壳组合结构的计算有着重要的意义，而且可以用来判定各种近似法，如变分法，差分法，有限元法的精度。

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The Exact Solution for the General Bending Problems of Conical Shells on the Elastic Foundation

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Abstract

The general bending problem of conical shells on the elastic foundation (Winkler Medium) is not solved. In this paper, the displacement solution method for this problem is presented. From the governing differential equations in displacement form of conical shell and by introducing a displacement function $U(s,\theta)$ the differential equations are changed into an eight-order soluble partial differential equation about the displacement function $U(s,\theta)$ in which the coefficients are variable. At the same time, the expressions of the displacement and internal force components of the shell are also given by the displacement function $U(s,\theta)$. As special cases of this paper, the displacement function introduced by V. S. Vlasov in circular cylindrical shell[5], the basic equation of the cylindrical shell on the elastic foundation and that of the circular plates on the elastic foundation are directly derived.

Under the arbitrary loads and boundary conditions, the general bending problem of the conical shell on the elastic foundation is reduced to find the displacement function $U(s, \theta)$. The general solution of the eight-order differential equation is obtained in series form. For the symmetric bending deformation of the conical shell on the elastic foundation, which has been widely used in practice, the detailed numerical results and boundary influence coefficients for edge loads have been obtained. These results have important meaning in analysis of conical shell combination construction on the elastic foundation and provide a valuable judgement for the numerical solution accuracy of some of the same existing type of problem.