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变厚度圆柱形薄壳轴对称问题的渐近解*

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摘　要

本文给出了变厚度圆柱形薄壳轴对称问题的一致有效渐近解.

圆柱形薄壳在工程上得到极其广泛应用．等厚圆柱形薄壳轴对称问题早已得到完满解决．壁厚按线性变化的圆柱形薄壳轴对称问题也已给出精确解[1]．研究变厚度圆柱形薄壳轴对称问题的论文很多[2][3][4]．本文不同于以上论文的方法，给出了变厚度圆柱形薄壳轴对称问题的一致有效渐近解，其中也包括等厚、壁厚按线性变化和壁厚按抛物线变化的圆柱形薄壳轴对称问题的精确解．

一、基 本 方 程

圆柱形薄壳轴对称问题的载荷、内力和中面角位移如图1.平衡方程：

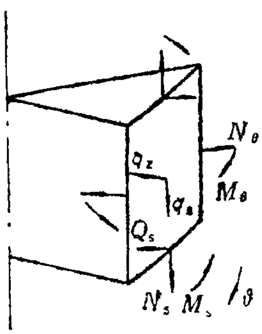

图 1

$$\left.\begin{aligned}&\frac{dN_s}{ds}+q_s=0\\&R\frac{dQ_s}{ds}+N_\theta+Rq_z=0\\&\frac{dM_s}{ds}-Q_s=0\end{aligned}\right\}\tag{1.1}$$

中面变形连续方程：

$$R\cdot\frac{d\varepsilon_\theta}{ds}=-\vartheta\tag{1.2}$$

内力与应变之间关系：

* 钱伟长推荐.

$$\left.\begin{aligned}
\varepsilon_s &= \frac{1}{Eh}(N_s-\nu N_\theta) \\
\varepsilon_\theta &= \frac{1}{Eh}(N_\theta-\nu N_s) \\
M_s &= -\frac{Eh^3}{12(1-\nu^2)}\frac{d\vartheta}{ds} \\
M_\theta &= \nu M_s
\end{aligned}\right\} \tag{1.3}$$

由式(1.1)得：

$$\left.\begin{aligned}
N_s &= -\int_{s^*}^{s} q_s ds + N_s^* \\
N_\theta &= -Rq_z - R\frac{dQ_s}{ds}
\end{aligned}\right\} \tag{1.4}$$

由以上各式可以推得：

$$\left.\begin{aligned}
&\frac{d^2\vartheta}{ds^2}+\frac{3}{h}\frac{dh}{ds}\frac{d\vartheta}{ds} = -\frac{12(1-\nu^2)}{Eh^3}Q_s \\
&\frac{d^2Q_s}{ds^2}-\frac{1}{h}\frac{dh}{ds}\frac{dQ_s}{ds} = \frac{Eh}{R^2}(\vartheta-\vartheta_m)
\end{aligned}\right\} \tag{1.5}$$

式中：

$$\vartheta_m = \frac{R}{Eh}\left[-\frac{R}{h}\frac{dh}{ds}q_z + R\frac{dq_z}{ds} + \nu q_s + \frac{\nu}{h}\frac{dh}{ds}\left(\int_{s^*}^{s} q_s ds - N_s^*\right)\right]$$

作以下变换：

$$\left.\begin{aligned}
\theta &= h^{5/4}\vartheta \\
V &= Rh^{-3/4}Q_s \\
\frac{dy}{ds} &= \frac{1}{\sqrt{Rh}}
\end{aligned}\right\} \tag{1.6}$$

由式(1.5)得：

$$\left.\begin{aligned}
&\frac{d^2\theta}{dy^2}-\left[\frac{15R}{16h}\left(\frac{dh}{ds}\right)^2+\frac{5R}{4}\frac{d^2h}{ds^2}\right]\theta = -\frac{4\beta^4}{E}V \\
&\frac{d^2V}{dy^2}-\left[\frac{15R}{16h}\left(\frac{dh}{ds}\right)^2-\frac{3R}{4}\frac{d^2h}{ds^2}\right]V = E(\theta-\theta_m)
\end{aligned}\right\} \tag{1.7}$$

式中：　$\beta=\sqrt[4]{3(1-\nu^2)}$，$\theta_m=h^{5/4}\vartheta_m$

二、几种变厚度圆柱形薄壳

2.1. $h=h_{\max}[1+g_1(x)]$的变厚度圆柱形薄壳

作以下变换：

$$z_1=\sqrt{\frac{h_{\max}}{R}}\,y=\int_{x^*}^{x}\frac{dx}{\sqrt{1+g_1(x)}} \tag{2.1}$$

由式(1.7)得：

$$\left.\begin{aligned}&\frac{d^2\theta}{dz_1^2}+P_{11}(z_1)\theta=-\frac{4\beta^4R}{Eh_{\max}}V\\&\frac{d^2V}{dz_1^2}+P_{12}(z_1)V=\frac{ER}{h_{\max}}(\theta-\theta_m)\end{aligned}\right\}\tag{2.2}$$

式中：

$$P_{11}(z_1)=-\frac{15}{16[1+g_1(x)]}\left[\frac{dg_1(x)}{dx}\right]^2-\frac{5}{4}\frac{d^2g_1(x)}{dx^2}$$

$$P_{12}(z_1)=-\frac{15}{16[1+g_1(x)]}\left[\frac{dg_1(x)}{dx}\right]^2+\frac{3}{4}\frac{d^2g_1(x)}{dx^2}$$

当满足下式： $g_1(x)=O(1)$

$$\frac{dg_1(x)}{dx}=O(1),\ \frac{d^2g_1(x)}{dx^2}=O(1)$$

我们有： $P_{11}(z_1)=O(1)$，$P_{12}(z_1)=O(1)$

对等厚圆柱形薄壳，有$P_{11}(z_1)\equiv 0$和$P_{12}(z_1)\equiv 0$。

2.2. $h=k_1[1+g_2(x_0)]x_0$ 的变厚度圆柱形薄壳

作以下变换：

$$\left.\begin{aligned}&\Theta_2=\left[\int_0^{x_0}\frac{dx_0}{\sqrt{1+g_2(x_0)}\sqrt{x_0}}\right]^{3/2}\theta\\&U_2=\left[\int_0^{x_0}\frac{dx_0}{\sqrt{1+g_2(x_0)}\sqrt{x_0}}\right]^{3/2}V\\&z_2=\left[\frac{1}{4}\int_0^{x_0}\frac{dx_0}{\sqrt{1+g_2(x_0)}\sqrt{x_0}}\right]^4\end{aligned}\right\}\tag{2.3}$$

由式 (1.7) 得：

$$\left.\begin{aligned}&\frac{d^2\Theta_2}{dz_2^2}+P_{21}(z_2)\Theta_2=-\frac{4\beta^4R}{Ek_1z_2^{3/2}}U_2\\&\frac{d^2U_2}{dz_2^2}+P_{22}(z_2)U_2=\frac{ER}{k_1z_2^{3/2}}(\Theta_2-\Theta_{2m})\end{aligned}\right\}\tag{2.4}$$

式中

$$P_{21}(z_2)=\frac{1}{z_2^{3/2}}\left\{-\frac{15}{16[1+g_2(x_0)]x_0}\left[1+g_2(x_0)+x_0\frac{dg_2(x_0)}{dx_0}\right]^2+\frac{15}{4}\left[\int_0^{x_0}\frac{dx_0}{\sqrt{1+g_2(x_0)}\sqrt{x_0}}\right]^{-2}-\frac{5}{4}\left[2\frac{dg_2(x_0)}{dx_0}+\frac{d^2g(x_0)}{dx_0^2}\right]\right\}$$

$$P_{22}(z_2)=\frac{1}{z_2^{3/2}}\left\{-\frac{15}{16[1+g_2(x_0)]x_0}\left[1+g_2(x_0)+x_0\frac{dg_2(x_0)}{dx_0}\right]^2+\frac{15}{4}\left[\int_0^{x_0}\frac{dx_0}{\sqrt{1+g_2(x_0)}\sqrt{x_0}}\right]^{-2}+\frac{3}{4}\left[2\frac{dg_2(x_0)}{dx_0}+\frac{d^2g_2(x_0)}{dx_0^2}\right]\right\}$$

$$\Theta_{2m}=8z_2^{3/8}\theta_m$$

当满足下式：

$$g_2(x_0)=O(1)$$

$$\frac{dg_2(x_0)}{dx_0}=O(1),\ \frac{d^2g_2(x_0)}{dx_0^2}=O(1)$$

我们有： $P_{21}(z_2)z_2^{3/2}=O(1)$和$P_{22}(z_2)z_2^{3/2}=O(1)$

2.3. $h=k_2x^2$ **的变厚度圆柱形薄壳**(即圆柱形薄壳的壁厚是按抛物线变化的)

作变换: $$z_3=\sqrt{\frac{k_2}{R}}y=\ln x_0 \tag{2.5}$$

由式(1.7)得:
$$\left.\begin{aligned}&\frac{d^2\theta}{dz_3^2}-\frac{25}{4}\theta=-\frac{4\beta^4R}{Ek_2}V\\&\frac{d^2V}{dz_3^2}-\frac{9}{4}V=\frac{ER}{k_2}(\theta-\theta_m)\end{aligned}\right\} \tag{2.6}$$

2.4. $h=k_2[1+g_4(x_0)]x_0^2$ **的变厚度圆柱形薄壳**

作以下变换:
$$z_4=\sqrt{\frac{k_2}{R}}y=\int_0^{x_0}\frac{dx_0}{x_0\sqrt{1+g_4(x_0)}} \tag{2.7}$$

由式(1.7)得:
$$\left.\begin{aligned}&\frac{d^2\theta}{dz_4^2}+\left[-\frac{15}{4}+P_{41}(z_4)\right]\theta=-\frac{4\beta^4R}{Ek_2}V\\&\frac{d^2V}{dz_4^2}+\left[-\frac{15}{4}+P_{42}(z_4)\right]V=\frac{ER}{k_2}(\theta-\theta_m)\end{aligned}\right\} \tag{2.8}$$

式中:
$$P_{41}(z_4)=-\frac{5}{2}-\frac{25}{4}g_4(x_0)-\frac{35}{4}x_0\frac{dg_4(x_0)}{dx_0}-\frac{15x_0^2}{16[1+g_4(x_0)]}\left[\frac{dg_4(x_0)}{dx_0}\right]^2-\frac{5}{4}x_0^2\frac{d^2g_4(x_0)}{dx_0^2}$$

$$P_{42}(z_4)=\frac{3}{2}-\frac{19}{4}g_4(x_0)-\frac{3}{4}x_0\frac{dg_4(x_0)}{dx_0}-\frac{15x_0^2}{16[1+g_4(x_0)]}\left[\frac{dg_4(x_0)}{dx_0}\right]^2+\frac{3}{4}x_0^2\frac{d^2g_4(x_0)}{dx_0^2}$$

2.5. $h=k_mx_0^m[1+g_5(x_0)]\,(m>2)$**的变厚度圆柱形薄壳**

作以下变换:
$$z_5=\sqrt{\frac{k_m}{R}}\,y=\int_0^{x_0}\frac{dx_0}{x_0^{m/2}\sqrt{1+g_5(x_0)}} \tag{2.9}$$

由式(1.7)得:
$$\left.\begin{aligned}&\frac{d^2\theta}{dz_5^2}+P_{51}(z_5)\theta=-\frac{4\beta^4R}{Ek_m}V\\&\frac{d^2V}{dz_5^2}+P_{52}(z_5)V=\frac{ER}{k_m}(\theta-\theta_m)\end{aligned}\right\} \tag{2.10}$$

式中:
$$P_{51}(z_5)=-mx_0^{m-2}\left(\frac{35}{16}m-\frac{5}{4}\right)[1+g_5(x_0)]-\frac{35}{8}mx_0^{m-1}\frac{dg_5(x_0)}{dx_0}-\frac{15x_0^m}{16[1+g_5(x_0)]}\left[\frac{dg_5(x_0)}{dx_0}\right]^2-\frac{5}{4}x_0^m\frac{d^2g_5(x_0)}{dx_0^2}$$

$$P_{52}(z_5)=-mx_0^{m-2}\left(\frac{35}{16}m+\frac{3}{4}\right)[1+g_5(x_0]-\frac{3}{8}mx_0^{m-1}\frac{dg_5(x_0)}{dx_0}-\frac{15x_0^m}{16[1+g_5(x_0)]}\left[\frac{dg_5(x_0)}{dx_0}\right]^2+\frac{3}{4}x_0^m\frac{d^2g_5(x_0)}{dx_0^2}$$

2.6. **具有变厚度$h=h_{max}t(x)$的圆柱形薄壳的边缘问题**

当边缘不在壁厚顶点附近，并满足：

$$\frac{1}{t(x)}\frac{dt(x)}{dx}=O(1)$$

$$\frac{d^2t(x)}{dx^2}=O(1)$$

作以下变换：

$$z_6=\sqrt{\frac{h_{max}}{R}}y=\int_{x^*}^{x}\frac{dx}{\sqrt{t(x)}} \tag{2.11}$$

由式(1.7)得：

$$\left.\begin{aligned}&\frac{d^2\theta}{dz_6^2}+P_{61}(z_6)\theta=-\frac{4\beta^4R}{Eh_{max}}V\\&\frac{d^2V}{dz_6^2}+P_{62}(z_6)V=\frac{ER}{h_{max}}(\theta-\theta_m)\end{aligned}\right\} \tag{2.12}$$

式中：

$$P_{61}(z_6)=-\frac{15}{16t(x)}\left[\frac{dt(x)}{dx}\right]^2-\frac{5}{4}\frac{d^2t(x)}{dx^2}=O(1)$$

$$P_{62}(z_6)=-\frac{15}{16t(x)}\left[\frac{dt(x)}{dx}\right]^2+\frac{3}{4}\frac{d^2t(x)}{dx^2}=O(1)$$

对上述各种变厚度圆柱薄壳，式(2.2)(2.4)(2.6)(2.8)(2.10)(2.12)都可以写成以下同一形式：

$$\left.\begin{aligned}&\frac{d^2\Theta}{dz^2}+[a_1z^n+P_1(z)]\Theta=-\frac{4\beta^4}{E}\lambda_0^2z^nU\\&\frac{d^2U}{dz^2}+[a_2z^n+P_2(z)]U=E\lambda_0^2z^n(\Theta-\Theta_m)\end{aligned}\right\} \tag{2.13}$$

当$P_1(z)\neq 0$，$P_2(z)\neq 0$时，有$a_1=a_2=a$。对上述变厚度圆柱形薄壳，若分别有$\frac{R}{h_{max}}\gg 1$、$\frac{R}{k}\gg 1$、$\frac{R}{k_2}\gg 1$、$\frac{R}{k_m}\gg 1$，那么，式(2.13)是一个含有一个大参数λ_0的二阶微分方程组。

三、一致有效渐近解

3.1 齐次解

式(2.13)的齐次方程是：

$$\left.\begin{aligned}&\frac{d^2\bar\Theta}{dz^2}+[a_1z^n+P_1(z)]\bar\Theta=-\frac{4\beta^4}{E}\lambda_0^2z^n\bar U\\&\frac{d^2\bar U}{dz^2}+[a_2z^n+P_2(z)]\bar U=E\lambda_0^2z^n\bar\Theta\end{aligned}\right\} \tag{3.1}$$

如果$P_1(z)$和$P_2(z)$满足下式：

$$\left.\begin{aligned} z^{-n}P_1(z)=O(1) \\ z^{-n}P_2(z)=O(1) \end{aligned}\right\} \tag{3.2}$$

那末，式(3.1)的比较方程是：

$$\left.\begin{aligned} \frac{d^2I}{dz^2}+a_1z^nI=-\frac{4\beta^4}{E}\lambda_0^2z^nW \\ \frac{d^2W}{dz^2}+a_2z^nW=E\lambda_0^2z^nI \end{aligned}\right\} \tag{3.3}$$

我们令：

$$\left.\begin{aligned} \tilde{X}=W-i\frac{E}{3\beta^2}\left[\sqrt{1-\frac{(a_2-a_1)^2}{16\beta^4\lambda_0^4}}-i\frac{a_2-a_1}{4\beta^2\lambda_0^2}\right]I \\ \lambda^2=-2i\beta^2\lambda_0^2\sqrt{1-\frac{(a_2-a_1)^2}{16\beta^4\lambda_0^4}}+\frac{a_2+a_1}{2} \end{aligned}\right\} \tag{3.4}$$

由式(2.3)得：
$$\frac{d^2\tilde{X}}{dz^2}+\lambda^2z^n\tilde{X}=0 \tag{3.5}$$

上式有解：
$$\tilde{X}=\sqrt{z}\,J_{\frac{1}{n+2}}\left(\frac{2\lambda}{n+2}z^{\frac{n}{2}+1}\right) \tag{3.6}$$

式中：　$J_{\frac{1}{n+2}}\left(\frac{2\lambda}{n+2}z^{\frac{n}{2}+1}\right)$是$\frac{1}{n+2}$阶Bessel函数。

那么，式(3.1)的一次渐近解是：

$$\begin{aligned} &\bar{U}_I-i\frac{E}{2\beta^2}\left[\sqrt{1-\frac{(a_2-a_1)^2}{16\beta^4\lambda_0^4}}-i\frac{a_2-a_1}{4\beta^2\lambda_0^2}\right]\bar{\Theta}_I \\ &=\sqrt{z}\left[\tilde{C}_1H^{(1)}_{\frac{1}{n+1}}\left(\frac{2\lambda}{n+2}z^{\frac{n}{2}+1}\right)+\tilde{C}_2H^{(2)}_{\frac{1}{n+1}}\left(\frac{2\lambda}{n+2}z^{\frac{n}{2}+1}\right)\right] \end{aligned} \tag{3.7}$$

式中：$H^{(1)}_{\frac{1}{n+2}}\left(\frac{2\lambda}{n+2}z^{\frac{n}{2}+1}\right)$和$H^{(2)}_{\frac{1}{n+2}}\left(\frac{2\lambda}{n+2}z^{\frac{n}{2}+1}\right)$分别是第一种和第二种的$\frac{1}{n+2}$阶 Hankel 函数。当$P_1(z)=P_2(z)=0$时，式(3.7)就是式(3.1)的精确解。

在本文中，对等厚的和壁厚按抛物线变化的圆柱形薄壳，有：

$$P_1(z)=P_2(z)=0$$
$$n=0$$

对壁厚按线性变化的圆柱形薄壳，有：

$$P_1(z)=P_2(z)=0$$
$$n=-3/2$$

当$P_1(z)\neq0$，$P_2(z)\neq0$时，对上述圆柱形薄壳，我们有$a_1=a_2=a$。由式(3.7)得：

$$\begin{aligned} \bar{U}_I-i\frac{E}{2\beta^2}\bar{\Theta}_I=&\sqrt{z}\left[\tilde{C}_1H^{(1)}_{\frac{1}{n+2}}\left(\frac{2\lambda}{n+2}z^{\frac{n}{2}+1}\right)\right. \\ &\left.+\tilde{C}_2H^{(2)}_{\frac{1}{n+2}}\left(\frac{2\lambda}{n+2}z^{\frac{n}{2}+1}\right)\right] \end{aligned}$$

式中：　$\lambda^2=-2i\beta^2\lambda_0^2+a$

那么，式(3.1)的高次近似的一致有效渐近解，本文给出以下形式：

$$\left.\begin{aligned}\bar{U}&=\operatorname{Re}\left(\tilde{\alpha}_2\tilde{X}+\tilde{\gamma}_2\frac{d\tilde{X}}{dz}\right)\\ \bar{\Theta}&=-\frac{2\beta^2}{E}\operatorname{Im}\left(\tilde{\alpha}_1\tilde{X}+\tilde{\gamma}_1\frac{d\tilde{X}}{dz}\right)\end{aligned}\right\}\tag{3.8}$$

式中：

$$\left.\begin{aligned}\tilde{\alpha}_1&=\sum_{i=0}^{\infty}\alpha_{1i}(z)\lambda^{-i}\\ \tilde{\alpha}_2&=\sum_{i=0}^{\infty}\alpha_{2i}(z)\lambda^{-i}\\ \tilde{\gamma}_1&=\sum_{i=0}^{\infty}\gamma_{1i}(z)\lambda^{-i}\\ \tilde{\gamma}_2&=\sum_{i=0}^{\infty}\gamma_{2i}(z)\lambda^{-i}\end{aligned}\right\}\tag{3.9}$$

将式(3.6)(3.8)代入式(3.1)中，得：

$$\left.\begin{aligned}&\left[\frac{d^2\tilde{\alpha}_1}{dz^2}-\lambda^2z^n(\tilde{\alpha}_1-\tilde{\alpha}_2)+P_1(z)\tilde{\alpha}_1-2\lambda^2z^n\frac{d\tilde{\gamma}_1}{dz}\right.\\ &\qquad\left.-n\lambda^{2n}z^{-1}\tilde{\gamma}_1\right]\tilde{X}+\left[\frac{d^2\tilde{\gamma}_1}{dz^2}-\lambda^2z^n(\tilde{\gamma}_1-\tilde{\gamma}_2)\right.\\ &\qquad\left.+P_1(z)\tilde{\gamma}_1+2\frac{d\tilde{\alpha}_1}{dz}\right]\frac{d\tilde{X}}{dz}=0\\ &\left[\frac{d^2\tilde{\alpha}_2}{dz^2}-\lambda^2z^n(\tilde{\alpha}_2-\tilde{\alpha}_1)+P_2(z)\tilde{\alpha}_2-2\lambda^2z^n\frac{d\tilde{\gamma}_2}{dz}\right.\\ &\qquad\left.-n\lambda^2z^{n-1}\tilde{\gamma}_2\right]\tilde{X}+\left[\frac{d^2\tilde{\gamma}_2}{dz^2}-\lambda^2z^n(\tilde{\gamma}_2-\tilde{\gamma}_1)\right.\\ &\qquad\left.+P_2(z)\tilde{\gamma}_2+2\frac{d\tilde{\alpha}_2}{dz}\right]\frac{d\tilde{X}}{dz}=0\end{aligned}\right\}\tag{3.10}$$

令$\tilde{X}$和$d\tilde{X}/dz$的系数分别为零，得：

$$\left.\begin{aligned}&\frac{d^2\tilde{\alpha}_1}{dz^2}-\lambda^2z^n(\tilde{\alpha}_1-\tilde{\alpha}_2)+P_1(z)\tilde{\alpha}_1-2\lambda^2z^n\frac{d\tilde{\gamma}_1}{dz}-n\lambda^2z^{n-1}\tilde{\gamma}_1=0\\ &\frac{d^2\tilde{\alpha}_2}{dz^2}-\lambda^2z^n(\tilde{\alpha}_2\sim\tilde{\alpha}_1)+P_2(z)\tilde{\alpha}_2-2\lambda^2z^n\frac{d\tilde{\gamma}_2}{dz}-n\lambda^2z^{n-1}\tilde{\gamma}_2=0\\ &\frac{d^2\tilde{\gamma}_1}{dz^2}-\lambda^2z^n(\tilde{\gamma}_1-\tilde{\gamma}_2)+P_1(z)\tilde{\gamma}_1+2\frac{d\tilde{\alpha}_1}{dz}=0\\ &\frac{d_2\tilde{\gamma}_2}{dz^2}-\lambda^2z^n(\tilde{\gamma}_2-\tilde{\gamma}_1)+P_2(z)\tilde{\gamma}_2(z)+2\frac{d\tilde{\alpha}_2}{dz}=0\end{aligned}\right\}\tag{3.11}$$

由上式得：

$$\left.\begin{aligned}&\frac{d^2(\tilde{\alpha}_1+\tilde{\alpha}_2)}{dz^2}+P_1(z)\tilde{\alpha}_1+P_2(z)\tilde{\alpha}_2-2\lambda^2z^n\frac{d(\tilde{\gamma}_1+\tilde{\gamma}_2)}{dz}\\&\qquad-n\lambda^2z^{n-1}(\tilde{\gamma}_1+\tilde{\gamma}_2)=0\\&\frac{d^2(\tilde{\gamma}_1+\tilde{\gamma}_2)}{dz^2}+P_1(z)\tilde{\gamma}_1+P_2(z)\tilde{\gamma}_2+2\frac{d(\tilde{\alpha}_1+\tilde{\alpha}_2)}{dz}=0\end{aligned}\right\}\tag{3.12}$$

由式(3.9)(3.11)(3.12)得:

$$\left.\begin{aligned}&\sum_{t=0}^{\infty}\left\{\frac{d^2[\alpha_{1t}(z)+\alpha_{2t}(z)]}{dz^2}+P_1(z))\alpha_{1t}(z)+P_2(z)\alpha_{2t}(z)\right.\\&\qquad-2z^n\frac{d}{dz}[\gamma_{1,t+2}(z)+\gamma_{2,t+2}(z)\\&\qquad\left.-nz^{n-1}[\gamma_{1,t+2}(z)+\gamma_{2,t+2}(z)]\right\}\lambda^{-t}=0\\&\sum_{t=0}^{\infty}\left\{\frac{d^2[\gamma_{1,t}(z)+\gamma_{2,t}(z)]}{dz^2}+P_1(z)\gamma_{1,t}(z)+P_2(z)\gamma_{2,t}(z)\right.\\&\qquad\left.+2\frac{d[\alpha_{1,t}(z)+\alpha_{2,t}(z)]}{dz}\right\}\lambda^{-t}=0\\&\sum_{t=0}^{\infty}\left\{\frac{d^2\alpha_{1t}(z)}{dz^2}-z^n[\alpha_{1,t+2}(z)-\alpha_{2,t+2}(z)]+P_1(z)\alpha_{1,t}(z)\right.\\&\qquad\left.-2z^n\frac{d\gamma_{1,t+2}(z)}{dz}-nz^{n-1}\gamma_{1,t+2}(z)\right\}\lambda^{-t}=0\\&\sum_{t=0}^{\infty}\left\{\frac{d^2\gamma_{1,t}(z)}{dz^2}-z^n[\gamma_{1,t+2}(z)-\gamma_{2,t+2}(z)]\right.\\&\qquad\left.+P_1(z)\gamma_{1,t}(z)+2\frac{d\alpha_{1,t}(z)}{dz}\right\}\lambda^{-t}=0\end{aligned}\right\}\tag{3.13}$$

由式(3.13)得:

$$\left.\begin{aligned}&\alpha_{1,2t+1}(z)=\alpha_{2,2t+1}(z)=\gamma_{1,2t+1}(z)=\gamma_{2,2t+1}(z)=0\\&\alpha_{1,0}(z)=\alpha_{2,0}(z)=1\\&\gamma_{1,0}(z)=\gamma_{2,0}(z)=0\\&\alpha_{1,2}(z)=-\frac{1}{4}\frac{d[\gamma_{1,2}(z)+\gamma_{2,2}(z)]}{dz}-\frac{1}{4}\int^z[P_1(z)\gamma_{1,2}(z)\\&\qquad+P_2(z)\gamma_{2,2}(z)]dz+\frac{1}{2}\left[P_1(z)-2z^n\frac{d\gamma_{1,2}(z)}{dz}-2z^{n-2}\gamma_{1,2}(z)\right]z^{-n}\\&\alpha_{22}(z)=\alpha_{1,2}(z)-\left[P_1(z)-2z^n\frac{d\gamma_{1,2}(z)}{dz}-nz^{n-1}\gamma_{1,2}(z)\right]z^{-n}\\&\gamma_{1,2}(z)=\gamma_{2,2}(z)=\frac{1}{4z^{n/2}}\int^z\frac{P_1(z)+P_2(z)}{z^{n/2}}dz\\&\alpha_{1,2t+2}(z)=-\frac{1}{4}\frac{d}{dz}[\gamma_{1,2t+2}(z)+\gamma_{2,2t+2}(z)]\end{aligned}\right.$$

$$
\left.\begin{aligned}
&-\frac{1}{4}\int^{z}[P_1(z)\gamma_{1,2i+2}(z)+P_2(z)\gamma_{2,2i+2}(z)]dz+\frac{1}{2}\Big[\frac{d^2\alpha_{1,2i}(z)}{dz^2}\\
&+P_1(z)\alpha_{1,2i}(z)-2z^n\frac{d\gamma_{1,2i+2}(z)}{dz}-nz^{n-1}\gamma_{1,2i+2}(z)\Big]z^{-n}\\
&\alpha_{2,2i+2}(z)=\alpha_{1,2i+2}(z)-\Big[\frac{d^2\alpha_{1,2i}(z)}{dz^2}+P_1(z)\alpha_{1,2i}(z)\\
&\qquad -2z^n\frac{d\gamma_{1,2i+2}(z)}{dz}-nz^{n-1}\gamma_{1,2i+2}(z)\Big]z^{-n}\\
&\gamma_{1,2i+2}(z)=\frac{1}{4z^{n/2}}\int^{z}\Big\{P_1(z)\alpha_{1,2i}(z)+P_2(z)\alpha_{2,2i}(z)\\
&\qquad +\frac{d^2}{dz^2}[\alpha_{1,2i}(z)+\alpha_{2,2i}(z)]\Big\}\frac{dz}{z^{n/2}}+\frac{1}{2}\Big[\frac{d^2\gamma_{1,2i}(z)}{dz^2}\\
&\qquad +P_1(z)\gamma_{1,2i}(z)+2\frac{d\alpha_{1,2i}(z)}{dz}\Big]z^{-n}\\
&\gamma_{2,2i+2}(z)=\gamma_{1,2i+2}(z)-\Big[\frac{d^2\gamma_{1,2i}(z)}{dz^2}+P_1(z)\gamma_{1,2i}(z)\\
&\qquad +2\frac{d\alpha_{1,2i}(z)}{dz}\Big]z^{-n}
\end{aligned}\right\}\quad (3.14)
$$

利用以下关系：

$$
\frac{dJ_{\frac{1}{n+2}}\left(\frac{2\lambda}{n+2}z^{\frac{n}{2}+1}\right)}{dz}=\lambda z^{n/2}J_{-\frac{n+1}{n+2}}\left(\frac{2\lambda}{n+2}z^{\frac{n}{2}+1}\right)-\frac{1}{2\lambda z}J_{\frac{1}{n+2}}\left(\frac{2\lambda}{n+2}z^{\frac{n}{2}+1}\right)\quad (3.15)
$$

式中：$J_{-\frac{n+1}{n+2}}\left(\frac{2\lambda}{n+2}z^{\frac{n}{2}+1}\right)$是$\left(-\frac{n+1}{n+2}\right)$阶Bessel函数。由式(3.6)(3.8)(3.15)得：

$$
\left.\begin{aligned}
&\bar U=\sqrt{z}\,\mathrm{Re}\Big\{\widetilde C_1\Big[\widetilde\alpha_2H^{(1)}_{\frac{1}{n+2}}\left(\frac{2\lambda}{n+2}z^{\frac{n}{2}+1}\right)\\
&\quad+\lambda\widetilde\gamma_2z^{n/2}H^{(1)}_{-\frac{n+1}{n+2}}\left(\frac{2\lambda}{n+2}z^{\frac{n}{2}+1}\right)\Big]+\widetilde C_2\Big[\widetilde\alpha_2H^{(2)}_{\frac{1}{n+2}}\left(\frac{2\lambda}{n+2}z^{\frac{n}{2}+1}\right)\\
&\quad+\lambda\widetilde\gamma_2z^{n/2}H^{(2)}_{-\frac{n+1}{n+2}}\left(\frac{2\lambda}{n+2}z^{\frac{n}{2}+1}\right)\Big]\Big\}\\
&\bar\Theta=-\frac{2\beta^2}{E}\sqrt{z}\,\mathrm{Im}\Big\{\widetilde C_1\Big[\widetilde\alpha_1H^{(1)}_{\frac{1}{n+2}}\left(\frac{2\lambda}{n+2}z^{\frac{n}{2}+1}\right)\\
&\quad+\lambda\widetilde\gamma_1z^{n/2}H^{(1)}_{-\frac{n+1}{2+n}}\left(\frac{2\lambda}{n+2}z^{\frac{n}{2}+1}\right)\Big]+\widetilde C_2\Big[\widetilde\alpha_1H^{(1)}_{\frac{1}{n+2}}\left(\frac{2\lambda}{n+2}z^{\frac{n}{2}+1}\right)\\
&\quad+\lambda\widetilde\gamma_1z^{n/2}H^{(2)}_{-\frac{n+1}{n+2}}\left(\frac{2\lambda}{n+2}z^{\frac{n}{2}+1}\right)\Big]\Big\}
\end{aligned}\right\}\quad (3.16)
$$

式中：$H^{(1)}_{-\frac{n+1}{n+2}}\left(\frac{2\lambda}{n+2}z^{\frac{n}{2}+1}\right)$和$H^{(2)}_{-\frac{n+1}{n+2}}\left(\frac{2\lambda}{n+2}z^{\frac{n}{2}+1}\right)$分别是第一种和第二种的$\left(-\frac{n+1}{n+2}\right)$阶 Hankel

函数。

3.2. 特解

3.2.1. $P_1(z)=P_2(z)=0$

在$n=0$时，由式(2.13)得：

$$\left.\begin{aligned}&\frac{d^2\Theta}{dz^2}+a_1\Theta=-\frac{4\beta^4}{E}\lambda_0^2U\\&\frac{d^2U}{dz^2}+a_2U=E\lambda_0^2(\Theta-\Theta_m)\end{aligned}\right\}\tag{3.17}$$

由上式得：

$$\frac{d^2\tilde{Y}_1}{dz^2}+\lambda^2Y_1=-E\lambda_0^2\Theta_m\tag{3.18}$$

式中：
$$\tilde{Y}_1=U-i\frac{E}{2\beta^2}\left[\sqrt{1-\frac{(a_2-a_1)^2}{16\beta^4\lambda_0^4}}-i\frac{a_2-a_1}{4\beta^2\lambda_0^2}\right]\Theta$$

上式有特解：

$$\begin{aligned}\tilde{Y}_1^p=-\frac{E\lambda_0^2}{\lambda}\Big(&\sin\lambda z\int^z\Theta_m\cos\lambda z dz\\&-\cos\lambda z\int^z\Theta_m\sin\lambda z dz\Big)\end{aligned}\tag{3.19}$$

对壁厚按线性变化的圆柱薄壳，有 $a_1=a_2=0$, $n=-\frac{3}{2}$，由式(2.13)得：

$$\left.\begin{aligned}&\frac{d^2\Theta_2}{dz^2}=-\frac{4\beta^4}{E}\lambda_0^2z_2^{-3/2}U_2\\&\frac{d^2U_2}{dz^2}=E\lambda_0^2z_2^{3/2}(\Theta_2-\Theta_{2m})\end{aligned}\right\}\tag{3.20}$$

由上式得：
$$\frac{d^2\tilde{Y}_2}{dz_2^2}+\lambda^2z_2^{-3/2}\tilde{Y}_2=-E\lambda_0^2z_2^{-3/2}\Theta_{2m}\tag{3.21}$$

式中：
$$\tilde{Y}_2=U_2-i\frac{E}{2\beta^2}\Theta_2,\quad \lambda^2=-2i\beta^2\frac{R}{k_1}$$

例如，在压力容器中重要的载荷是气压p，即：$q_z=-p$，$q_s=0$

$$N_s^*=\frac{pR}{2}$$

$$\Theta_m=\frac{\sqrt{8}}{E}p\left(1-\frac{\nu}{2}\right)Rk_1^{1/4}$$

那末，式(3.21)的特解是：

$$\tilde{Y}_2^p=-i\frac{\sqrt{8}}{2\beta^2}p\left(1-\frac{\nu}{2}\right)Rk_1^{1/4}$$

对一些工程上重要载荷，如果可以写成：

$$\Theta_m=c[1+f(z_2)]\tag{3.22}$$

式中：c是常数；$f(z_2)=O(1)$，$\frac{d^2f(z_2)}{dz_2^2}=O(1)$将式(3.21)的特解展为$\lambda$的负次幂级数：

$$\tilde{Y}_2^p=\sum_{i=0}^{\infty} y_{2i}(z_2)\lambda^{-i} \tag{3.23}$$

将式(3.22)和(3.23)代入式(3.21)中，由λ的同次幂的系数相等，可以求得一次和二次渐近特解：

$$\tilde{Y}_{2,\mathrm{I}}^p=\tilde{Y}_{2,\mathrm{I}}^p=-i\frac{E}{2\beta^2}\Theta_m \tag{3.24}$$

3.2.2. $P_1(z)\neq 0$，$P_2(z)\neq 0$，且$a_1=a_2=a$：

当$n=0$时，由式(2.13)得：

$$\left.\begin{aligned}&\frac{d^2\Theta}{dz^2}+[a+P_1(z)]\Theta=-\frac{4\beta^4}{E}\lambda_0^2U\\&\frac{d^2U}{dz^2}+[a+P_2(z)]U=E\lambda_0^2(\Theta-\Theta_m)\end{aligned}\right\} \tag{3.25}$$

如果有$P_1(z)=O(1)$，$P_2(z)=O(1)$，那么，第一次渐近特解满足下式：

$$\left.\begin{aligned}&\frac{d^2\Theta_\mathrm{I}^p}{dz^2}+a\Theta_\mathrm{I}^p=-\frac{4\beta^4}{E}\lambda_0^2U_\mathrm{I}^p\\&\frac{d^2U_\mathrm{I}^p}{dz^2}+aU_\mathrm{I}^p=E\lambda_0^2(\Theta_\mathrm{I}^p-\Theta_m)\end{aligned}\right\} \tag{3.26}$$

上式有特解：
$$U_\mathrm{I}^p-i\frac{E}{2\beta^2}\Theta_\mathrm{I}^p=-\frac{E\lambda_0^2}{\lambda}\left(\sin\lambda z\int^z\Theta_m\cos\lambda z dz-\cos\lambda z\int^z\Theta_m\sin\lambda z dz\right) \tag{3.27}$$

我们可以用常数变量法来求得高阶近似的特解。

如果我们有$\dfrac{d^2(E\Theta_m)}{dz^2}=O(1)$，那么，一次和三次渐近特解是：

$$U_\mathrm{I}^p-i\frac{E}{2\beta^2}\Theta_\mathrm{I}^p=U_\mathrm{I}^p-i\frac{E}{2\beta^2}\Theta_\mathrm{I}^p=-\frac{E\lambda_0^2}{\lambda^2}\Theta_m \tag{3.28}$$

当$n=-\dfrac{3}{2}$，且$a=0$时，式(2.13)变成：

$$\left.\begin{aligned}&\frac{d^2\Theta_2}{dz^2}+P_1(z_2)\Theta_2=-\frac{4\beta^4}{E}\lambda_0^2z_2^{-3/2}U_2\\&\frac{d^2U_2}{dz^2}+P_2(z_2)U_2=E\lambda_0^2z_2^{-3/2}(\Theta_2-\Theta_{2m})\end{aligned}\right\} \tag{3.29}$$

如果Θ_{2m}也可以写成式(3.22)，且$P_1(z_2)=O(1)$，$P_2(z_2)=O(1)$。我们将式(3.29)的特解展为λ的负次幂级数：

$$\left.\begin{aligned}&\Theta_2^p=\sum_{i=0}^{\infty}A_i(z_2)\lambda^{-i}\\&U_2^p=\sum_{i=0}^{\infty}B_i(z_2)\lambda^{-i}\end{aligned}\right\} \tag{3.30}$$

将式(3.22)和(3.30)代入式(3.29)中，给出一次和二次渐近特解是：

$$U_{2,I}^{*}-i\frac{E}{2\beta^{2}}\Theta_{2,I}^{*}=U_{2,I}^{*}-i\frac{E}{2\beta^{2}}\Theta_{2,I}^{*}=-i\frac{E\Theta_{2m}}{2\beta^{2}} \tag{3.31}$$

本文给出的等厚和壁厚按线性变化的圆柱形薄壳轴对称问题的解是与S. Timoshnko 在[1]中给出的相应解完全一致的。

符 号 说 明

a,a_1,a_2：实常数，

$\widetilde{C}_1,\widetilde{C}_2$：复常数，

E：弹性模量，

h：壁厚，

h_{max}：壁厚的最大值，

k_1,k_2,k_m：实常量

M_s,M_Q：经向和环向弯矩

N_S,N_θ：经向和环向内力

Q_s：横剪力

q_s：单位壳面面积上经向载荷力

q_z：单位壳面面积上法向载荷力

R：圆柱壳的中面半径

s：从壁厚最大值处计算起经线长度

s_0：从壁厚顶点处计算起的经线长度

x：$x=\frac{s}{R}$

x_0：$x_0=\frac{s_0}{R}$

$\varepsilon_s,\varepsilon_\theta$：经向和环向应变

ϑ：经线切线方向的辅角

ν：泊松比

s^*,x^*,N_s^*：分别是s,x,N_s在上边界处的该值

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The Asymptotic Solutions of Axisymmetrical Problems for the Cylindrical Shells with Varying Wall Thickness

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Abstract

In this paper, the uniformly valid asymptotic solutions of axisymmetrical problems for the cylindrical shells with varying wall thickness are given.