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横观各向同性板的弹性理论和弹性改进理论及一种新的厚板理论*

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摘　要

本文从横观各向同性体弹性力学位移形式的基本方程出发，考虑板面承受横向荷载，建立了横观各向同性板弯曲的弹性理论．并由此建立了一个在板的每边能满足三个边界条件的弹性改进理论和一种新的厚板理论．文中求得了周边简支多边形板的弹性改进理论解，数值结果与三维弹性理论精确解的结果非常接近．新的厚板理论和以往的中厚板理论的系统比较表明，我们提出的厚板理论最靠近弹性理论的结果．

一、引　言

为了修正经典的薄板理论，人们建立了各种考虑剪切变形的中厚板理论．这些实用理论的解不满足三维弹性理论的全部方程，当板的厚跨比较大时，与三维弹性理论解比较，中厚板理论存在一定的误差．所以人们一方面寻找更高阶的理论，另一方面致力于用三维弹性理论分析板的弯曲问题．研究各种中厚板理论与三维弹性理论之间的联系和差别，是一件既有理论意义又有实际价值的工作．

本文采用算子符号法[1~3]，直接从横观各向同性体三维弹性理论基本方程出发，导出了三维弹性理论的形式解．利用恰当解[4]的构造方法，建立了横观各向同性板弯曲的弹性理论．这个理论考虑了板面所受的横向荷载，包含三个基本方程和一个特解方程．对于横向刚性材料，由这个理论能导出板弯曲的经典理论．文中证明了第三个基本方程的解不产生横向剪力．考虑到第三个基本解的这个特性，并以满足板每边的三个边界条件为前提，建立了一个弹性改进理论．对于周边简支多边形板的弯曲问题，我们利用各向同性板经典解的已知结果，直接求得了横观各向同性板弹性改进理论的解．由于弹性改进理论的解满足三维弹性理论的全部方程，所以我们得到的数值结果与三维弹性理论精确解的数值结果非常接近．考虑到板的尺寸特征，对特解方程进行量级分析和简化，我们在弹性改进理论的基础上建立了一个新的厚板理论．这个理论考虑了横向剪切变形、横向正应力以及横向正应变的影响．文中将常见的中厚板理论进行了解的结构分解，以我们提出的理论为尺度，进行了系统的比较，

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能定量地指出各种中厚板理论与新理论之间的差别。新理论的退化理论能包含所比较的中厚板理论。比较清楚地揭示了中厚板理论与弹性理论之间的联系和差别。

二、三维弹性理论的形式解

横观各向同性体弹性理论的应力应变关系为

$$\sigma_x = A_{11}\frac{\partial U}{\partial x} + A_{12}\frac{\partial V}{\partial y} + A_{13}\frac{\partial W}{\partial z} \tag{2.1a}$$

$$\sigma_y = A_{12}\frac{\partial U}{\partial x} + A_{11}\frac{\partial V}{\partial y} + A_{13}\frac{\partial W}{\partial z} \tag{2.1b}$$

$$\sigma_z = A_{13}\frac{\partial U}{\partial x} + A_{13}\frac{\partial V}{\partial y} + A_{33}\frac{\partial W}{\partial z} \tag{2.1c}$$

$$\tau_{xz} = A_{44}\left(\frac{\partial W}{\partial x} + \frac{\partial U}{\partial z}\right) \tag{2.1d}$$

$$\tau_{yz} = A_{44}\left(\frac{\partial W}{\partial y} + \frac{\partial V}{\partial z}\right) \tag{2.1e}$$

$$\tau_{xy} = A_{66}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) \tag{2.1f}$$

式中

$$A_{66} = \frac{1}{2}(A_{11} - A_{12}) \tag{2.2}$$

以位移表示的平衡方程（无体力）为

$$\mathbf{A}[U\ \ V\ \ W]^T = 0 \tag{2.3}$$

其中线性偏微分算子矩阵 **A** 为

$$\begin{bmatrix} B_{11}\frac{\partial^2}{\partial x^2} + B_{66}\frac{\partial^2}{\partial y^2} + B_{44}\frac{\partial^2}{\partial z^2} & B_{12}\frac{\partial^2}{\partial x\partial y} & B_{13}\frac{\partial^2}{\partial x\partial z} \\ B_{12}\frac{\partial^2}{\partial x\partial y} & B_{66}\frac{\partial^2}{\partial x^2} + B_{11}\frac{\partial^2}{\partial y^2} + B_{44}\frac{\partial^2}{\partial z^2} & B_{13}\frac{\partial^2}{\partial y\partial z} \\ B_{13}\frac{\partial^2}{\partial x\partial z} & B_{13}\frac{\partial^2}{\partial y\partial z} & B_{44}\frac{\partial^2}{\partial x^2} + B_{44}\frac{\partial^2}{\partial y^2} + B_{33}\frac{\partial^2}{\partial z^2} \end{bmatrix} \tag{2.4}$$

此处

$$B_{11} = A_{11},\ B_{33} = A_{33},\ B_{44} = A_{44},\ B_{66} = A_{66},\ B_{12} = A_{12} + A_{44},\ B_{13} = A_{13} + A_{44} \tag{2.5}$$

按照文献[1]提出的方法，我们可以把偏微分方程组(2.3)看成以z为自变量的常微分方程组，求出(2.3)的形式解。

对于板的弯曲问题，我们只考虑反对称变形，并引入下列函数

$$U_0'(x,y) = \left.\frac{\partial U}{\partial z}\right|_{z=0},\quad V_0'(x,y) = \left.\frac{\partial V}{\partial z}\right|_{z=0},\quad W_0(x,y) = W|_{z=0} \tag{2.6}$$

以U_0'，V_0'，W_0表示的三维弹性理论反对称变形的形式解为

$$U=\frac{\sin s_0\nabla z}{s_0\nabla}U_0'-\frac{1}{\nabla^2}\left[\frac{\sin s_0\nabla z}{s_0\nabla}-\frac{1}{s_1^2-s_2^2}\left(s_1^2\frac{\sin s_1\nabla z}{s_1\nabla}-s_2^2\frac{\sin s_2\nabla z}{s_2\nabla}\right)\right]$$

$$\cdot\frac{\partial}{\partial x}\left(\frac{\partial U_0'}{\partial x}+\frac{\partial V_0'}{\partial y}\right)-\frac{1}{s_1^2-s_2^2}\frac{B_{13}}{B_{33}}\frac{1}{\nabla^2}\left(\frac{\sin s_1\nabla z}{s_1\nabla}-\frac{\sin s_2\nabla z}{s_2\nabla}\right)\frac{\partial\theta_0'}{\partial x} \quad (2.7a)$$

$$V=\frac{\sin s_0\nabla z}{s_0\nabla}V_0'-\frac{1}{\nabla^2}\left[\frac{\sin s_0\nabla z}{s_0\nabla}-\frac{1}{s_1^2-s_2^2}\left(s_1^2\frac{\sin s_1\nabla z}{s_1\nabla}-s_2^2\frac{\sin s_2\nabla z}{s_2\nabla}\right)\right]$$

$$\cdot\frac{\partial}{\partial y}\left(\frac{\partial U_0'}{\partial x}+\frac{\partial V_0'}{\partial y}\right)-\frac{1}{s_1^2-s_2^2}\frac{B_{13}}{B_{33}}\frac{1}{\nabla^2}\left(\frac{\sin s_1\nabla z}{s_1\nabla}-\frac{\sin s_2\Delta z}{s_2\nabla}\right)\frac{\partial\theta_0'}{\partial y} \quad (2.7b)$$

$$W=\frac{1}{s_1^2-s_2^2}(s_1^2\cos s_2\nabla z-s_2^2\cos s_1\nabla z)W_0+\frac{1}{s_1^2-s_2^2}\frac{1}{\nabla^2}(\cos s_1\nabla z-\cos s_2\nabla z)$$

$$\cdot\left[\frac{B_{44}}{B_{33}}\nabla^2W_0+\frac{B_{13}}{B_{33}}\left(\frac{\partial U_0'}{\partial x}+\frac{\partial V_0'}{\partial y}\right)\right] \quad (2.7c)$$

式中

$$\left.\begin{aligned}&\theta_0'=\alpha\left(\frac{\partial U_0'}{\partial x}+\frac{\partial V_0'}{\partial y}\right)+\nabla^2W_0,\quad \alpha=\frac{B_{44}}{B_{13}},\quad \nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\quad s_0^2=\frac{B_{66}}{B_{44}}\\&s_1^2=[B_{44}^2+B_{11}B_{33}-B_{13}^2+\sqrt{(B_{44}^2+B_{11}B_{33}-B_{13}^2)^2-4B_{11}B_{33}B_{44}^2}]/2B_{33}B_{44}\\&s_2^2=[B_{44}^2+B_{11}B_{33}-B_{13}^2-\sqrt{(B_{44}^2+B_{11}B_{33}-B_{13}^2)^2-4B_{11}B_{33}B_{44}^2}]/2B_{33}B_{44}\end{aligned}\right\} \quad (2.8)$$

三、板弯曲的弹性理论

考虑等厚度平板，材料的各向同性面与板的中面平行，如图1所示取定坐标系$Oxyz$，使Oxy平面与板的中面重合。

考虑板的弯曲问题，则上、下表面的边界条件为

$$\tau_{xz}\Big|_{z=\pm\frac{h}{2}}=\tau_{yz}\Big|_{z=\pm\frac{h}{2}}=0,\quad \sigma_z\Big|_{z=\pm\frac{h}{2}}=\pm\frac{1}{2}q(x,y) \quad (3.1)$$

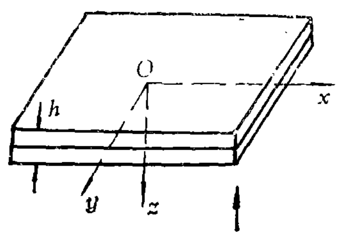

图 1

将(2.7)代入(2.1c,d,e)得

$$\left.\begin{aligned}&\frac{\tau_{xz}}{A_{44}}=\left(B_0+\frac{\partial^2L_1}{\partial x^2}\right)U_0'+\frac{\partial^2L_1}{\partial x\partial y}V_0'+\frac{\partial L_2}{\partial x}W_0\\&\frac{\tau_{yz}}{A_{44}}=\frac{\partial^2L_1}{\partial x\partial y}U_0'+\left(B_0+\frac{\partial^2L_1}{\partial y^2}\right)V_0'+\frac{\partial L_2}{\partial y}W_0\\&\sigma_z=\frac{\partial L_3}{\partial x}U_0'+\frac{\partial L_3}{\partial y}V_0'+\nabla^2L_4W_0\end{aligned}\right\} \quad (3.2)$$

式中

$$
\left.\begin{aligned}
&B_0=\cos(s_0\nabla z)\\
&L_1=\frac{1}{\nabla^2}\left(\frac{s_1^2\cos s_1\nabla z-s_2^2\cos s_2\nabla z}{s_1^2-s_2^2}+\frac{A_{13}}{A_{33}}\frac{\cos s_1\nabla z-\cos s_2\nabla z}{s_1^2-s_2^2}-B_0\right)\\
&L_2=\frac{1}{s_1^2-s_2^2}\left[s_1^2\cos s_2\nabla z-s_2^2\cos s_1\nabla z-\frac{A_{13}}{A_{33}}(\cos s_1\nabla z-\cos s_2\nabla z)\right]\\
&L_3=-\frac{A_{44}}{s_1^2-s_2^2}\left[\frac{s_1^2\sin s_1\nabla z}{s_1\nabla}-\frac{s_2^2\sin s_2\nabla z}{s_2\nabla}+\frac{A_{13}}{B_{13}}\left(\frac{\sin s_1\nabla z}{s_1\nabla}-\frac{\sin s_2\nabla z}{s_2\nabla}\right)\right]\\
&L_4=\frac{1}{s_1^2-s_2^2}\left[\left(B_{11}-\frac{A_{13}B_{13}}{B_{33}}\right)\left(\frac{\sin s_1\nabla z}{s_1\nabla}-\frac{\sin s_2\nabla z}{s_2\nabla}\right)-B_{44}\left(\frac{s_1^2\sin s_1\nabla z}{s_1\nabla}-\frac{s_2^2\sin s_2\nabla z}{s_2\nabla}\right)\right]
\end{aligned}\right\}
\tag{3.3}
$$

应用边界条件(3.1)，可得

$$
\begin{bmatrix}
\bar{B}_0+\dfrac{\partial^2\bar{L}_1}{\partial x^2} & \dfrac{\partial^2\bar{L}_1}{\partial x\partial y} & \dfrac{\partial\bar{L}_2}{\partial x}\\
\dfrac{\partial^2\bar{L}_1}{\partial x\partial y} & \bar{B}_0+\dfrac{\partial^2\bar{L}_1}{\partial y^2} & \dfrac{\partial\bar{L}_2}{\partial y}\\
\dfrac{\partial\bar{L}_3}{\partial x} & \dfrac{\partial\bar{L}_3}{\partial y} & \nabla^2\bar{L}_4
\end{bmatrix}
\begin{bmatrix}U_0'\\ V_0'\\ W_0\end{bmatrix}
=\begin{bmatrix}0\\ 0\\ \dfrac{q}{2}\end{bmatrix}
\tag{3.4}
$$

式中 $\bar{B}_0=B_0|_{z=h/2}$, $\bar{L}_i=L_i|_{z=h/2}$ $(i=1,2,3,4)$ (3.5)

用Q表示(3.4)中的方阵，Q_{ij}为Q的代数余子式，则

$$
\left.\begin{aligned}
&Q_{31}=-\bar{B}_0\frac{\partial\bar{L}_2}{\partial x},\quad Q_{32}=-\bar{B}_0\frac{\partial\bar{L}_2}{\partial y},\quad Q_{33}=\bar{B}_0(\bar{B}_0+\nabla^2\bar{L}_1)\\
&Q_{13}=-\bar{B}_0\frac{\partial\bar{L}_3}{\partial x},\quad Q_{23}=-\bar{B}_0\frac{\partial\bar{L}_3}{\partial y}\\
&\det\mathbf{Q}=\bar{B}_0\nabla^2[\bar{L}_4(\bar{B}_0+\nabla^2\bar{L}_1)-\bar{L}_2\bar{L}_3]
\end{aligned}\right\}
\tag{3.6}
$$

上式中各量有最大公因子$\bar{B}_0$，参照文献[4]，我们可构造(3.4)的一般解为

$$
\begin{bmatrix}U_0'\\ V_0'\\ W_0\end{bmatrix}
=\begin{bmatrix}
0 & \dfrac{\partial^2\bar{L}_1}{\partial x\partial y} & -\dfrac{\partial\bar{L}_2}{\partial x}\\
0 & -\bar{B}_0-\dfrac{\partial^2\bar{L}_1}{\partial x^2} & -\dfrac{\partial\bar{L}_2}{\partial y}\\
0 & 0 & \bar{B}_0+\nabla^2\bar{L}_1
\end{bmatrix}
\begin{bmatrix}0\\ F\\ \varphi\end{bmatrix}
\tag{3.7}
$$

其中F，φ满足下列方程

$$\bar{B}_0F=0 \tag{3.8}$$

$$\nabla^4L^*\varphi=q/D \tag{3.9}$$

式中

$$L^*=\frac{2}{(s_1^2-s_2^2)\nabla^2}\left(\cos s_2\frac{\nabla h}{2}\cdot\frac{\sin s_1\dfrac{\nabla h}{2}}{s_1\nabla}-\cos s_1\frac{\nabla h}{2}\,\frac{\sin s_2\dfrac{\nabla h}{2}}{s_2\nabla}\right) \tag{3.10}$$

$$J=\frac{h^3}{12},\quad D=\frac{Eh^3}{12(1-\mu^2)} \tag{3.11}$$

E，μ分别为各向同性面的杨氏模量和泊松比。

设φ^*为满足(3.9)的一个特解，并考虑到算子∇^4与L^*互质，令

$$\varphi=\bar{\varphi}_1+\bar{\varphi}_2+\varphi^* \tag{3.12}$$

我们可写出横观各向同性板弯曲的弹性理论的基本方程为

$$\nabla^4\bar{\varphi}_1=0,\ \bar{B}_0F=0,\ L^*\bar{\varphi}_2=0,\ \nabla^4L^*\varphi^*=q/D \tag{3.13a～d}$$

第一个方程为双调和方程；第二个方程为剪切方程；第三个方程我们可称之为第三基本方程；第四个方程为特解方程，它对应于板面横向荷载的影响；这四个方程构成了完整的板弯曲的弹性理论方程。全部满足这些方程的解就是三维弹性理论的精确解。这个理论解应满足板边侧面上精确给出的各种应力边界条件和位移边界条件。

再由(3.7)、(2.7)和(2.1)，我们就可以写出各个基本方程的解所对应的位移分量和应力分量。

1. 双调和解

设$\bar{\varphi}_1$所产生的W_0记为$\overline{W}_0$，由(3.7)可得

$$\overline{W}_0=(\bar{B}_0+\nabla^2L_1)\bar{\varphi}_1=\left[1-\frac{h^2}{8}\left(s_1^2+s_2^2+\frac{A_{13}}{A_{33}}\right)\nabla^2\right]\bar{\varphi}_1$$

于是(3.13a)变换为如下的基本方程

$$\nabla^4\overline{W}_0=0 \tag{3.14}$$

经计算，得到

$$U_0'=D_1\overline{W}_0,\ V_0'=D_2\overline{W}_0,\ U=D_3\overline{W}_0,\ V=D_4\overline{W}_0,\ W=D_5\overline{W}_0 \tag{3.15}$$

$$\sigma_x=D_6\overline{W}_0,\ \sigma_y=D_7\overline{W}_0,\ \sigma_z=0,\ \tau_{xy}=D_8\overline{W}_0,\ \tau_{xz}=D_9\overline{W}_0,\ \tau_{yz}=D_{10}\overline{W}_0 \tag{3.16}$$

$$M_x=D_{11}\overline{W}_0,\ M_y=D_{12}\overline{W}_0,\ M_{xy}=D_{13}\overline{W}_0,\ Q_x=D_{14}\overline{W}_0,\ Q_y=D_{15}\overline{W}_0 \tag{3.17}$$

式中算子记号定义如下

$$\left.\begin{aligned}
&D_1=-\left(1+\frac{h^2}{4}s\nabla^2\right)\frac{\partial}{\partial x},\ D_2=-\left(1+\frac{h^2}{4}s\nabla^2\right)\frac{\partial}{\partial y}\\
&D_3=-z\left[1+\left(\frac{h^2}{4}s-\frac{z^2}{6}s'\right)\nabla^2\right]\frac{\partial}{\partial x}\\
&D_4=-z\left[1+\left(\frac{h^2}{4}s-\frac{z^2}{6}s'\right)\nabla^2\right]\frac{\partial}{\partial y},\ D_5=\left(1+\frac{1}{2}\frac{A_{13}}{A_{33}}z^2\nabla^2\right)
\end{aligned}\right\} \tag{3.18a～e}$$

$$\left.\begin{aligned}
&D_6=-\frac{Ez}{1-\mu^2}\left[\frac{\partial^2}{\partial x^2}+\mu\frac{\partial^2}{\partial y^2}-\frac{1}{2}\left(\frac{h^2}{2}k_1-\frac{2k_1-k_2\mu'}{3}z^2\right)\frac{\partial^2}{\partial y^2}\nabla^2\right]\\
&D_7=-\frac{Ez}{1-\mu^2}\left[\frac{\partial^2}{\partial y^2}+\mu\frac{\partial^2}{\partial x^2}-\frac{1}{2}\left(\frac{h^2}{2}k_1-\frac{2k_1-k_2\mu'}{3}z^2\right)\frac{\partial^2}{\partial x^2}\nabla^2\right]\\
&D_8=-\frac{Ez}{1-\mu^2}\frac{\partial^2}{\partial x\partial y}\left[1-\mu+\frac{1}{2}\left(\frac{h^2}{2}k_1-\frac{2k_1-k_2\mu'}{3}z^2\right)\nabla^2\right]\\
&D_9=-\frac{E}{2(1-\mu^2)}\left(\frac{h^2}{4}-z^2\right)\frac{\partial}{\partial x}\nabla^2,\ D_{10}=-\frac{E}{2(1-\mu^2)}\left(\frac{h^2}{4}-z^2\right)\frac{\partial}{\partial y}\nabla^2
\end{aligned}\right\}$$

(3.19a～e)

$$
\left.\begin{aligned}
D_{11}&=-D\left(\frac{\partial^2}{\partial x^2}+\mu\frac{\partial^2}{\partial y^2}-\frac{8k_1+k_2\mu'}{40}h^2\frac{\partial^2}{\partial y^2}\nabla^2\right)\\
D_{12}&=-D\left(\frac{\partial^2}{\partial y^2}+\mu\frac{\partial^2}{\partial x^2}-\frac{8k_1+k_2\mu'}{40}h^2\frac{\partial^2}{\partial x^2}\nabla^2\right)\\
D_{13}&=-D\left[(1-\mu)\frac{\partial^2}{\partial x\partial y}+\frac{8k_1+k_2\mu'}{40}h^2\frac{\partial^2}{\partial x\partial y}\nabla^2\right]\\
D_{14}&=-D\frac{\partial}{\partial x}\nabla^2,\quad D_{15}=-D\frac{\partial}{\partial y}\nabla^2
\end{aligned}\right\}\qquad(3.20\text{a}\sim\text{e})
$$

符号k_1，k_2，s，s'，μ'的意义见后。

2. 剪切解

令

$$\psi=\frac{\partial L_1}{\partial x}F \qquad (3.21)$$

由(3.7)得

$$U_0'=\frac{\partial\psi}{\partial y},\quad V_0'=-\frac{\partial\psi}{\partial x},\quad W_0=0 \qquad (3.22)$$

剪切解的基本方程为

$$\cos\left(s_0\frac{h}{2}\nabla\right)\psi=0 \qquad (3.23)$$

上式的解等价于下面方程的解

$$\left[\nabla^2-\left(\frac{n\pi}{s_0h}\right)^2\right]\psi_n=0\qquad(n=1,3,5,\cdots) \qquad (3.24)$$

由(2.7)，可得

$$U_n=\frac{h}{n\pi}\sin\frac{n\pi}{h}z\cdot\frac{\partial\psi_n}{\partial y},\quad V_n=-\frac{h}{n\pi}\sin\frac{n\pi}{h}z\cdot\frac{\partial\psi_n}{\partial x},\quad W_n=0 \qquad (3.25)$$

应用(2.1)，即得

$$
\left.\begin{aligned}
&\sigma_{xn}=-\sigma_{yn}=\frac{2Gh}{n\pi}\sin\frac{n\pi z}{h}\frac{\partial^2\psi_n}{\partial x\partial y},\quad \tau_{xyn}=\frac{Gh}{n\pi}\sin\frac{n\pi z}{h}\left(\frac{\partial^2\psi_n}{\partial y^2}-\frac{\partial^2\psi_n}{\partial x^2}\right)\\
&\tau_{xzn}=\frac{G}{k_1}\cos\frac{n\pi z}{h}\frac{\partial\psi_n}{\partial y},\quad \tau_{yzn}=-\frac{G}{k_1}\cos\frac{n\pi z}{h}\frac{\partial\psi_n}{\partial x},\quad \sigma_{zn}=0\\
&M_{xn}=-M_{yn}=\frac{4Gh^3}{n^3\pi^3}(-1)^{\frac{n-1}{2}}\frac{\partial^2\psi_n}{\partial x\partial y}\\
&M_{xyn}=\frac{2Gh^3}{n^3\pi^3}(-1)^{\frac{n-1}{2}}\left(\frac{\partial^2\psi_n}{\partial y^2}-\frac{\partial^2\psi_n}{\partial x^2}\right)\\
&Q_{xn}=\frac{2G}{k_1}\frac{h}{n\pi}(-1)^{\frac{n-1}{2}}\frac{\partial\psi_n}{\partial y},\quad Q_{yn}=-\frac{2G}{k_1}\frac{h}{n\pi}(-1)^{\frac{n-1}{2}}\frac{\partial\psi_n}{\partial x}
\end{aligned}\right\}\qquad(3.26)
$$

3. 第三基本方程的解

$$U_0'=-\frac{\partial L_2}{\partial x}\bar{\varphi}_2,\quad V_0'=-\frac{\partial L_2}{\partial y}\bar{\varphi}_2,\quad W_0=(\bar{B}_0+\nabla^2L_1)\bar{\varphi}_2 \qquad (3.27)$$

对应的位移U，V，W与应力σ_x，σ_y，τ_{xy}，τ_{xz}，τ_{yz}，σ_z很容易自(2.7)、(2.1)及(3.2)求得。下面我们证明第三基本方程的解不产生横向剪力。由(3.2)与(3.27)，易知

$$\left.\begin{aligned}\tau_{xz}&=A_{44}[L_2(\bar{B}_0+\nabla^2\bar{L}_1)-\bar{L}_2(B_0+\nabla^2L_1)]\frac{\partial\bar{\varphi}_2}{\partial x}\\ \tau_{yz}&=A_{44}[L_2(\bar{B}_0+\nabla^2\bar{L}_1)-\bar{L}_2(B_0+\nabla^2L_1)]\frac{\partial\bar{\varphi}_2}{\partial y}\end{aligned}\right\}\tag{3.28}$$

又可证

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}(B_0+\nabla^2L_1)dz=-2\frac{\bar{L}_3}{A_{44}},\quad \int_{-\frac{h}{2}}^{\frac{h}{2}}L_2dz=-2\frac{\bar{L}_4}{A_{44}}\tag{3.29}$$

$$\begin{aligned}&\int_{-\frac{h}{2}}^{\frac{h}{2}}[L_2(\bar{B}_0+\nabla^2\bar{L}_1)-\bar{L}_2(B_0+\nabla^2L_1)]\bar{\varphi}_2dz\\ &=-\frac{2}{A_{44}}[\bar{L}_4(\bar{B}_0+\nabla^2\bar{L}_1)-\bar{L}_2\bar{L}_3]\bar{\varphi}_2\\ &=-\frac{1}{A_{44}}\frac{E}{1-\mu^2}J\nabla^2L^*\bar{\varphi}_2=0\end{aligned}\tag{3.30}$$

故

$$Q_x=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{xz}dz=0,\quad Q_y=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{yz}dz=0\tag{3.31}$$

既然第三基本方程的解不产生横向剪力，所以讨论横向剪力对变形的影响时，只需考虑双调和解和剪切解。

4. 特解

关于方程(3.13d)的特解的求法，可参考文献[1]。为了便于在第四节推导厚板理论，将特解方程(3.13d)写成进行了量级处理的形式

$$\nabla^4\varphi^*=\frac{q}{D}+O(\nabla^2h^2)\nabla^4\varphi^*\tag{3.32}$$

与特解φ^*所对应的挠度W_0^*为

$$W_0^*=(\bar{B}_0+\nabla^2\bar{L}_1)\varphi^*=\varphi^*-\frac{h^2}{8}\left(s_1^2+s_2^2+\frac{A_{13}}{A_{33}}\right)\nabla^2\varphi^*+O(h^4)\nabla^4\varphi^*$$

于是

$$\nabla^4W_0^*=q/D+O(\nabla^2h^2)\nabla^4\varphi^*\tag{3.33}$$

$$U_0'^*=D_1W_0^*+D_1^*q+h^4O(\nabla^2h^2)\frac{\partial}{\partial x}\nabla^4\varphi^*\tag{3.34a}$$

$$V_0'^*=D_2W_0^*+D_2^*q+h^4O(\nabla^2h^2)\frac{\partial}{\partial y}\nabla^4\varphi^*\tag{3.34b}$$

$$U^*=D_3W_0^*+D_3^*q+h^5O(\nabla^2h^2)\frac{\partial}{\partial x}\nabla^4\varphi^*\tag{3.35a}$$

$$V^*=D_4W_0^*+D_4^*q+h^5O(\nabla^2h^2)\frac{\partial}{\partial y}\nabla^4\varphi^*\tag{3.35b}$$

$$W^*=D_5W_0^*+D_5^*q+h^4O(\nabla^2h^2)\nabla^4\varphi^*\tag{3.35c}$$

$$\begin{cases} \sigma_x^* = D_6 W_0^* + D_6^* q + h^3 O(\nabla^2 h^2) \nabla^4 \varphi^* & (3.36a) \\ \sigma_y^* = D_7 W_0^* + D_6^* q + h^3 O(\nabla^2 h^2) \nabla^4 \varphi^* & (3.36b) \\ \sigma_z^* = D_7^* q + h^3 O(\nabla^2 h^2) \nabla^4 \varphi^* & (3.36c) \\ \tau_{xy}^* = D_8 W_0^* + h^3 O(\nabla^2 h^2) \nabla^4 \varphi^* & (3.36d) \\ \tau_{xz}^* = D_9 W_0^* + D_9^* q + h^4 O(\nabla^2 h^2) \dfrac{\partial}{\partial x} \nabla^4 \varphi^* & (3.36e) \\ \tau_{yz}^* = D_{10} W_0^* + D_{10}^* q + h^4 O(\nabla^2 h^2) \dfrac{\partial}{\partial y} \nabla^4 \varphi^* & (3.36f) \end{cases}$$

$$\begin{cases} M_x^* = D_{11} W_0^* + D_{11}^* q + h^5 O(\nabla^2 h^2) \nabla^4 \varphi^* & (3.37a) \\ M_y^* = D_{12} W_0^* + D_{11}^* q + h^5 O(\nabla^2 h^2) \nabla^4 \varphi^* & (3.37b) \\ M_{xy}^* = D_{13} W_0^* + h^5 O(\nabla^2 h^2) \nabla^4 \varphi^* & (3.37c) \\ Q_x^* = D_{14} W_0^* + D_{14}^* q + h^5 O(\nabla^2 h^2) \nabla^4 \varphi^* & (3.37d) \\ Q_y^* = D_{15} W_0^* + D_{15}^* q + h^5 O(\nabla^2 h^2) \nabla^4 \varphi^* & (3.37e) \end{cases}$$

式中

$$\begin{cases} D_1^* = \dfrac{h^4}{32} s \left[\dfrac{1}{6} (s_1^2 + s_2^2) - s' \right] \dfrac{1}{D} \dfrac{\partial}{\partial x} & (3.38a) \\ D_2^* = \dfrac{h^4}{32} s \left[\dfrac{1}{6} (s_1^2 + s_2^2) - s' \right] \dfrac{1}{D} \dfrac{\partial}{\partial y} & (3.38b) \\ D_3^* = z \left\{ \dfrac{1}{24} h^2 z^2 \left(s_1^2 + s_2^2 - \dfrac{B_{44}}{B_{33}} \right) s + \dfrac{h^4}{32} s \left[\dfrac{1}{6} (s_1^2 + s_2^2) - s' \right] \right. & \\ \qquad \left. - \dfrac{z^4}{120} \left[s_1^4 + s_2^4 + s_1^2 s_2^2 + \dfrac{A_{13}}{A_{33}} (s_1^2 + s_2^2) \right] \right\} \dfrac{1}{D} \dfrac{\partial}{\partial x} & (3.38c) \\ D_4^* = z \left\{ \dfrac{1}{24} h^2 z^2 \left(s_1^2 + s_2^2 - \dfrac{B_{44}}{B_{33}} \right) s + \dfrac{h^4}{32} s \left[\dfrac{1}{6} (s_1^2 + s_2^2) - s' \right] \right. & \\ \qquad \left. - \dfrac{z^4}{120} \left[s_1^4 + s_2^4 + s_1^2 s_2^2 + \dfrac{A_{13}}{A_{33}} (s_1^2 + s_2^2) \right] \right\} \dfrac{1}{D} \dfrac{\partial}{\partial y} & (3.38d) \\ D_5^* = \dfrac{1}{8} \dfrac{B_{13}}{B_{33}} h^2 z^2 \dfrac{s}{D} - \dfrac{z^3}{24} \left[\dfrac{A_{13}}{A_{33}} (s_1^2 + s_2^2) + s_1^2 s_2^2 \right] \dfrac{1}{D} & (3.38e) \end{cases}$$

$$\begin{cases} D_6^* = -\dfrac{Ez}{1-\mu^2} \dfrac{1}{2D} \left[\dfrac{h^2}{4} s' + \dfrac{2}{3} z^2 \left(\dfrac{A_{13}}{A_{33}} - s \right) \right] & (3.39a) \\ D_7^* = \dfrac{z}{2J} \left(\dfrac{h^2}{4} - \dfrac{z^2}{3} \right) & (3.39b) \\ D_9^* = \left[\dfrac{h^2}{16J} s' \left(z^2 - \dfrac{h^2}{4} \right) + \dfrac{1}{24J} \left(\dfrac{h^2}{4} - z^2 \right) \left(\dfrac{h^2}{4} + z^2 \right) (s_1^2 + s_2^2) \right] \dfrac{\partial}{\partial x} & (3.39c) \\ D_{10}^* = \left[\dfrac{h^2}{16J} s' \left(z^2 - \dfrac{h^2}{4} \right) + \dfrac{1}{24J} \left(\dfrac{h^2}{4} - z^2 \right) \left(\dfrac{h^2}{4} + z^2 \right) (s_1^2 + s_2^2) \right] \dfrac{\partial}{\partial y} & (3.39d) \end{cases}$$

$$\begin{cases} D^*_{11}=-\frac{h^2}{8}\left[s'+\frac{2}{5}\left(\frac{A_{13}}{A_{33}}-s\right)\right] & (3.40a) \\ D^*_{14}=-\frac{h^2}{8}\left[s'+\frac{2}{5}\left(\frac{A_{13}}{A_{33}}-s\right)\right]\frac{\partial}{\partial x} & (3.40b) \\ D^*_{16}=-\frac{h^2}{8}\left[s'+\frac{2}{5}\left(\frac{A_{13}}{A_{33}}-s\right)\right]\frac{\partial}{\partial y} & (3.40c) \end{cases}$$

且

$$G=\frac{E}{2(1+\mu)},\quad k_1=\frac{G}{G'},\quad k_2=\frac{E}{E'},\ s=\frac{k_1}{1-\mu},\quad s'=\frac{2k_1-k_2\mu'}{1-\mu} \tag{3.41a～e}$$

G为各向同性面内的剪切模量，G'，E'，μ'分别为横向剪切模量、杨氏模量、泊松比。

上面我们写出了三个基本解和一个特解的有关力学量的表达式，这些表达式构成板弯曲的弹性理论。

在横观各向同性材料中，令$E'\to\infty$，$G'\to\infty$就得到一种横向刚性材料。将本节理论应用于这种材料的板弯曲问题，我们可以证明剪切方程和第三个基本方程的解为零，并求得横向刚性材料板弯曲的弹性理论解为

$$U=-z\frac{\partial W_0}{\partial x},\quad V=-z\frac{\partial W_0}{\partial y},\quad W=W_0 \tag{3.42a～c}$$

W_0满足下列方程

$$D\nabla^4 W_0=q \tag{3.43}$$

这就是 Kirchhoff 假定下板弯曲的经典理论的结果。可见，对变形的 Kirchhoff 假定和对材料的横向刚性假定是等价的。

四、横观各向同性板的弹性改进理论和厚板理论

首先我们将建立一个在板的每边能规定三个边界条件的弹性改进理论。在以上推导的板弯曲的弹性理论中，由于第三个基本解不产生横向剪力，故放弃第三个基本方程。首先取双调和方程(3.14)

$$\nabla^4 W_0=0 \tag{4.1}$$

它的解在板的每边能满足两个边界条件。与式(3.24)～(3.26)相应的各量可看作剪切解的富里叶展式的单项。为了构成每边能满足三个边界条件的板弯曲的弹性理论，在考虑双调和解后，自然取剪切解展式的第一项($n=1$)来满足剩下的一个边界条件。令

$$\psi=2G\frac{h}{\pi}\psi_1 \tag{4.2}$$

则(3.24)～(3.26)化为（令$n=1$）

$$\left[\nabla^2-\left(\frac{\pi}{s_0 h}\right)^2\right]\psi=0 \tag{4.3}$$

$$U=\frac{1+\mu}{E}\sin\frac{\pi}{h}z\frac{\partial\psi}{\partial y},\quad V=-\frac{1+\mu}{E}\sin\frac{\pi}{h}z\frac{\partial\psi}{\partial x},\quad W=0 \tag{4.4a,b,c}$$

$$\begin{cases} \sigma_x=-\sigma_y=\sin\frac{\pi}{h}z\frac{\partial^2\psi}{\partial x\partial y},\ \tau_{xy}=\frac{1}{2}\sin\frac{\pi}{h}z\left(\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right) & (4.5a,b) \\ \sigma_z=0,\ \tau_{xz}=\frac{1}{2k_1}\frac{\pi}{h}\cos\frac{\pi}{h}z\frac{\partial\psi}{\partial y},\ \ \tau_{yz}=-\frac{1}{2k_1}\frac{\pi}{h}\cos\frac{\pi}{h}z\frac{\partial\psi}{\partial x} & (4.5c,d,e) \end{cases}$$

$$\begin{cases} M_x=-M_y=2\left(\frac{h}{\pi}\right)^2\frac{\partial^2\psi}{\partial x\partial y},\ \ M_{xy}=\left(\frac{h}{\pi}\right)^2\left(\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right) & (4.6a,b) \\ Q_x=\frac{1}{k_1}\frac{\partial\psi}{\partial y},\ \ Q_y=-\frac{1}{k_1}\frac{\partial\psi}{\partial x} & (4.6c,d) \end{cases}$$

基本方程(4.1)、(4.3)，再加上特解方程(3.13d)就构成了板弯曲的弹性改进理论的基本方程。它的解满足弹性理论的全部方程，只是所满足的三个边界条件是以应力的合力、中面位移、平均转角来描述的。

在弹性改进理论的基础上，考虑到板的尺寸特征，可以对特解方程进行简化。将(3.33)中量级为$O(\nabla^2h^2)\nabla^4\varphi^*$的量以及(3.34)～(3.37)中量级为$h^3O(\nabla^2h^2)\nabla^4\varphi^*$以上的微量略去不计。特解方程简化为

$$\nabla^4W_0^{**}=q/D \tag{4.7}$$

令

$$W_0=\overline{W}_0+W_0^{**}$$

由(4.1)、(4.7)、(4.3)，我们可以建立一个新的厚板理论，其基本方程为

$$\nabla^4W_0=q/D \tag{4.8}$$

$$\left[\nabla^2-\left(\frac{\pi}{s_0h}\right)^2\right]\psi=0 \tag{4.9}$$

与W_0有关的力学量，由(3.15)～(3.17)与简化后的(3.34)～(3.37)迭加得到

$$\left.\begin{array}{l} U_0'=D_1W_0+D_1^*q,\ V_0'=D_2W_0+D_2^*q \\ U=D_3W_0+D_3^*q,\ V=D_4W_0+D_4^*q,\ W=D_6W_0+D_5^*q \end{array}\right\} \tag{4.10a～e}$$

$$\left.\begin{array}{l} \sigma_x=D_6W_0+D_6^*q,\ \sigma_y=D_7W_0+D_6^*q,\ \sigma_z=D_7^*q \\ \tau_{xy}=D_8W_0,\ \tau_{xz}=D_9W_0+D_9^*q,\ \tau_{yz}=D_{10}W_0+D_{10}^*q \end{array}\right\} \tag{4.11a～f}$$

$$\left.\begin{array}{l} M_x=D_{11}W_0+D_{11}^*q,\ M_y=D_{12}W_0+D_{11}^*q,\ M_{xy}=D_{13}W_0 \\ Q_x=D_{14}W_0+D_{14}^*q,\ Q_y=D_{15}W_0+D_{15}^*q \end{array}\right\} \tag{4.12a～e}$$

与ψ有关的力学量由(4.4)～(4.6)确定。新的厚板理论不但考虑了横向剪切变形的影响，而且考虑了横向正应变和横向正应力的影响，它在板的每边能满足三个边界条件，在q为x，y的线性函数时，这个理论就与弹性改进理论一致，完全满足弹性理论的全部方程。

下面我用弹性改进理论求解周边简支多边形板的弯曲问题(图2)。只考虑板面承受反对称均布荷载，周边受均布边缘弯矩，边界条件为

$$W_0|_\Gamma=0,\ M_n|_\Gamma=\overline{M}_n,\ \omega_\tau|_\Gamma=0 \tag{4.13}$$

式中ω_τ为平均切向转角

$$\omega_\tau=\frac{12}{h^2}\int_{-\frac{h}{2}}^{\frac{h}{2}}u_\tau\frac{z}{h}dz \tag{4.14}$$

假定板的挠度W_0满足下列边界条件

$$W_0|_\Gamma=0,\ -D\frac{\partial^2W_0}{\partial n^2}\Big|_\Gamma=\overline{M}_n+\frac{qh^2}{40}\left(4s'+\frac{A_{13}}{A_{33}}\right) \tag{4.15a,b}$$

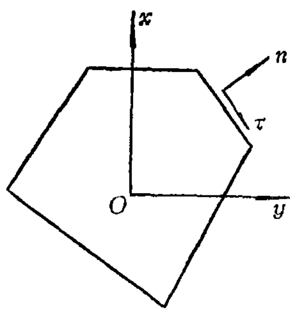

图 2

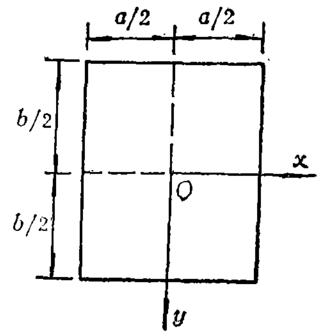

图 3

则在$\overline{M}_n$，q为常数的条件下，很容易验证(4.13)的三个边界条件已得到满足。于是把剪切解取为零。此时W_0的求解等价于边界条件为(4.15)的周边简支板的按经典理论的求解。此时横向剪切效应由算子符号及(4.15b)中的s，s'，k_1，k_2得到反映。

下面以受均布荷载的矩形板为例求解(图3)。设

$$W_0=W_1+W_2+W_3 \tag{4.16}$$

W_0的求解可分解为

（1） $D\nabla^4W_1=q$ (4.17a)

$$\left.\begin{aligned}&W_1|_{x=\pm\frac{a}{2}}=0,\quad -D\frac{\partial^2W_1}{\partial x^2}\Big|_{x=\pm\frac{a}{2}}=0\\&W_1|_{y=\pm\frac{b}{2}}=0,\quad -D\frac{\partial^2W_1}{\partial y^2}\Big|_{y=\pm\frac{b}{2}}=0\end{aligned}\right\} \tag{4.17b～e}$$

（2） $\nabla^4W_2=0$ (4.18a)

$$\left.\begin{aligned}&W_2|_{x=\pm\frac{a}{2}}=0,\quad -D\frac{\partial^2W_2}{\partial x^2}\Big|_{x=\pm\frac{a}{2}}=0\\&W_2|_{y=\pm\frac{b}{2}}=0,\quad -D\frac{\partial^2W_2}{\partial y^2}\Big|_{y=\pm\frac{b}{2}}=\frac{qh^2}{40}\left(4s'+\frac{A_{13}}{A_{33}}\right)\end{aligned}\right\} \tag{4.18b～e}$$

（3） $\nabla^4W_3=0$ (4.19a)

$$\left.\begin{aligned}&W_3|_{x=\pm\frac{a}{2}}=0,\quad -D\frac{\partial^2W_3}{\partial x^2}\Big|_{x=\pm\frac{a}{2}}=\frac{qh^2}{40}\left(4s'+\frac{A_{13}}{A_{33}}\right)\\&W_3|_{y=\pm\frac{b}{2}}=0,\quad -D\frac{\partial^2W_3}{\partial y^2}\Big|_{y=\pm\frac{b}{2}}=0\end{aligned}\right\} \tag{4.19b～e}$$

表1 受均布荷载的四边简支方板$\mu=0.3$

	$\frac{h}{a}$	经典板	Reissner	Srinivas at al	初始函数法(MIF)	本文
$\left.\frac{2GW_0}{qh}\right\|_{x=0,y=0}$	0.14	88.827	96.497	96.801	96.800	96.8333
	0.10	341.22	356.27	356.90	356.87	356.931
	0.05	5459.5	5519.9	5522.5	5523.0	5522.58
$\left.\frac{\sigma_x}{q}\right\|_{x=0,y=0,z=h/2}$	0.14	14.659	14.749	14.946	14.910	14.8779
	0.10	28.732	25.822	28.998	28.983	28.9507
	0.05	114.93	115.02	115.26	115.20	115.147

利用(4.17)、(4.18)、(4.19)的已知经典理论解，就得到了这个问题的横观各向同性解。为了与已知结果比较，我们计算了各向同性方形板的数值结果。将板的最大挠度和最大应力分量列于表1。比较结果表明，我们的解与三维弹性理论精确解[5]、[6]的结果非常接近。

五、对中厚板理论的讨论

为了着重讨论各种中厚板理论之间的差别，限定横向荷载 $q(x,y)$ 为均布荷载。我们可以将各种中厚板理论的解分解为两部分，并写成统一的表达式如下

1. $$D\nabla^4 W=q \tag{5.1}$$

$$\omega_x=-\frac{\partial W}{\partial x}-\frac{k_1}{1-\mu}\frac{h^2}{5}\frac{\partial}{\partial x}\nabla^2 W \tag{5.2a}$$

$$\omega_y=-\frac{\partial W}{\partial y}-\frac{k_1}{1-\mu}\frac{h^2}{5}\frac{\partial}{\partial y}\nabla^2 W \tag{5.2b}$$

$$M_x=-D\left(\frac{\partial^2 W}{\partial x^2}+\mu\frac{\partial^2 W}{\partial y^2}+k_1\frac{h^2}{5}\frac{\partial^2}{\partial x^2}\nabla^2 W\right)-k_2\frac{\mu}{1-\mu}\frac{qh^2}{10} \tag{5.2c}$$

$$M_y=-D\left(\frac{\partial^2 W}{\partial y^2}+\mu\frac{\partial^2 W}{\partial x^2}+k_1\frac{h^2}{5}\frac{\partial^2}{\partial y^2}\nabla^2 W\right)-k_2\frac{\mu}{1-\mu}\frac{qh^2}{10} \tag{5.2d}$$

$$M_{xy}=-D\left[(1-\mu)\frac{\partial^2 W}{\partial x\partial y}+k_1\frac{h^2}{5}\frac{\partial^2}{\partial x\partial y}\nabla^2 W\right] \tag{5.2e}$$

$$Q_x=-D\frac{\partial}{\partial x}\nabla^2 W,\quad Q_y=-D\frac{\partial}{\partial y}\nabla^2 W \tag{5.2f,g}$$

2. $$\psi-k_3\frac{h^2}{10}\nabla^2\psi=0 \tag{5.3}$$

$$\omega_x=k_3\frac{12}{5}\frac{1+\mu}{E}\frac{\partial\psi}{\partial y},\quad \omega_y=-k_3\frac{12}{5}\frac{1+\mu}{E}\frac{\partial\psi}{\partial x} \tag{5.4a,b}$$

$$M_x=-M_y=k_3\frac{h^2}{5}\frac{\partial^2\psi}{\partial x\partial y},\quad M_{xy}=k_3\frac{h^2}{10}\left(\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right) \tag{5.4c,d}$$

$$Q_x=\frac{\partial\psi}{\partial y},\quad Q_y=-\frac{\partial\psi}{\partial x} \tag{5.4e,f}$$

各种中厚板理论的系数k_1，k_2，k_3值见表2。

将本文提出的各向同性厚板理论按与 Reissner 理论[7]相同的形式沿 z 方向进行加权平均，就得到了退化理论Ⅰ。我们发现 Reissner 理论与退化理论Ⅰ关于挠度解W的系数完全

表2 各种中厚板理论的 k_1，k_2，k_3值

	本文厚板理论	退化理论Ⅰ	Reissner 理论	退化理论Ⅱ	Mindlin 理论	Panc等一类理论	Hencky理论
k_1	$1+\frac{\mu}{8}$	1	1	1	$\frac{10}{\pi^2}$	1	$\frac{5}{6}$
k_2	$1+\frac{\mu}{4}$	1	1	2	$2\left(\frac{10}{\pi^2}\right)$	2	$2\left(\frac{5}{6}\right)$
k_3	$\frac{10}{\pi^2}$	$\frac{10}{\pi^2}$	1	$\frac{10}{\pi^2}$	$\frac{10}{\pi^2}$	1	$\frac{5}{6}$

相同；而对剪切解ψ，前者的系数是1，后者的系数是$10/\pi^2$，略有差异。所以 Reissner 理论与退化理论Ⅰ基本相同。

除Reissner理论外，还有Hencky理论[8]、Mindlin理论[9]和Panc理论[10]、Власов理论[11]、修正的Hencky理论[10]、修正的Reissner理论[12]。它们的共同特点是忽略ε_z，σ_z的影响，但在考虑横向剪切变形的影响时有不同的假设。在本文提出的厚板理论中，令$E'\to\infty$，$G'=G$，即有

$$k_1=1,\ k_2=0$$

就得到了退化理论Ⅱ，它退化到了各向同性的情况，考虑了横向剪切变形的影响，但未计及ε_z，σ_z的影响。退化理论Ⅱ为这一大类中厚板理论提供了数学上的依据。对于Panc等一类理论，其系数与退化理论Ⅱ相比较，解的第一部分完全相同，只是在解的第二部分中存在1与$10/\pi^2$的差别。非常有趣，Mindlin理论与退化理论Ⅱ相比，正好相反，解的第二部分系数相同，解的第一部分，对应系数存在$10/\pi^2$与1的差别。Hencky理论与退化理论Ⅱ相比，其对应系数的差别将比上面提到的理论大一些。总之，退化理论Ⅱ包含这一大类中厚板理论。

当横向荷载为均布时，本文提出的厚板理论解满足弹性理论的全部方程，只是边界条件是以应力的合力、中面位移、平均转角来描述的。所以它作为比较的尺度是合理的。从表2可以看出，Reissner理论是比较好的中厚板理论，其次是Panc等一类理论、Mindlin理论，再次是Hencky理论。当q为一般函数时，厚板理论与弹性理论相比，差别在于对荷载有关的特解项作了量级简化。考虑到板的尺寸特征，厚板理论的解在满足全部弹性力学基本方程方面，这种简化所带来的误差很小，所以本文提出的厚板理论最靠近弹性理论。另外应指出的是对一般荷载，当厚跨比较大时，要精确的分析应力情况，关于中面的正对称变形将不能忽略。

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Theory and Refined Theory of Elasticity for Transversely Isotropic Plates and a New Theory for Thick Plates

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Abstract

A theory of elasticity for the bending of transversely isotropic plates has been developed from the basic equations of elasticity in terms of displacements for transversely isotropic bodies, which takes into account the loads distributed over the surfaces of the plates. Based on this theory, a refined theory of plates which can satisfy three boundary conditions along each edge of the plates and a new theory of thick plates are established. The solution of the refined theory for simply supported polygonal plates has been obtained; and its numerical result is very close to the exact solution of the three-dimensional theory of elasticity. A systematic comparison with the former theories of thick plates shows that the present theory of thick plates is closest to the result of the theory of elasticity.