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对称铺设各向异性叠层矩形板的非线性弯曲*

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摘 要

本文研究了对称铺设各向异性矩形叠层板在各种支承条件下的非线性弯曲．应用奇异摄动方法[1]导出了挠度和应力函数的一致有效的N阶渐近解．对承受边缘张力和侧向载荷的简支矩形叠层板应用奇摄动方法和改进了的迦辽金方法（一种加权残数法）进行了分析和计算．

一、问 题 的 提 出

对于对称叠层板，所有的耦合刚度(即B_{ij})为零．在弯曲和拉伸之间无耦合产生．因此，对称叠层板普遍地被采用．对于多重特殊正交对称铺设叠层板的非线性弯曲笔者已在文[3]中进行了研究．本文则研究多重各向异性、对称铺设的叠层板（包括多重一般正交铺设，例如正规对称角铺设叠层板）的非线性弯曲．由于剪切和扭转耦合刚度A_{16}，A_{26}和D_{16}，D_{26}的存在，使分析复杂化了．对于早期工作，读者可参考文献[4～6]．本文则应用奇摄动方法[1,2]，使问题得以简化．

考虑一在x方向取长度为a、y方向取宽度为b以及z方向取厚度为t的矩形薄板．取板未变形前的中面为xy平面．假设板由N层均匀的各向异性单层胶合而成，每层可有任意的厚度和弹性性质；且其正交轴关于板的坐标轴可有任意的定向．

对于对称叠层板的合力和弯矩是[7]

$$\begin{Bmatrix} N \\ M \end{Bmatrix}=\begin{bmatrix} A & \\ & D \end{bmatrix}\begin{Bmatrix} \varepsilon^0 \\ k \end{Bmatrix} \tag{1.1}$$

式中

$$\left.\begin{aligned} &\{N\}:\ 合力=[N_x,N_y,N_{xy}]^T \\ &\{M\}:\ 合力矩=[M_x,M_y,M_{xy}]^T \\ &\{\varepsilon^0\}:\ 中面应变=\{\varepsilon_x^0,\ \varepsilon_y^0,\ \gamma_{xy}^0\}^T \\ &\{k\}:\ 曲率=[k_x,k_y,k_{xy}]^T \end{aligned}\right|$$

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$$\left.\begin{aligned}&\{A_{ij}\}：拉伸刚度=\sum_{k=1}^{n}(\bar{Q}_{ij})_k(z_k-z_{k-1})\\&\{D_{ij}\}：弯曲刚度=\frac{1}{3}\sum_{k=1}^{n}(\bar{Q}_{ij})_k(z_k^3-z_{k-1}^3)\\&\qquad\qquad(i,j=1,2,6)\end{aligned}\right\}\tag{1.2}$$

或

$$\begin{Bmatrix}\varepsilon^0\\M\end{Bmatrix}=\begin{bmatrix}A^* & \\ & D\end{bmatrix}\begin{Bmatrix}N\\k\end{Bmatrix}\tag{1.3}$$

假设

$$A_{ij}^*=a_{ij}^*/t；\ D_{ij}=d_{ij}\cdot t^3\tag{1.4}$$

引入应力函数φ，使

$$N_x=\varphi_{,yy}\qquad N_y=\varphi_{,xx}\qquad N_{xy}=-\varphi_{,xy}\tag{1.5}$$

则基本非线性微分方程为

$$\begin{bmatrix}L_1^* & \\ & L_2^*\end{bmatrix}\begin{Bmatrix}w\\\varphi\end{Bmatrix}=\begin{Bmatrix}L(\varphi,w)+q(x,y)\\-\frac{1}{2}L(w,w)\end{Bmatrix}\tag{1.6}$$

式中

$$\left.\begin{aligned}L_1^*=&D_{11}\frac{\partial^4}{\partial x^4}+4D_{16}\frac{\partial^4}{\partial x^3\partial y}+2(D_{12}\\&+2D_{66})\frac{\partial^4}{\partial x^2\partial y^2}+4D_{26}\frac{\partial^4}{\partial x\partial y^3}+D_{22}\frac{\partial^4}{\partial y^4}\\L_2^*=&A_{22}^*\frac{\partial^4}{\partial x^4}-2A_{26}^*\frac{\partial^4}{\partial x^3\partial y}+(2A_{12}^*+A_{66}^*)\frac{\partial^4}{\partial x^2\partial y^2}\\&-2A_{16}^*\frac{\partial^4}{\partial x\partial y^3}+A_{11}^*\frac{\partial^4}{\partial y^4}\\L=&\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial y^2}\frac{\partial^2}{\partial x^2}-2\frac{\partial^2}{\partial x\partial y}\frac{\partial^2}{\partial x\partial y}\end{aligned}\right\}\tag{1.7}$$

引入下列无量纲量 $\tilde{w}=\dfrac{w}{a}$，$\tilde{x}=\dfrac{x}{a}$，$\tilde{y}=\dfrac{y}{a}$，$\tilde{\varphi}=\dfrac{\varphi A_{11}^*}{a^2}$，$\tilde{q}=aA_{11}^*q$

和小参数

$$\varepsilon=\sqrt{a_{11}^*d_{11}}\,t/a\tag{1.8}$$

于是方程组(1.6)化为（略去了字母上的"～"号）：

$$\begin{bmatrix}\varepsilon^2L_1 & \\ & L_2\end{bmatrix}\begin{Bmatrix}w\\\varphi\end{Bmatrix}=\begin{Bmatrix}L(w,\varphi)+q(x,y)\\-\frac{1}{2}L(w,w)\end{Bmatrix}\tag{1.9}$$

式中

$$\left.\begin{aligned}&L_i=a_i\frac{\partial^4}{\partial x^4}+b_i\frac{\partial^4}{\partial x^3\partial y}+c_i\frac{\partial^4}{\partial x^2\partial y^2}+d_i\frac{\partial^4}{\partial x\partial y^3}+e_i\frac{\partial^4}{\partial y^4}\\&a_1=1,\ b_1=\frac{4D_{16}}{D_{11}},\ c_1=\frac{2(D_{12}+2D_{66})}{D_{11}},\ d_{11}=\frac{4D_{26}}{D_{11}},\ e_1=\frac{D_{22}}{D_{11}}\\&a_2=\frac{A_{22}^*}{A_{11}^*},\ b_2=-\frac{2A_{26}^*}{A_{11}^*},\ c_2=\frac{2A_{12}^*+A_{66}^*}{A_{11}^*},\ d_2=\frac{2A_{16}^*}{A_{11}^*},\ e_2=1\end{aligned}\right\}\tag{1.10}$$

假设挠度w的边界条件为

$$\left.\begin{aligned}
&w\Big|_{x=0}=f_1(y),\quad \frac{\partial w}{\partial x}\Big|_{x=0}=g_1(y)\\
&w\Big|_{x=1}=f_2(y),\quad \frac{\partial w}{\partial x}\Big|_{x=1}=g_2(y)\\
&w\Big|_{y=0}=f_3(x),\quad -\frac{D_{21}}{D_{22}}\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial x^2}-2\frac{D_{26}}{D_{22}}\frac{\partial^2 w}{\partial x\partial y}\Big|_{y=0}=g_3(x)\\
&w\Big|_{y=\frac{b}{a}}=f_4(x),\quad -\frac{D_{21}}{D_{22}}\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial y^2}-2\frac{D_{26}}{D_{22}}\frac{\partial^2 w}{\partial x\partial y}\Big|_{y=\frac{b}{a}}=g_4(x)
\end{aligned}\right\}\tag{1.11}$$

应力函数φ的边界条件为

$$\left.\begin{aligned}
&\frac{\partial^2\varphi}{\partial y^2}\Big|_{x=0}=h_1(y),\quad \frac{\partial^2\varphi}{\partial x\partial y}\Big|_{x=0}=I_1(y)\\
&\frac{\partial^2\varphi}{\partial y^2}\Big|_{x=1}=h_2(y),\quad \frac{\partial^2\varphi}{\partial x\partial y}\Big|_{x=1}=I_2(y)\\
&\frac{\partial^2\varphi}{\partial x^2}\Big|_{y=0}=h_3(x),\quad \frac{\partial^2\varphi}{\partial x\partial y}\Big|_{y=0}=I_3(x)\\
&\frac{\partial^2\varphi}{\partial x^2}\Big|_{y=\frac{b}{a}}=h_4(x),\quad \frac{\partial^2\varphi}{\partial x\partial y}\Big|_{y=\frac{b}{a}}=I_4(x)
\end{aligned}\right\}\tag{1.12}$$

或

$$\int_0^{b/a}t\frac{\partial^2\varphi}{\partial y^2}dy=\bar{P}_x\cdot t\cdot\frac{b}{a},\quad \int_0^1 t\frac{\partial^2\varphi}{\partial x^2}dx=\bar{P}_y\cdot t\tag{1.13}$$

$$\left.\begin{aligned}
&\int_0^1\left[\frac{\partial^2\varphi}{\partial y^2}+\frac{A^*_{12}}{A^*_{11}}\frac{\partial^2\varphi}{\partial x^2}-\frac{A^*_{16}}{A^*_{11}}\frac{\partial^2\varphi}{\partial x\partial y}-\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]dx=\delta_x\\
&\int_0^{b/a}\left[\frac{A^*_{21}}{A^*_{11}}\frac{\partial^2\varphi}{\partial y^2}+\frac{A^*_{22}}{A^*_{11}}\frac{\partial^2\varphi}{\partial x^2}-\frac{A^*_{26}}{A^*_{11}}\frac{\partial^2\varphi}{\partial x\partial y}-\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]dy=\delta_y
\end{aligned}\right\}\tag{1.14}$$

式中$\bar{P}_x tb/a$和$\bar{P}_y t$分别为板在板平面内，在x方向和在y方向所承受的载荷；δ_x和δ_y则分别为板在x方向和y方向的伸长。

二、微 分 算 子 展 开 式

我们分别在边界$x=0$和$x=1$的邻域内引进新变量

$$\xi=\frac{u(x,y)}{\varepsilon},\quad \eta=x\tag{2.1}$$

和

$$\tilde{\xi}=\frac{\tilde{u}(x,y)}{\varepsilon},\quad \tilde{\eta}=x\tag{2.2}$$

式中$u(x,y)$和$\tilde{u}(x,y)$是待定函数。则微分算子$L_i(i=1,2)$在$x=0$和$x=1$的展开式分别为

$$L_i\equiv\varepsilon^{-4}\sum_{j=0}^{4}\varepsilon^i D_{ij}\tag{2.3}$$

$$\tilde{L}_i\equiv\varepsilon^{-4}\sum_{j=0}^{4}\varepsilon^i\tilde{D}_{ij}\tag{2.4}$$

式中

$$\left.\begin{aligned}&D_{i0}=a_i\delta_{4,0},\quad D_{i1}=a_i\delta_{4,1}+b_i\delta_{3,0}\frac{\partial}{\partial y}\\&D_{i2}=a_i\delta_{4,2}+b_i\delta_{3,1}\frac{\partial}{\partial y}+c_i\delta_{2,0}\frac{\partial^2}{\partial y^2}\\&D_{i3}=a_i\delta_{4,3}+b_i\delta_{3,2}\frac{\partial}{\partial y}+c_i\delta_{2,1}\frac{\partial^2}{\partial y^2}+d_i\delta_{1,0}\frac{\partial^3}{\partial y^3}\\&D_{i4}=a_i\delta_{4,4}+b_i\delta_{3,3}\frac{\partial}{\partial y}+c_i\delta_{2,2}\frac{\partial^2}{\partial y^2}+d_i\delta_{1,1}\frac{\partial^3}{\partial y^3}+e_i\frac{\partial^4}{\partial y^4}\end{aligned}\right\}\tag{2.5}$$

$$\left.\begin{aligned}&\tilde{D}_{i0}=a_i\tilde{\delta}_{4,0}\\&\cdots\cdots\\&\tilde{D}_{i4}=a_i\tilde{\delta}_{4,4}+b_i\tilde{\delta}_{3,3}\frac{\partial}{\partial y}+c_i\tilde{\delta}_{2,2}\frac{\partial^2}{\partial y^2}+d_i\tilde{\delta}_{1,1}\frac{\partial^3}{\partial y^3}+e_i\frac{\partial^4}{\partial y^4}\end{aligned}\right\}\tag{2.6}$$

同样地，在边界$y=0$和$y=b/a$的邻域内分别引进新变量

$$\alpha=p(x,y)/\varepsilon,\quad \beta=y\tag{2.7}$$

和

$$\tilde{\alpha}=\tilde{p}(x,y)/\varepsilon,\quad \tilde{\beta}=y\tag{2.8}$$

式中$p(x,y)$和$\tilde{p}(x,y)$是待定函数。

则算子L_i在边界$y=0$的展开式为

$$L_i'\equiv\varepsilon^{-4}\sum_{j=0}^{4}\varepsilon^{i}D_{ij}'\tag{2.9}$$

在边界$y=b/a$的展开式为

$$\tilde{L}_i'\equiv\varepsilon^{-4}\sum_{j=0}^{4}\varepsilon^{i}\tilde{D}_{ij}'\tag{2.10}$$

式中

$$\left.\begin{aligned}&D_{i0}'=e_i\gamma_{4,0},\quad D_{i1}'=e_i\gamma_{4,1}+d_i\gamma_{3,0}\frac{\partial}{\partial x}\\&D_{i2}'=e_i\gamma_{4,2}+d_i\gamma_{3,1}\frac{\partial}{\partial x}+c_i\gamma_{2,0}\frac{\partial^2}{\partial x^2}\\&D_{i3}'=e_i\gamma_{4,3}+d_i\gamma_{3,2}\frac{\partial}{\partial x}+c_i\gamma_{2,1}\frac{\partial^2}{\partial x^2}+b_i\gamma_{1,0}\frac{\partial^3}{\partial x^3}\\&D_{i4}'=e_i\gamma_{4,4}+d_i\gamma_{3,3}\frac{\partial}{\partial x}+c_i\gamma_{2,2}\frac{\partial^2}{\partial x^2}+b_i\gamma_{1,1}\frac{\partial^3}{\partial x^3}+a_i\frac{\partial^4}{\partial x^4}\end{aligned}\right\}\tag{2.11}$$

$$\left.\begin{aligned}&\tilde{D}_{i0}'=e_i\tilde{\gamma}_{4,0}\qquad\cdots\cdots\\&\tilde{D}_{i4}'=e_i\tilde{\gamma}_{4,4}+d_i\tilde{\gamma}_{3,3}\frac{\partial}{\partial x}+c_i\tilde{\gamma}_{2,2}\frac{\partial^2}{\partial x^2}+b_i\tilde{\gamma}_{1,1}\frac{\partial^3}{\partial x}+a_i\frac{\partial^4}{\partial x^4}\end{aligned}\right\}\tag{2.12}$$

对于微分算子L的展开式，建议读者参考文献[9]。

三、递推公式和边界条件

假设挠度w和应力函数φ的N阶渐近展开式分别为

$$W_N(x,y,\varepsilon)=\sum_{n=0}^{N}\varepsilon^n w_n(x,y)+\sum_{n=0}^{N}\varepsilon^{n+\alpha_1}v_n^{(1)}(\xi,\eta,y)+\sum_{n=0}^{N}\varepsilon^{n+\alpha_2}v_n^{(2)}(\tilde{\xi},\tilde{\eta},y)$$

$$+\sum_{n=0}^{N}\varepsilon^{n+\alpha_3}v_n^{(3)}(x,\alpha,\beta)+\sum_{n=0}^{N}\varepsilon^{n+\alpha_4}v_n^{(4)}(x,\tilde{\alpha},\tilde{\beta}) \tag{3.1}$$

$$\Phi_N(x,y,\varepsilon)=\sum_{n=0}^{N}\varepsilon^n\varphi_n(x,y)+\sum_{n=0}^{N}\varepsilon^{n+\beta_1}h_n^{(1)}(\xi,,\eta y)+\sum_{n=0}^{N}\varepsilon^{n+\beta_2}h_n^{(2)}(\tilde{\xi},\tilde{\eta},y)$$

$$+\sum_{n=0}^{N}\varepsilon^{n+\beta_3}h_n^{(3)}(x,\alpha,\beta)+\sum_{n=0}^{N}\varepsilon^{n+\beta_4}h_n^{(4)}(x,\tilde{\alpha},\tilde{\beta}) \tag{3.2}$$

式中$v_n^{(1)}$，$h_n^{(1)}$和$v_n^{(2)}$，$h_n^{(2)}$分别为在$x=0$和$x=1$邻域内的边界层函数；$v_n^{(3)}$，$h_n^{(3)}$和$v_n^{(4)}$，$h_n^{(4)}$分别为在$y=0$和$y=b/a$邻域内的边界层函数。$\alpha_1,\cdots,\alpha_4$；$\beta_1,\cdots,\beta_4$则为待定常数。

将方程(3.1)和(3.2)代入方程组(1.9)以及边界条件(1.11)和(1.12)，立即得出$\alpha_1=\alpha_2=1$，$\alpha_3=\alpha_4=2$，$\beta_1=\beta_2=3$，$\beta_3=\beta_4=4$和关于w_n，φ_n以及$v_n^{(i)}$和$h_n^{(i)}(i=1,\cdots,4)$[9]的递推公式和边界条件。对于w_n，φ_n的递推公式为

$$L(w_0,\varphi_0)+q=0,\ L_2\varphi_0+2^{-1}L(w_0,w_0)=0 \tag{3.3}$$

$$\left.\begin{aligned}&L(w_0,\varphi_n)+L(w_n,\varphi_0)=L_1w_{n-2}-\sum_{i=1}^{n-1}L(w_i,\varphi_{n-i})\\&L_2\varphi_n+L(w_0,w_n)=-\frac{1}{2}\sum_{i=1}^{n-1}L(w_i,w_{n-i})\quad(n=1,2,\cdots,N)\end{aligned}\right\} \tag{3.4}$$

$v_n^{(i)}$和$h_n^{(i)}$的递推公式的导出已在文[9]中给出。其边界条件是

$$\left.\begin{aligned}
&w_0|_{x=0}=f_1(y) \qquad w_n|_{x=0}+v_{n-1}^{(1)}|_{\eta=0}=0\\
&w_{0,x}|_{x=0}+\delta_{1,0}v_0^{(1)}|_{\eta=0}=g_1(y) \qquad w_{n,x}|_{x=0}+(\delta_{1,0}v_n^{(1)}+\delta_{1,1}v_{n-1}^{(1)})|_{\eta=0}=0\\
&w_0|_{x=1}=f_2(y) \qquad w_n|_{x=1}+v_{n-1}^{(2)}|_{\tilde{\eta}=1}=0\\
&w_{0,x}|_{x=1}+\tilde{\delta}_{1,0}v_0^{(2)}|_{\tilde{\eta}=1}=g_2(y) \qquad w_{n,x}|_{x=1}+(\tilde{\delta}_{1,0}v_n^{(2)}+\tilde{\delta}_{1,1}v_{n-1}^{(2)})|_{\tilde{\eta}=1}=0\\
&w_0|_{y=0}=f_3(x) \qquad w_n|_{y=0}+v_{n-2}^{(3)}|_{\beta=0}=0\\
&-\frac{D_{21}}{D_{22}}w_{0,xx}-w_{0,yy}-2\frac{D_{26}}{D_{22}}w_{0,xy}|_{y=0}-\gamma_{2,0}v_0^{(3)}|_{\beta=0}=g_3(x)\\
&-\frac{D_{21}}{D_{22}}w_{n,xx}-w_{n,yy}-2\frac{D_{26}}{D_{22}}w_{n,xy}|_{y=0}-\Big[\frac{D_{21}}{D_{22}}v_{n-2,xx}^{(3)}+\gamma_{2,0}v_n^{(3)}+\gamma_{2,1}v_{n-1}^{(3)}\\
&\qquad+\gamma_{2,2}v_{n-2}^{(3)}+2\frac{D_{26}}{D_{22}}(\gamma_{1,0}v_{n-1,x}^{(3)}+\gamma_{1,1}v_{n-2,x}^{(3)})\Big]_{\beta=0}=0\\
&w_0|_{y=b/a}=f_4(x) \qquad w_n|_{y=b/a}+v_{n-2}^{(4)}|_{\tilde{\beta}=b/a}=0\\
&-\frac{D_{21}}{D_{22}}w_{0,xx}-w_{0,yy}-2\frac{D_{26}}{D_{22}}w_{0,xy}|_{y=b/a}-\tilde{\gamma}_{2,0}v_0^{(4)}|_{\tilde{\beta}=b/a}=g_4(x)\\
&-\frac{D_{21}}{D_{22}}w_{n,xx}-w_{n,yy}-2\frac{D_{26}}{D_{22}}w_{n,xy}|_{y=b/a}-\Big[\frac{D_{21}}{D_{22}}v_{n-2,xx}^{(4)}+\tilde{\gamma}_{2,0}v_n^{(4)}+\tilde{\gamma}_{2,1}v_{n-1}^{(4)}\\
&\qquad+\tilde{\gamma}_{2,2}v_{n-2}^{(4)}+2\frac{D_{26}}{D_{22}}(\tilde{\gamma}_{1,0}v_{n-1,x}^{(4)}+\tilde{\gamma}_{1,1}v_{n-2,x}^{(4)})|_{\tilde{\beta}=b/a}=0\\
&\qquad\qquad(n=1,2,\cdots,N)
\end{aligned}\right\} \tag{3.5}$$

$$\left.\begin{array}{ll}
\varphi_{0,yy}|_{x=0}=h_1(y) & \varphi_{n,yy}|_{x=0}+h^{(1)}_{n-3,yy}|_{\eta=0}=0 \\
\varphi_{0,xy}|_{x=0}=I_1(y) & \varphi_{n,xy}|_{x=0}+[\delta_{1,0}h^{(1)}_{n-2,y}+\delta_{1,1}\,h^{(1)}_{n-3,y}]_{\eta=0}=0 \\
\varphi_{0,yy}|_{x=1}=h_2(y) & \varphi_{n,yy}|_{x=1}+h^{(2)}_{n-3,yy}|_{\tilde\eta=1}=0 \\
\varphi_{0,xy}|_{x=1}=I_2(y) & \varphi_{n,xy}|_{x=1}+[\tilde\delta_{1,0}\,h^{(2)}_{n-2,y}+\tilde\delta_{1,1}\,h^{(2)}_{n-3,y}]_{\tilde\eta=1}=0 \\
\varphi_{0,xx}|_{y=0}=h_3(x) & \varphi_{n,xx}|_{y=0}+h^{(3)}_{n-4,xx}|_{\beta=0}=0 \\
\varphi_{0,xy}|_{y=0}=I_3(x) & \varphi_{n,xy}|_{y=0}+[\gamma_{1,0}\,h^{(3)}_{n-3,x}+\gamma_{1,1}\,h^{(3)}_{n-4,x}]_{\beta=0}=0 \\
\varphi_{0,xx}|_{y=b/a}=h_4(x) & \varphi_{n,xx}|_{y=b/a}+h^{(4)}_{n-4,xx}|_{\tilde\beta=b/a}=0 \\
\varphi_{0,xy}|_{y=b/a}=I_4(x) & \varphi_{n,xy}|_{y=b/a}+[\tilde\gamma_{1,0}h^{(4)}_{n-3,x}+\tilde\gamma_{1,1}h^{(4)}_{n-4,x}]_{\tilde\beta=b/a}=0 \\
 & (n=1,2,\cdots,N)
\end{array}\right\} \quad (3.6)$$

或

$$\left.\begin{array}{ll}
\int_0^{b/a} t\varphi_{0,yy}dy=\bar P_x\cdot\frac{b}{a}t & \int_0^{b/a} t[\varphi_{n,yy}+h^{(1)}_{n-3,yy}]dy=0 \\
 & \int_0^{b/a} t[\varphi_{n,yy}+h^{(2)}_{n-3,yy}]dy=0 \\
\int_0^1 t\varphi_{0,xx}dy=\bar P_y\cdot t & \int_0^1 t[\varphi_{n,xx}+h^{(3)}_{n-4,xx}]dx=0 \\
 & \int_0^1 t[\varphi_{n,xx}+h^{(4)}_{n-4,xx}]dx=0
\end{array}\right\} \quad (3.7)$$

$$\left.\begin{array}{l}
\int_0^1\left[\varphi_{0,yy}+\frac{A^*_{12}}{A^*_{11}}\varphi_{0,xx}-\frac{A^*_{16}}{A^*_{11}}\varphi_{0,xy}-\frac{1}{2}(w_{0,x})^2\right]dx=\delta_x \\
\int_0^1\Big\{(\varphi_{n,yy}+h^{(3)}_{n-4,yy})+\frac{A^*_{12}}{A^*_{11}}(\varphi_{n,xx}+h^{(3)}_{n-4,xx})-\frac{A^*_{16}}{A^*_{11}}(\varphi_{n,xy}+h^{(3)}_{n-4,xy}) \\
\qquad -\frac{1}{2}\Big[w^2_{n/2,x}+2\sum_{i=0}^{n}(w_{i,x}w_{n-i,x}+w_{i,x}v^{(3)}_{n-2-i,x})\Big]\Big\}dx=0 \\
\int_0^1\Big\{(\varphi_{n,yy}+h^{(4)}_{n-4,yy})+\frac{A^*_{12}}{A^*_{11}}(\varphi_{n,xx}+h^{(4)}_{n-4,xx})-\frac{A^*_{16}}{A^*_{11}}(\varphi_{n,xy}+h^{(4)}_{m-4,xy}) \\
\qquad -\frac{1}{2}\Big[w^2_{n/2,x}+2\sum_{i=0}^{n}(w_{i,x}w_{n-i,x}+w_{i,x}v^{(4)}_{n-2-i,x})\Big]\Big\}dx=0 \\
\int_0^{b/a}\left[\frac{A^*_{21}}{A^*_{11}}\varphi_{0,yy}+\frac{A^*_{22}}{A^*_{11}}\varphi_{0,xx}-\frac{A^*_{26}}{A^*_{11}}\varphi_{0,xy}-\frac{1}{2}(w_{0,y})^2\right]dy=\delta_y \\
\int_0^{b/a}\Big\{\frac{A^*_{21}}{A^*_{11}}(\varphi_{n,yy}+h^{(1)}_{n-3,yy})+\frac{A^*_{22}}{A^*_{11}}\varphi_{n,xx}+h^{(1)}_{n-3,xx})\frac{A^*_{26}}{A^*_{11}}(\varphi_{n,xy} \\
\qquad +h^{(1)}_{n-3,xy})-\frac{1}{2}\Big[w^2_{n/2,y}+2\sum_{i=0}^{n}(w_{i,y}w_{n-i,y}+w_{i,y}v^{(1)}_{n-1-i,y})\Big]\Big\}dy=0 \\
\int_0^{b/a}\Big\{\frac{A^*_{21}}{A^*_{11}}(\varphi_{n,yy}+h^{(2)}_{n-3,yy})+\frac{A^*_{22}}{A^*_{11}}(\varphi_{n,xx}+h^{(2)}_{n-3,xx})-\frac{A^*_{26}}{A^*_{11}}(\varphi_{n,xy} \\
\qquad +h^{(2)}_{n-3,xy})-\frac{1}{2}\Big[w^2_{n/2,y}+2\sum_{i=0}^{n}(w_{i,y}w_{n-i,y}+w_{i,y}v^{(2)}_{n-1-i,y})\Big]\Big\}dy=0 \\
\qquad (n=1,2,\cdots,N)
\end{array}\right\} \quad (3.8)$$

注意在本文中所有带负下标的量都取为零。

四、形式渐近解的导出

利用以上的递推方程和边界条件，我们可连续地导出板在各种支承条件下的N阶形式渐近解。例如，对于两对边简支、两对边固支的对称铺设的各向异性叠层板，利用方程(3.3)和相应的边界条件求得薄膜解w_0，φ_0后，边界层函数$v_0^{(i)}$和$h_0^{(i)}(i=1,2,3,4)$即可求得[9]。而待定函数$C_0^{(i)}(i=1,2,3,4)$则可从下列微分方程

$$\left.\begin{aligned}
&2A\frac{\partial C_0^{(1)}}{\partial\eta}+(2\varphi_{0,xy}+b_1A)\frac{\partial C_0^{(1)}}{\partial y}+\left[\frac{5}{2}A_{,x}+\varphi_{1,yy}A^{1/2}\right]C_0^{(1)}=0\\
&2A\frac{\partial C_0^{(2)}}{2\tilde{\eta}}+(2\varphi_{0,xy}+b_1A)\frac{\partial C_0^{(2)}}{\partial y}+\left[\frac{5}{2}A_{,x}+\varphi_{1,yy}A^{1/2}\right]C_0^{(2)}=0\\
&2B\frac{\partial C_0^{(3)}}{\partial\beta}+\left(2\varphi_{0,xy}+\frac{d_1}{e_1}B\right)\frac{\partial C_0^{(3)}}{\partial x}+\left[\frac{5}{2}B_{,y}+\varphi_{1,xx}\left(\frac{B}{e_1}\right)^{1/2}\right]C_0^{(3)}=0\\
&2B\frac{\partial C_0^{(4)}}{\partial\tilde{\beta}}+\left(2\varphi_{0,xy}+\frac{d_1}{e_1}B\right)\frac{\partial C_0^{(4)}}{\partial x}+\left[\frac{5}{2}B_{,y}+\varphi_{1,xx}\left(\frac{B}{e_1}\right)^{1/2}\right]C_0^{(4)}=0
\end{aligned}\right\}\tag{4.1}$$

和边界条件

$$\left.\begin{aligned}
&C_0^{(1)}(\eta,y)|_{\eta=0}=-h_1^{-1/2}(y)[g_1(y)-w_{0,x}(0,y)]\\
&C_0^{(2)}(\tilde{\eta},y)|_{\tilde{\eta}=1}=-h_2^{-1/2}(y)[g_2(y)-w_{0,x}(1,y)]\\
&C_0^{(3)}(x,\beta)|_{\beta=0}=-\frac{e_1}{D_{22}h_3(x)}[D_{22}g_3(x)+D_{12}w_{0,xx}(x,0)\\
&\qquad+D_{22}w_{0,yy}(x,0)+2D_{26}w_{0,xy}(x,0)]\\
&C_0^{(4)}(x,\tilde{\beta})|_{\tilde{\beta}=b/a}=-\frac{e_1}{D_{22}h_4(x)}\left[D_{22}g_4(x)+D_{12}w_{0,xx}\left(x,\frac{b}{a}\right)\right.\\
&\qquad\left.+D_{22}w_{0,yy}\left(x,\frac{b}{a}\right)+2D_{26}w_{0,xy}\left(x,\frac{b}{a}\right)\right]
\end{aligned}\right\}\tag{4.2}$$

求得。

将零阶渐近解代入递推方程和相应的边界条件（取$n=1$），则一阶渐近解w_1,φ_1和相应的边界层函数可求得为

$$\left.\begin{aligned}
&v_1^{(1)}(\xi,\eta,y)=C_1^{(1)}(\eta,y)\exp(-\xi)\\
&v_1^{(2)}(\tilde{\xi},\tilde{\eta},y)=C_1^{(2)}(\tilde{\eta},y)\exp(-\tilde{\xi})\\
&v_1^{(3)}(x,\alpha,\beta)=C_1^{(3)}(x,\beta)\exp(-\alpha)\\
&v_1^{(4)}(x,\tilde{\alpha},\tilde{\beta})=C_1^{(4)}(x,\tilde{\beta})\exp(-\tilde{\alpha})
\end{aligned}\right\}\tag{4.3}$$

$$\begin{aligned}
h_1^{(1)}(\xi,\eta,y)=&-\frac{1}{a_2A}\left\{w_{0,yy}C_1^{(1)}+\left[w_{1,yy}+\frac{5}{2}w_{0,yy}A^{-3/2}A_{,x}\right]C_0^{(1)}\right.\\
&+2w_{0,yy}A^{-1/2}\frac{\partial C_0^{(1)}}{\partial\eta}+\left(2w_{0,xy}+\frac{b_2}{a_2}w_{0,yy}\right)A^{-1/2}\frac{\partial C_0^{(1)}}{\partial y}\\
&\left.-\frac{1}{16}\left[\left(\frac{\partial C_0^{(1)}}{\partial y}\right)^2-\frac{\partial^2C_0^{(1)}}{\partial y^2}C_0^{(1)}\right]\exp[-\xi]\right\}\exp[-\xi]
\end{aligned}$$

$$\left.\begin{aligned}
h_1^{(2)}(\tilde{\xi},\tilde{\eta},y)=&-\frac{1}{a_2A}\Big\{w_{0,yy}C_1^{(2)}+\Big[w_{1,yy}+\frac{5}{2}w_{0,yy}A^{-3/2}A_{,x}\Big]C_0^{(2)}\\
&+2w_{0,yy}A^{-1/2}\frac{\partial C_0^{(2)}}{\partial\tilde{\eta}}+\Big(2w_{0,xy}+\frac{b_2}{a_2}w_{0,yy}\Big)A^{-1/2}\frac{\partial C_0^{(2)}}{\partial y}\\
&-\frac{1}{16}\Big[\Big(\frac{\partial C_0^{(2)}}{\partial y}\Big)^2-\frac{\partial^2C_0^{(2)}}{\partial y^2}C_0^{(2)}\Big][\exp[-\tilde{\xi}]\Big\}\exp[-\tilde{\xi}]\\
h_1^{(3)}(x,\alpha,\beta)=&-\frac{e_1}{B}\Big\{w_{0,xx}C_1^{(3)}+\Big[w_{1,xx}+\frac{5}{2}w_{0,xx}e_1^{1/2}B^{-3/2}B_{,y}\Big]C_0^{(3)}\\
&+2w_{0,xx}\Big(\frac{B}{e_1}\Big)^{-\frac{1}{2}}\frac{\partial C_0^{(3)}}{\partial\beta}+(2w_{0,xy}+d_2w_{0,xx})\Big(\frac{B}{e_1}\Big)^{-1/2}\frac{\partial C_0^{(3)}}{\partial x}\Big\}\exp[-\alpha]\\
h_1^{(4)}(x,\tilde{\alpha},\tilde{\beta})=&-\frac{e_1}{B}\Big\{w_{0,xx}C_1^{(4)}+\Big[w_{1,xx}+\frac{5}{2}w_{0,xx}e_1^{1/2}B^{-3/2}B_{,y}\Big]C_0^{(4)}\\
&+2w_{0,xx}\Big(\frac{B}{e_1}\Big)^{-\frac{1}{2}}\frac{\partial C_0^{(4)}}{\partial\tilde{\beta}}+(2w_{0,xy}+d_2w_{0,xx})\Big(\frac{B}{e_1}\Big)^{-1/2}\frac{\partial C_0^{(4)}}{\partial x}\Big\}\exp[-\tilde{\alpha}]
\end{aligned}\right\}\quad(4.4)$$

式中待定函数$C_1^{(i)}(i=1,\cdots,4)$由下列偏微分方程

$$\left.\begin{aligned}
&2A\frac{\partial C_1^{(1)}}{\partial\eta}+(2\varphi_{0,xy}+b_1A)\frac{\partial C_1^{(1)}}{\partial y}+\Big(\frac{5}{2}A_{,x}+\varphi_{1,yy}A^{1/2}\Big)C_1^{(1)}=f_1\\
&2A\frac{\partial C_1^{(2)}}{\partial\tilde{\eta}}+(2\varphi_{0,xy}+b_1A)\frac{\partial C_1^{(2)}}{\partial y}+\Big(\frac{5}{2}A_{,x}+\varphi_{1,yy}A^{1/2}\Big)C_1^{(2)}=f_2\\
&2B\frac{\partial C_1^{(3)}}{\partial\beta}+\Big(2\varphi_{0,xy}+\frac{d_1}{e_1}B\Big)\frac{\partial C_1^{(3)}}{\partial x}+\Big[\frac{5}{2}B_{,y}+\varphi_{1,xx}\Big(\frac{B}{e_1}\Big)^{1/2}\Big]C_1^{(3)}=f_3\\
&2B\frac{\partial C_1^{(4)}}{\partial\tilde{\beta}}+\Big(2\varphi_{0,xy}+\frac{d_1}{e_1}B\Big)\frac{\partial C_1^{(4)}}{\partial x}+\Big[\frac{5}{2}B_{,y}+\varphi_{1,xx}\Big(\frac{B}{e_1}\Big)^{1/2}\Big]C_1^{(4)}=f_4
\end{aligned}\right\}\quad(4.5)$$

和边界条件

$$\left.\begin{aligned}
&C_1^{(1)}(\eta,y)|_{\eta=0}=h_1^{-1/2}(y)\Big[w_{1,x}(0,y)+\frac{\partial C_1^{(1)}}{\partial\eta}(0,y)\Big]\\
&C_1^{(2)}(\tilde{\eta},y)|_{\tilde{\eta}=1}=h_2^{-1/2}(y)\Big[w_{1,x}(1,y)+\frac{\partial C_0^{(2)}}{\partial\tilde{\eta}}(1,y)\Big]\\
&C_1^{(3)}(x,\beta)|_{\beta=0}=-\frac{e_1}{D_{22}h_3(x)}[D_{12}w_{1,xx}(x,0)+D_{22}w_{1,yy}(x,0)\\
&\quad+2D_{26}w_{1,xy}(x,0)]+2e_1^{1/2}\Big[\Big(\frac{\partial C_0^{(3)}}{\partial\beta}+\frac{D_{26}}{D_{22}}\frac{\partial C_0^{(3)}}{\partial x}\Big)h_3^{-1/2}(x)\\
&\quad+\frac{1}{4}h_3^{-3/2}(x)B_{,y}C_0^{(3)}\Big]_{\beta=0}\\
&C_1^{(4)}(x,\tilde{\beta})|_{\tilde{\beta}=b/a}=-\frac{e_1}{D_{22}h_4(x)}\Big[D_{12}w_{1,xx}\Big(x,\frac{b}{a}\Big)+D_{22}w_{1,yy}\Big(x,\frac{b}{a}\Big)\\
&\quad+2D_{26}w_{1,xy}\Big(x,\frac{b}{a}\Big)\Big]+2e_1^{1/2}\Big[\Big(\frac{\partial C_0^{(4)}}{\partial\tilde{\beta}}+\frac{D_{26}}{D_{22}}\frac{\partial C_0^{(4)}}{\partial x}\Big)h_4^{-1/2}(x)\\
&\quad+\frac{1}{4}h_4^{-3/2}(x)B_{,y}C_0^{(4)}\Big]_{\tilde{\beta}=b/a}
\end{aligned}\right\}\quad(4.6)$$

定出。式中$f_i(i=1,\cdots,4)$分别为$C_0^{(i)}$的已知函数。

重复以上解的步骤，我们可逐次得到任意阶的形式渐近解。

五、承受边缘拉力和侧向载荷的矩形、对称铺设的各向异性的叠层板

对于对称铺设的各向异性矩形叠层板，在本构方程和边界条件中由于含有扭转耦和刚度D_{16}，D_{26}和剪切刚度A_{16}，A_{26}，致使不可能得到封闭解。这里，利用奇摄动方法连同改进了的伽辽金方法（一种加权残数法），对在边缘拉伸和侧向载荷联合作用下的四边简支的对称铺设各向异性叠层矩形板作了具体分析。

本构微分方程是

$$\begin{cases}\varepsilon^2 L_1 w - L(w,\varphi)=q(x,y) & (5.1a)\\ L_2\varphi + L(w,w)/2=0 & (5.1b)\end{cases}$$

其边界条件为

$$x=0,1:\ w=0,M_x=-\frac{\partial^2 w}{\partial x^2}-\frac{D_{12}}{D_{11}}\frac{\partial^2 w}{\partial y^2}-2\frac{D_{16}}{D_{11}}\frac{\partial^2 w}{\partial x\partial y}=0 \tag{5.2}$$

$$y=0,b/a:\ w=0,M_y=-\frac{D_{21}}{D_{22}}\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial y^2}-2\frac{D_{26}}{D_{22}}\frac{\partial^2 w}{\partial x\partial y}=0 \tag{5.3}$$

$$\int_0^{b/a} t\frac{\partial^2\varphi}{\partial y^2}dy=\bar{P}_x,\qquad \int_0^1 t\frac{\partial^2\varphi}{\partial x^2}dx=\bar{P}_y \tag{5.4}$$

零阶渐近解$(\varepsilon=0)$可从下列方程和边界条件确定

$$\begin{cases}L(w_0,\varphi_0)+q=0 & (5.5a)\\ L_2\varphi_0+L(w_0,w_0)/2=0 & (5.5b)\end{cases}$$

$$w_0(0,y)=w_0(1,y)=w_0(x,0)=w_0\left(x,\ \frac{b}{a}\right)=0 \tag{5.6}$$

$$\int_0^{b/a} t\varphi_{0,yy}dy=\bar{P}_x\qquad \int_0^1 t\varphi_{0,xx}dx=\bar{P}_y \tag{5.7}$$

选取挠度和应力函数分别为

$$w_0(x,y)=\sum_{r=1}^{R}\sum_{s=1}^{S} w_0^{rs}\sin r\pi x\sin\frac{sa\pi}{b} \tag{5.8}$$

$$\varphi_0(x,y)=\frac{\bar{P}_y x^2}{2}+\frac{\bar{P}_x y^2}{2}+\sum_{p=1}^{P}\sum_{q=1}^{Q}\varphi_0^{pq}(1-\cos 2p\pi x)\left(1-\cos\frac{2qa\pi}{b}y\right) \tag{5.9}$$

式中R，s，p和Q是整数，可取得足够大直至得到所需要的精确解。

显然，级数表达式(5.8)是满足边界条件(5.6)的；同样，在级数表达式(5.9)中的每一项是满足边界条件(5.7)的对于方程(5.5a)和(5.5b)，权函数分别为

$$\sin r'\pi x\sin\frac{s'2\pi}{b}y \tag{5.10}$$

$$(r'=1,\cdots,R;\ s'=1,\cdots,S)$$

和

$$(1-\cos 2p'\pi x)\left(1-\cos\frac{2q'\pi a}{b}y\right) \tag{5.11}$$

$$(p'=1,\cdots,P;\ q'=1,\cdots,Q)$$

按照伽辽金程序，下列积分（伽辽金积分）必须化为零

$$\int_0^{b/a}\int_0^1[L(w_0,\varphi_0)+q]\left(\sin r'\pi x\sin\frac{s'a\pi}{b}y\right)dxdy=0 \tag{5.12a}$$

$$\int_0^{b/a}\int_0^1\left[L_2\varphi_0+\frac{1}{2}L(w_0,w_0)\right](1-\cos 2p'\pi x)\left(1-\cos\frac{2q'a\pi}{b}y\right)dxdy=0 \tag{5.12b}$$

这里，我们省略了中间步骤，仅给出解的最后形式

$$\sum_{p=1}^{P}\sum_{q=1}^{Q}\varphi_0^{pq}\left\{\begin{array}{ll}12A_{22}^{*}p^4+4(2A_{12}^{*}+A_{66}^{*})\,p^2q^2\dfrac{a^2}{b^2}+12A_{11}^{*}q^4\dfrac{a^4}{b^4} & \text{for } p=p',\ q=q'\\ 8A_{22}^{*}p^4 & \text{for } p=p',\ q\neq q'\\ 8A_{11}^{*}q^4\dfrac{a^4}{b^4} & \text{for } p\neq p',\ q=q'\\ 0 & \text{for } p\neq p',\ q\neq q'\end{array}\right\}$$

$$-\sum_{r=1}^{R}\sum_{s=1}^{S}\sum_{r'=1}^{R}\sum_{s'=1}^{S}w_0^{rs}w_0^{r's'}\frac{a^2}{b^2}\frac{rs'}{16}\left\{r's\left[\begin{array}{rl}2; & \text{for } r'=r\neq p'\\ 1; & \text{for } r'=r=p'\\ -1; & \text{for } r\pm r'=2p'\\ & (r\neq r')\\ -1; & \text{for } r+2p'=r'\\ 0; & \text{otnerwise}\end{array}\right]\right.$$

$$\times\left[\begin{array}{rl}2; & \text{for } s'=s\neq q'\\ 1; & \text{for } s'=s=q'\\ -1; & \text{for } s\pm s'=2q'\\ & (s\neq s')\\ -1; & \text{for } s+2q'=s'\\ 0; & \text{otherwise}\end{array}\right]-rs'\left[\begin{array}{rl}2; & \text{for } r'=r\neq p'\\ 3; & \text{for } r'=r=p'\\ 1; & \text{for } r+r'=2p'\\ & (r'\neq r)\\ -1; & \text{for } r-r'=2p'\\ -1; & \text{for } r+2p'=r'\\ 0; & \text{otherwise}\end{array}\right]$$

$$\left.\times\left[\begin{array}{rl}2; & \text{for } s'=s\neq q'\\ 3; & \text{for } s'=s=q'\\ 1; & \text{for } s+s'=2q'\\ & (s'\neq s)\\ -1; & \text{for } s-s'=2q'\\ -1; & \text{for } s+2q'=s'\\ 0; & \text{otherwise}\end{array}\right]\right\}=0 \qquad (p'=1,\cdots,P;\ q'=1,\cdots,Q) \tag{5.13a}$$

$$\sum_{p=1}^{P}\sum_{q=1}^{Q}\sum_{r=1}^{R}\sum_{s=1}^{S}\varphi_0^{pq}w_0^{rs}\frac{a^2}{b^2}\left\{\frac{p^2s^2}{4}\left[\begin{array}{rl}-1; & \text{for } r'=r=p\\ -1; & \text{for } r'=2p-r\\ & (r'\neq r)\\ 1; & \text{for } r'=r\pm 2p\\ 0; & \text{otherwise}\end{array}\right]\times\left[\begin{array}{rl}3; & \text{for } s'=s=q\\ 2; & \text{for } s'=s\neq q\\ 1; & \text{for } s'=2q-s;\ s'\neq s\\ -1; & \text{for } s'=s\pm 2q\\ 0; & \text{otherwise}\end{array}\right]\right.$$

$$+\frac{pqrs}{2}\begin{bmatrix} 1; & \text{for } r'=r=p \\ 1; & \text{for } r'=2p-r \\ & (r'\neq r) \\ 1; & \text{for } r'=r+2p \\ -1; & \text{for } r'=r-2p \\ 0; & \text{otherwise}\end{bmatrix}\times\begin{bmatrix} 1; & \text{for } s'=s=q \\ 1; & \text{for } s'=2q-s \\ & (s'\neq s) \\ 1; & \text{for } s'=s+2q \\ -1; & \text{for } s'=s-2q \\ 0; & \text{otherwise}\end{bmatrix}$$

$$+\frac{q^2r^2}{4}\begin{bmatrix} 3; & \text{for } r'=r=p \\ 2; & \text{for } r'=r\neq p \\ 1; & \text{for } r'=2p-r \\ & (r'\neq r) \\ -1; & \text{for } r'=r\pm 2p \\ 0; & \text{otherwise}\end{bmatrix}\times\begin{bmatrix} -1; & \text{for } s'=s=q \\ -1; & \text{for } s'=2q-s \\ & (s'\neq s) \\ 1; & \text{for } s'=s\pm 2q \\ 0; & \text{otherwise}\end{bmatrix}\Bigg]\Bigg\}$$

$$+\sum_{r=1}^{R}\sum_{s=1}^{S} w_0^{rs}\left(\frac{P_x r^2}{4\pi^2}+\frac{P_y s^2}{4\pi^2}\frac{a^2}{b^2}\right)$$

$$(\text{for } r'=r,\ s'=s)$$

$$-\frac{1}{\pi^4}\frac{a}{b}\int_0^{b/a}\int_0^1 q(x,y)\sin r'\pi x\sin\frac{s'2\pi}{b}y\,dx\,dy=0 \tag{5.13b}$$

$$(r'=1,\cdots,R;\ s'=1,\cdots,S)$$

w_0^{rs}和φ_0^{pq}可由解联立方程(5.13a)和(5.13b)得到。其边界层函数分别为

$$\left.\begin{aligned}
v_0^{(1)}(\xi,\eta,y)&=C_0^{(1)}(\eta,y)\exp\left(-\frac{1}{\varepsilon}\int_0^x\sqrt{\varphi_{0,yy}}\,dx\right)\\
v_0^{(2)}(\tilde{\xi},\tilde{\eta},y)&=C_0^{(2)}(\tilde{\eta},y)\exp\left(-\frac{1}{\varepsilon}\int_x^1\sqrt{\varphi_{0,yy}}\,dx\right)\\
v_0^{(3)}(x,\alpha,\beta)&=C_0^{(3)}(x,\beta)\exp\left(-\frac{1}{\varepsilon\sqrt{e_1}}\int_0^y\sqrt{\varphi_{0,xx}}\,dy\right)\\
v_0^{(4)}(x,\tilde{\alpha},\tilde{\beta})&=C_0^{(4)}(x,\tilde{\beta})\exp\left(-\frac{1}{\varepsilon\sqrt{e_1}}\int_y^{b/a}\sqrt{\varphi_{0,xx}}\,dy\right)
\end{aligned}\right\} \tag{5.14}$$

$$\left.\begin{aligned}
h_0^{(1)}(\xi,\eta,y)&=-\frac{1}{a_2A(\eta,y)}w_{0,yy}C_0^{(1)}(\eta,y)\exp(-\xi)\\
h_0^{(2)}(\tilde{\xi},\tilde{\eta},y)&=-\frac{1}{a_2A(\tilde{\eta},y)}w_{0,yy}C_0^{(2)}(\tilde{\eta},y)\exp(-\tilde{\xi})\\
h_0^{(3)}(x,\alpha,\beta)&=-\frac{1}{e_1B(x,\alpha)}w_{0,xx}C_0^{(3)}(x,\alpha)\exp(-\alpha)\\
h_0^{(4)}(x,\tilde{\alpha},\tilde{\beta})&=-\frac{1}{e_1B(x,\tilde{\alpha})}w_{0,xx}C_0^{(4)}(x,\tilde{\alpha})\exp(-\tilde{\alpha})
\end{aligned}\right\} \tag{5.15}$$

式中，对于确定系数$C_0^{(i)}(i=1,\cdots,4)$的边界条件为

$$C_0^{(1)}(\eta,y)\big|_{\eta=0}=-\frac{2D_{16}}{D_{12}P_y}\left(\frac{a\pi^2}{b}\right)\sum_{r=1}^{R}\sum_{s=1}^{S}w_0^{rs}(rs)\cos\frac{s a\pi}{b}y$$

$$
\left.\begin{aligned}
&C_0^{(2)}(\tilde{\eta},y)|_{\tilde{\eta}=1}=-\frac{2D_{16}}{D_{12}\overline{P}_y}\left(\frac{a\pi^2}{b}\right)\sum_{r=1}^{R}\sum_{s=1}^{S}w_0^{rs}(rs)(-1)^r\cos\frac{sa\pi}{b}y\\
&C_0^{(3)}(x,\beta)|_{\beta=0}=-\frac{2e_1D_{26}}{D_{22}\overline{P}_x}\left(\frac{a\pi^2}{b}\right)\sum_{r=1}^{R}\sum_{s=1}^{S}w_0^{rs}(rs)\cos r\pi x\\
&C_0^{(4)}(x,\tilde{\beta})|_{\tilde{\beta}=b/a}=-\frac{2e_1D_{26}}{D_{22}\overline{P}_x}\left(\frac{a\pi^2}{b}\right)\sum_{r=1}^{R}\sum_{s=1}^{S}w_0^{sr}(rs)(-1)^s\cos r\pi x
\end{aligned}\right\}\tag{5.16}
$$

一阶渐近解($\varepsilon=1$)确定于下列方程和边界条件

$$
\begin{cases}
L(w_1,\varphi_0)+L(w_0,\varphi_1)=0 & (5.17a)\\
L_2\varphi_1+L(w_0,w_1)=0 & (5.17b)
\end{cases}
$$

$$
w_1(0,y)=w_1(1,y)=w_1(x,0)=w_1\left(x,\ \frac{b}{a}\right)=0\tag{5.18}
$$

$$
\int_0^{b/a}t\varphi_{1,yy}dy=0,\ \int_0^1 t\varphi_{1,xx}dx=0\tag{5.19}
$$

显然，$w_1=0$，$\varphi_1=0$。对于确定$C_0^{(i)}(i=1,\cdots,4)$的偏微分方程成为

$$
\left.\begin{aligned}
&2A\frac{\partial C_0^{(1)}}{\partial\eta}+(2\varphi_{0,xy}+b_1A)\frac{\partial C_0^{(1)}}{\partial y}+\frac{5}{2}A_{,x}C_0^{(1)}=0\\
&2A\frac{\partial C_0^{(2)}}{\partial\tilde{\eta}}+(2\varphi_{0,xy}+b_1A)\frac{\partial C_0^{(2)}}{\partial y}+\frac{5}{2}A_{,x}C_0^{(2)}=0\\
&2B\frac{\partial C_0^{(3)}}{\partial\beta}+\left(2\varphi_{0,xy}+\frac{d_1}{e_1}B\right)\frac{\partial C_0^{(3)}}{\partial x}+\frac{5}{2}B_{,y}C_0^{(3)}=0\\
&2B\frac{\partial C_0^{(4)}}{\partial\tilde{\beta}}+\left(2\varphi_{0,xy}+\frac{d_1}{e_1}B\right)\frac{\partial C_0^{(4)}}{\partial x}+\frac{5}{2}B_{,y}C_0^{(4)}=0
\end{aligned}\right\}\tag{5.20}
$$

利用边界条件(5.16)，$C_0^{(i)}$可以求得。例如，取w_0的级数表达式中的第一项计算，我们有

$$
\left.\begin{aligned}
&C_0^{(1)}(\eta,y)=-\frac{2D_{16}}{D_{12}\overline{P}_y}\left(\frac{2\pi^2}{b}\right)w_0^{11}\cos\frac{a\pi y}{b}\exp\left\{\left[\left(\varphi_{0,xy}A^{-1}+\frac{b_1}{2}\right)\right.\right.\\
&\qquad\left.\left.\cdot\left(\frac{a}{b}\pi\right)\mathrm{tg}\frac{a\pi}{b}y-\frac{5}{4}A_{,x}A^{-1}\right]\eta\right\}\\
&C_0^{(2)}(\tilde{\eta},y)=\frac{2D_{16}}{D_{12}\overline{P}_y}\left(\frac{2\pi^2}{b}\right)w_0^{11}\cos\frac{a\pi y}{b}\exp\left\{\left[\left(\varphi_{0,xy}A^{-1}+\frac{b_1}{2}\right)\right.\right.\\
&\qquad\left.\cdot\left(\frac{a}{b}\pi\right)\mathrm{tg}\frac{a\pi}{b}y-\frac{5}{4}A_{,x}A^{-1}\right]\\
&C_0^{(3)}(x,\beta)=-\frac{2e_1D_{26}}{D_{22}\overline{P}_x}\left(\frac{a\pi^2}{b}\right)w_0^{11}\cos\pi x\exp\left\{\left[\left(\varphi_{0,xy}B^{-1}+\frac{d_1}{2e_1}\right)\right.\right.\\
&\qquad\left.\left.\cdot(\pi)\mathrm{tg}\,\pi x-\frac{5}{4}B_{,y}B^{-1}\right]\beta\right\}\\
&C_0^{(4)}(x,\tilde{\beta})=\frac{2e_1D_{26}}{D_{22}\overline{P}_x}\left(\frac{a\pi^2}{b}\right)w_0^{11}\cos\pi x\exp\left\{\left[\left(\varphi_{0,xy}B^{-1}+\frac{d_1}{2e_1}\right)\right.\right.\\
&\qquad\left.\cdot(\pi)\ \mathrm{tg}\ \pi x-\frac{5}{4}B_{,y}B^{-1}\right]
\end{aligned}\right\}\tag{5.21}
$$

对于胶合顺序为 $+15^\circ/-15^\circ/15^\circ$ 的三层叠层板我们作了数值计算。其材料为高模量石墨环氧[3]。

图 1 给出了简单支承下，具有不同长宽比的对称铺设三层角叠层板在边缘位移为零的条件下，其无量纲的载荷-中心挠度曲线图。

与对称正交铺设矩形叠层板相比较，在同样的长宽比下，对于固定的载荷值，挠度增加了[3]。

图 2 展示了在板的中心处无量纲载荷和弯矩之间的关系。随着长宽比 a/b 的增加，弯矩 M_x 显著地减少了（如图 2 所示）。可是，M_y 和 M_{xy} 的减少并不明显（未在图中示出）。

因为参数值 ε 是小量（$\varepsilon=0.3064t/a$），解收敛得很快。对 ε 阶解，以 $R=S=P=Q=2$ 所得到的解是令人满意的。正如我们所看到的，因为方程和边界条件都变换成了奇摄动类型，计算是大大地简化了。

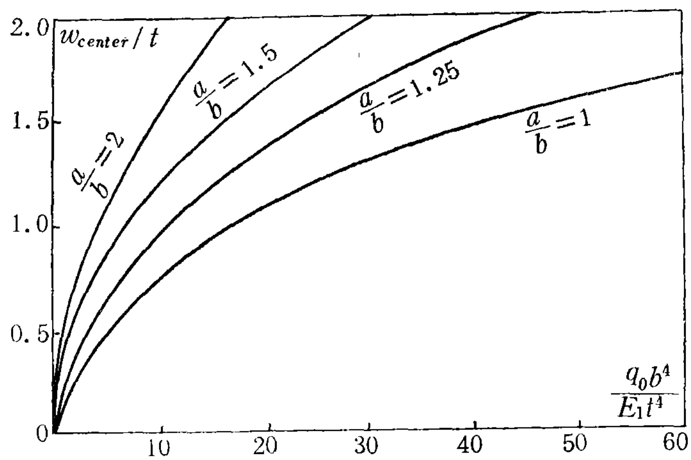

图 1

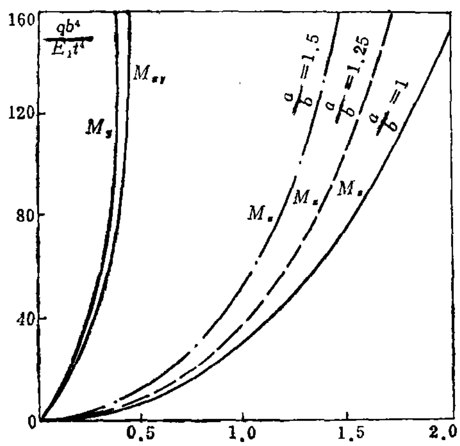

图 2

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Nonlinear Bendings of Symmetrically Layered Anisotropic Rectangular Plates

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Abstract

This paper has studied the nonlinear bendings of symmetrically layered anisotropic rectangular plates under various supports. The uniformly valid N-order asymptotic solutions of the deflection and stress function are derived by the singular perturbation method offered in [1]. The analysis and calculations are given for simply and clamped supported, rectangular plates subjected to combined edge tensions and lateral loading in conjunction with the modified Galerkin procedure (a method of weighted residuals).