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关于具有初始挠度的圆薄板的跳跃问题*

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摘　　要

本文利用江福汝提出的“两变量法”[3][4]与正则摄动法研究了具有初始挠度的圆薄板的跳跃问题[本文中(1.1)，(1.2)]．我们得到了这一问题的N阶一致有效渐近解[(1.66)，(1.67)]．当初始挠度变为零时，该解变为圆薄板非线性弯曲问题的解[6]；如果初始挠度较大而初始挠度与载荷强度符号相反时，载荷强度达到某一值时，将发生跳跃现象．

钱伟长早在1948年就研究了关于夹支圆板在均匀法向压力作用下的非线性弯曲问题[1]，而且得到了与实验符合的弯曲薄板的渐近解，并奠定了利用薄膜解研究薄板大挠度问题的基础．

1982年江福汝[2]利用他提出的方法[3][4]研究了在各种支承条件下的圆形和环形薄板的非线性和非对称弯曲问题．他构造了这类问题的直到N阶的形式渐近解．我们利用江福汝提出的方法和正则摄动法[7]，研究了具有初始挠度的圆薄板的跳跃问题．

一、具有初始挠度的圆形薄板

具有初始挠度的圆薄板，在均匀法向压力下，如果它的周边夹支，弯曲荷板的挠度函数W和应力函数Φ满足以下的方程和边界条件[8]：

$$\left.\begin{aligned} &D\frac{d}{dr}[\nabla^2 W]=\Psi+\frac{h}{r}\frac{d\Phi}{dr}\left(\frac{dW}{dr}+\frac{dW_{IN}}{dr}\right)\\ &\frac{d}{dr}[\nabla^2\Phi]=-\frac{E}{r}\left[\frac{1}{2}\left(\frac{dW}{dr}\right)^2\frac{dW_{IN}}{dr}\frac{dW}{dr}\right]\end{aligned}\right\}\tag{1.1}$$

$$\left.\begin{aligned} &W|_{r=c}=0,\qquad W_{,r}|_{r=c}=0,\qquad W_{,r}|_{r=0}=0\\ &\frac{d\Phi}{dr}\Big|_{r=0}=0,\quad \left(\frac{d^2\Phi}{dr^2}-\frac{\mu}{r}\frac{d\Phi}{dr}\right)\Big|_{r=c}=0\end{aligned}\right\}\tag{1.2}$$

其中，E是弹性模量，h是板的厚度，$D=Eh^3/12(1-\mu^2)$是抗弯刚度，μ是泊松比，q是法向压力强度，∇^2是二维Laplace 算子，而W_{IN}是初始挠度函数，假定为：

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* 江福汝推荐．

$$W_{IN}=\overline{W}_0\left(1+\cos\frac{\pi}{c}r\right) \tag{1.3}$$

其中，$\overline{W}_0$是最大初始挠度的一半，c为圆板半径，而Ψ是载荷函数：

$$\begin{aligned}\Psi&=\frac{1}{2\pi r}\int_{\mathscr{S}}q\,ds\\&=\frac{q}{r}\int_0^r r\left[1+\frac{\pi^2\overline{W}_0^2}{c^2}\sin^2\left(\frac{\pi}{c}r\right)\right]^{1/2}dr\end{aligned} \tag{1.4}$$

令$\overline{W}_0/c=\delta\ll 1$，并将$[1/\pi^2+\delta^2\sin^2(\pi r/c)]^{1/2}$展为$\delta^2\sin^2(\pi r/c)$的幂级数，积分(1.4)式，则得：

$$\begin{aligned}\Psi=&\frac{qr}{2}+\frac{q}{8r}\delta^2\left[\pi^2r^2-c\pi r\sin\left(\frac{2\pi}{c}r\right)\right.\\&\left.+\frac{c^2}{2}\left(1-\cos\left(\frac{2\pi}{c}r\right)\right)\right]+O(\delta^4)\end{aligned} \tag{1.5}$$

将(1.3)和(1.5)代入(1.1)，并无量纲化则得：

$$\left.\begin{aligned}&\varepsilon^2\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dW}{dr}\right)\right]=\frac{qr}{2}+\frac{q}{8r}\delta^2\left[\pi^2cr^2\right.\\&\qquad\left.-\pi cr\sin 2\pi r+\frac{c}{2}(1-\cos 2\pi r)\right]\\&\qquad+\frac{1}{r}\frac{d\Phi}{dr}\left(\frac{dW}{dr}-\pi\delta\sin\pi r\right)+O(\delta^4)\\&\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right)\right]=-\frac{1}{r}\left[\frac{1}{2}\left(\frac{dW}{dr}\right)^2-\pi\delta\sin\pi r\right]\end{aligned}\right\} \tag{1.6}$$

其中$\varepsilon^2=h^2/12(1-\mu^2)c^2\ll 1$。边界条件(1.2)变为：

$$\left.\begin{aligned}&W|_{r=1}=0,\quad W_{,r}|_{r=1}=0,\quad W_{,r}|_{r=0}=0\\&\left.\frac{d\Phi}{dr}\right|_{r=0}=0,\ \left.\left(\frac{d^2\Phi}{dr^2}-\frac{\mu}{r}\frac{d\Phi}{dr}\right)\right|_{r=1}=0\end{aligned}\right\} \tag{1.7}$$

为校正边界条件，按江福汝提出的“两变量法”作微分算子的展开式[3][4]。

在$r=1$的边界处引入变量ξ和η：

$$\xi=\frac{u(r)}{\varepsilon},\qquad \eta=r$$

把关于对r的导数变换为对ξ和η的偏导数，并认为ξ和η是独立的变量[5]，即：

$$\left.\begin{aligned}&\frac{d}{dr}=\varepsilon^{-1}\left(u_r\frac{\partial}{\partial\xi}+\varepsilon\frac{\partial}{\partial\eta}\right)\\&\cdots\cdots\cdots\cdots\cdots\end{aligned}\right\} \tag{1.8}$$

为得到递推方程和边界条件，我们假定挠度函数W和应力函数Φ的N阶渐近表达式为：

$$W(r,\varepsilon)=\sum_{n=0}^{N}\varepsilon^nW_n(r)+\sum_{n=0}^{N}\varepsilon^{n+\alpha}v_n(\xi,\eta) \tag{1.9a}$$

$$\Phi(r,\varepsilon)=\sum_{n=0}^{N}\varepsilon^n\varphi_n(r)+\sum_{n=0}^{N}\varepsilon^{n+\beta}\psi_n(\xi,\eta) \tag{1.9b}$$

将(1.8)和(1.9)分别代入(1.6)和(1.7)，考虑到边界层型函数v_n，ψ_n的性质以及φ_n，w_n

($n=0$, $1,\cdots,N$) 仅是 r 的函数。为得到递推方程和边界条件比较 ε^n 的最低幂次项我们应取 $\alpha=1$, $\beta=3$[3]。然后比较的 ε^n 系数，可得 w_n, v_n, φ_n, ψ_n($n=0$, 1, $\cdots$, N) 的递推方程和边界条件：

$$\frac{1}{r}\frac{d\varphi_0}{dr}\left[\frac{dw_0}{dr}-\pi\delta\sin\pi r\right]=-\left[\frac{qr}{2}+\frac{q}{8r}\delta^2\left(\pi^2 cr^2-\pi cr\sin 2\pi r+\frac{c}{2}-\frac{c}{2}\cos 2\pi r\right)\right] \tag{1.10}$$

$$\frac{d}{dr}\left[\frac{1}{r}\left(r\frac{d\varphi_0}{dr}\right)\right]+\frac{1}{r}\left[\frac{1}{2}\left(\frac{dw_0}{dr}\right)^2-\pi\delta\frac{dw_0}{dr}\sin\pi r\right]=0 \tag{1.11}$$

$$\frac{1}{r}\frac{d\varphi_n}{dr}\frac{dw_0}{dr}-\frac{1}{r}\pi\delta\frac{d\varphi_n}{dr}\sin\pi r+\frac{1}{r}\frac{d\varphi_0}{dr}\frac{dw_n}{dr}=\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw_{n-2}}{dr}\right)\right]-\frac{1}{r}\sum_{i=1}^{n-1}\frac{d\varphi_i}{dr}\frac{dw_{n-i}}{dr} \tag{1.12}$$

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\varphi_n}{dr}\right)\right]-\frac{1}{r}\pi\delta\frac{dw_n}{dr}\sin\pi r+\frac{1}{r}\frac{dw_0}{dr}\frac{dw_n}{dr}=-\frac{1}{2r}\sum_{i=1}^{n-1}\left(\frac{dw_i}{dr}\frac{dw_{n-i}}{dr}\right) \tag{1.13}$$

$$D_0v_0-\frac{u_r}{\eta}\frac{\partial\varphi_0}{\partial\eta}\frac{\partial v_0}{\partial\xi}=0 \tag{1.14}$$

$$D_0\psi_0+\frac{u_r^2}{2\eta}\left(\frac{\partial v_0}{\partial\xi}\right)^2+\frac{u_r}{\eta}\frac{\partial w_0}{\partial\eta}\frac{\partial v_0}{\partial\xi}-\pi\delta\frac{u_r}{\eta}\frac{\partial v_0}{\partial\xi}\sin\pi\eta=0 \tag{1.15}$$

$$\begin{aligned}D_0v_n-\frac{u_r}{\eta}\frac{\partial\varphi_0}{\partial\eta}\frac{\partial v_n}{\partial\xi}=&-\sum_{i=1}^{3}D_iv_{n-i}\\&+\frac{v_r}{\eta}\sum_{i=1}^{n}\frac{\partial\varphi_i}{\partial\eta}\frac{\partial v_{n-i}}{\partial\xi}+\frac{1}{\eta}\sum_{i=0}^{n}\frac{\partial\varphi_i}{\partial\eta}\frac{\partial v_{n-1-i}}{\partial\eta}\\&+\frac{u_r^2}{\eta}\sum_{i=0}^{n}\frac{\partial\psi_i}{\partial\xi}\frac{\partial v_{n-2-i}}{\partial\xi}+\frac{u_r}{\eta}\sum_{i=0}^{n}\frac{\partial\psi_i}{\partial\xi}\frac{\partial v_{n-3-i}}{\partial\eta}\\&+\frac{u_r}{\eta}\sum_{i=0}^{n}\frac{\partial\psi_i}{\partial\eta}\frac{\partial v_{n-3-i}}{\partial\xi}+\frac{1}{\eta}\sum_{i=0}^{n}\frac{\partial\psi_i}{\partial\eta}\frac{\partial v_{n-4-i}}{\partial\eta}\\&+\frac{u_r}{\eta}\sum_{i=0}^{n}\frac{\partial\psi_{n-2-i}}{\partial\xi}\frac{\partial w_i}{\partial\eta}+\frac{1}{\eta}\sum_{i=0}^{n}\frac{\partial\psi_{n-3-i}}{\partial\eta}\frac{\partial w_i}{\partial\eta}\\&-\frac{1}{\eta}\pi\delta\frac{\partial\psi_{n-2}}{\partial\xi}-\frac{1}{\eta}\pi\delta\frac{\partial\psi_{n-3}}{\partial\eta}\end{aligned} \tag{1.16}$$

$$D_0\psi_n=-\sum_{i=1}^{3}D_i\psi_{n-i}-\frac{u_r^2}{2\eta}\sum_{i=0}^{n}\frac{\partial v_i}{\partial \xi}\frac{\partial v_{n-i}}{\partial \xi}$$

$$-\frac{1}{2\eta}\sum_{i=0}^{n}\frac{\partial v_i}{\partial \eta}\frac{\partial v_{n-2-i}}{\partial \eta}-\frac{u_r}{\eta}\sum_{i=0}^{n}\frac{\partial v_i}{\partial \eta}\frac{\partial v_{n-1-i}}{\partial \eta}$$

$$-\frac{u_r}{\eta}\sum_{i=0}^{n}\frac{\partial w_i}{\partial \eta}\frac{\partial v_{n-i}}{\partial \xi}-\frac{1}{\eta}\sum_{i=0}^{n}\frac{\partial w_i}{\partial \eta}\frac{\partial v_{n-1-i}}{\partial \eta}$$

$$+\frac{u_r}{\eta}\pi\delta\frac{\partial v_n}{\partial \xi}\sin\pi\eta+\frac{1}{\eta}\pi\delta\frac{\partial v_{n-1}}{\partial \eta}\sin\pi\eta \tag{1.17}$$

而递推边界条件为：

$$w_0|_{r=1}=0 \tag{1.18}$$

$$w_{0,r}\Big|_{r=1}+u_r\frac{\partial v_0}{\partial \xi}\Big|_{\eta=1}=0 \tag{1.19}$$

$$w_{0,r}\Big|_{r=0}+u_r\frac{\partial v_0}{\partial \xi}\Big|_{\eta=0}=0 \tag{1.20}$$

$$\varphi_{0,r}|_{r=0}=0 \tag{1.21}$$

$$\varphi_{0,rr}\Big|_{r=1}-\frac{\mu}{r}\varphi_{0,r}\Big|_{r=1}=0 \tag{1.22}$$

$$w_n|_{r=1}+v_{n-1}|_{\eta=1}=0 \tag{1.23}$$

$$w_{n,r}\Big|_{r=1}+u_r\frac{\partial v_n}{\partial \xi}\Big|_{\eta=1}+\frac{\partial v_{n-1}}{\partial \eta}\Big|_{\eta=1}=0 \tag{1.24}$$

$$w_{n,r}\Big|_{r=0}+u_r\frac{\partial v_n}{\partial \xi}\Big|_{\eta=0}+\frac{\partial v_{n-1}}{\partial \eta}\Big|_{\eta=0}=0 \tag{1.25}$$

$$\varphi_{n,r}\Big|_{r=0}+u_r\frac{\partial \psi_{n-2}}{\partial \xi}\Big|_{\eta=0}+\frac{\partial \psi_{n-3}}{\partial \eta}\Big|_{\eta=0}=0 \tag{1.26}$$

$$\varphi_{n,rr}\Big|_{r=1}-\frac{\mu}{r}\varphi_{n,r}\Big|_{r=1}+u_r^2\frac{\partial^2\psi_{n-1}}{\partial \xi^2}\Big|_{\eta=1}$$

$$+\left(2u_r\frac{\partial^2}{\partial\xi\partial\eta}+u_{rr}\frac{\partial}{\partial\xi}\right)\psi_{n-2}\Big|_{\eta=1}+\frac{\partial^2\psi_{n-3}}{\partial\eta^2}\Big|_{\eta=1}$$

$$-\frac{\mu u_r}{\eta}\frac{\partial\psi_{n-2}}{\partial\xi}\Big|_{\eta=1}-\frac{\mu}{\eta}\frac{\partial\psi_{n-3}}{\partial\eta}\Big|_{\eta=1}=0 \tag{1.27}$$

其中 $D_0=u_r^3\dfrac{\partial^3}{\partial\xi^3}$,

$$D_1=3u_r^2\frac{\partial^3}{\partial\xi^2\partial\eta}+\left(\frac{u_r^2}{\eta}+3u_ru_{rr}\right)\frac{\partial^2}{\partial\xi^2},$$

$$D_2=\left(\frac{2u_r}{\eta}+3u_{rr}\right)\frac{\partial^2}{\partial\xi\partial\eta}+3u_r\frac{\partial^3}{\partial\xi\partial\eta^2}+\left(\frac{u_{rr}}{\eta}+u_{rrr}-\frac{u_r}{\eta^2}\right)\frac{\partial}{\partial\xi},$$

$$D_3=\frac{\partial^3}{\partial\eta^3}+\frac{1}{\eta}\frac{\partial^2}{\partial\eta^2}-\frac{1}{\eta^2}\frac{\partial}{\partial\eta}.$$

今后，具有负下标的量都取为零。

以下我们来求挠度函数W和应力函数Φ的形式渐近解。

由方程(1.10)和(1.11)和边界条件(1.18)，(1.21)和(1.22)，利用正则摄动法，可决定w_0和$d\varphi_0/dr$。假设w_0和$d\varphi_0/dr$的M阶渐近展开式为：

$$\begin{cases} w_0(r,\delta)=\sum_{n=0}^{M}\delta^n w_{0n} & (1.28a)\\ \dfrac{d\varphi_0(r,\delta)}{dr}=\sum_{n=0}^{M}\delta^n \dfrac{d\varphi_{0n}}{dr} & (1.28b)\end{cases}$$

把(1.28)式代入方程(1.10)和(1.11)，并比较δ^n $(n=0,\ \cdots,\ M)$的系数，使δ^0的系数等于$-qr/2$,而δ的其他幂次的系数都分别相等,则得关于w_{0n}，φ_{0n},$(n=0,1,\cdots,M)$的递推方程：

$$\frac{1}{r}\frac{d\varphi_{00}}{dr}\frac{dw_{00}}{dr}=-\frac{qr}{2} \tag{1.29}$$

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\varphi_{00}}{dr}\right)\right]=-\frac{1}{2r}\left(\frac{dw_{00}}{dr}\right)^2 \tag{1.30}$$

$$\frac{1}{r}\frac{d\varphi_{00}}{dr}\left(\frac{dw_{01}}{dr}-\pi\sin\pi r\right)+\frac{1}{r}\frac{d\varphi_{01}}{dr}\frac{dw_{00}}{dr}=0 \tag{1.31}$$

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\varphi_{01}}{dr}\right)\right]+\frac{1}{r}\left[\frac{dw_{00}}{dr}\frac{dw_{01}}{dr}-\pi\frac{dw_{00}}{dr}\sin\pi r\right]=0 \tag{1.32}$$

$$\frac{1}{r}\frac{d\varphi_{02}}{dr}\frac{dw_{00}}{dr}+\frac{1}{r}\frac{d\varphi_{01}}{dr}\left(\frac{dw_{01}}{dr}-\pi\sin\pi r\right)+\frac{1}{r}\frac{d\varphi_{00}}{dr}\frac{dw_{02}}{dr}$$
$$=-\frac{q}{8r}\left[\pi^2cr^2-\pi cr\sin2\pi r+\frac{c}{2}(1-\cos2\pi r)\right. \tag{1.33}$$

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\varphi_{02}}{dr}\right)\right]+\frac{1}{r}\frac{dw_{02}}{dr}\frac{dw_{00}}{dr}$$
$$+\frac{1}{2r}\left(\frac{dw_{01}}{dr}\right)^2-\frac{\pi}{r}\frac{dw_{01}}{dr}\sin\pi r=0 \tag{1.34}$$

. . . .

把(1.28)式代入边界条件(1.18),(1.21)和(1.22)，并比较$\delta^n(n=0,\ 1,\cdots,M)$的系数，使$\delta^n$的同次幂的系数等于零，则得$w_{0n}$，$\varphi_{0n}$的边界条件：

$$w_{00}|_{r=1}=0 \tag{1.35}$$

$$\left.\frac{d\varphi_{00}}{dr}\right|_{r=0}=0 \tag{1.36}$$

$$\varphi_{00,rr}\Big|_{r=1}-\frac{\mu}{r}\varphi_{00,r}\Big|_{r=1}=0 \tag{1.37}$$

$$w_{0n}|_{r=1}=0 \tag{1.38}$$

$$\varphi_{0n,r}|_{r=0}=0 \tag{1.39}$$

$$\varphi_{0n,rr}\Big|_{r=1}-\frac{\mu}{r}\varphi_{0n,r}\Big|_{r=1}=0 \tag{1.40}$$

由方程(1.29)，(1.30)和边界条件(1.35)～(1.37)，利用幂级数解法[6]可得：

$$\frac{d\varphi_{00}}{dr}=a_1r+a_3r^3+\cdots\cdots \tag{1.41}$$

其中，

$$a_1=\frac{1}{4}\left(\frac{3-\mu}{1-\mu}q^2\right)^{1/3},\quad a_3=-\frac{1}{4}\left(\frac{1-\mu}{3-\mu}q\right)^{2/3}$$

把(1.41)代入(1.29)，并积分，由边界条件(1.35)，则得：

$$w_{00}=-\frac{q}{4a_3}\ln\left(\frac{1-(1-\mu)r^2/(3-\mu)}{1-(1-\mu)/(3-\mu)}\right) \tag{1.42}$$

我们把(1.41)和(1.42)代入(1.31)和(1.32)，并根据边界条件(1.38)，(1.39)和(1.40)（分别取$n=1$）则得：

$$\frac{d\varphi_{01}}{dr}=0 \tag{1.43}$$

$$w_{01}=-(1+\cos\pi r) \tag{1.44}$$

我们把(1.43)和(1.44)式代入方程(1.33)和(1.34)，并利用边界条件(1.38)～(1.40)（分别取$n=2$），则可确定w_{02}和$d\varphi_{02}/dr$：

$$\frac{d\varphi_{02}}{dr}=e_1r+e_3r^3+e_5r^5+\cdots \tag{1.45}$$

$$w_{02}=-\frac{q\pi^2c}{8}\left[\frac{1}{2a_3}\ln\left(\frac{a_1+a_3r^3}{a_1+a_3}\right)\right]+\frac{q\pi c}{8}\int_1^r\frac{\sin 2\pi r}{a_1+a_3r^2}dr$$
$$+\frac{qc}{16}\int_1^r\frac{(\cos 2\pi r-1)}{r(a_1+a_3r^2)}dr+\frac{q}{2}\int_1^r\frac{r(e_1+e_3r^2+e_5r^4)}{(a_1+a_3r^2)^2} \tag{1.46}$$

其中
$$e_1=\frac{-\pi^4/16}{(1-\mu)+(3-\mu)\frac{q^3}{32a_1^2}+\frac{(5-\mu)q^2(q^2-96a_1^2a_3)}{3072a_1^6}}\left\{(3-\mu)\right.$$
$$\left.+\frac{(5-\mu)(q^2-96a_1^2a_3)}{96a_1^3}+\frac{(5-\mu)}{72}\left(72\left(\frac{a_3}{a_1}\right)-\frac{3q_c^2}{a_1^2}-8\pi^2\right)\right\}$$

$$e_3=\frac{q^2}{32a_1^3}e_1+\frac{\pi^4}{16}$$

$$e_5=\frac{1}{96a_1^3}(q^2-96a_1^2a_3)e_3+\frac{\pi^4}{1152}\left(72\frac{a_3}{a_1}-\frac{3q_c^2}{a_1^2}-8\pi^2\right)$$

按以上步骤，我们可以逐次确定w_{0n}，$d\varphi_{0n}/dr$ ($n=0,1,\cdots,M$)。对于w_{0n}和$d\varphi_{0n}/dr$限于求到2阶近似解，把(1.41)，(1.42)，(1.43)，(1.44)，(1.45)和(1.46)式分别代入(1.28a)和(1.28b)，则得：

$$w_0(r,\delta)=-\frac{q}{4a_3}\ln\left(\frac{1-(1-\mu)r^2/(3-\mu)}{1-(1-\mu)/(3-\mu)}\right)-\delta(1+\cos\pi r)$$
$$-\delta^2\left[\frac{q\pi_c^2}{16a_3}\ln\left(\frac{a_1+a_3r^2}{a_1+a_3}\right)-\frac{qc}{16}\int_1^r\frac{(\cos 2\pi r-1)}{r(a_1+a_3r^2)}dr\right.$$
$$\left.-\frac{q\pi c}{8}\int_1^r\frac{\sin 2\pi r}{(a_1+a_3r^2)}dr-\frac{q}{2}\int_1^r\frac{r(e_1+e_3r^2+e_5r^4)}{(a_1+a_3r^2)^2}dr\right]+O(\delta^3) \tag{1.47}$$

$$\frac{d\varphi_0(r,\delta)}{dr}=(a_1r+a_3r^3)+\delta^2(e_1r+e_3r^3+e_5r^5)+O(\delta^3) \tag{1.48}$$

决定w_0和$d\varphi_0/dr$后，把(1.47)和(1.48)代入(1.44)，可得关于v_0的微分方程：

$$u_r^3\frac{\partial^2 v_0}{\partial\xi^3}-\frac{u_r}{\eta}[(a_1r+a_3r^3)+\delta^2(e_1r+e_3r^3+e_5r^5)]\frac{\partial v_0}{\partial\xi}=0 \tag{1.49}$$

如果选取，$u_r^2=[(a_1r+a_3r^3)+\delta^2(e_1r+e_3r^3+e_5r^5)]/\eta$

即：
$$u_r(r)=\int_r^1[(a_1+a_3\eta^2)+\delta^2(e_1+e_3\eta^2+e_5\eta^4)]^{1/2}d\eta \tag{1.50}$$

此时，方程(1.49)变为齐次方程：

$$\frac{\partial^3 v_0}{\partial\xi^3}-\frac{\partial v_0}{\partial\xi}=0 \tag{1.51}$$

由方程(1.51)，可得v_0的边界层型的解：

$$\begin{aligned}v_0&=c_0(\eta)\exp[-\xi]\\&=c_0(r)\exp\left[-\varepsilon^{-1}\int_r^1[(a_1+a_3\eta^2)+\delta^2(e_1+e_3\eta^2+e_5\eta^4)]^{1/2}d\eta\right]\end{aligned} \tag{1.52}$$

其中$c_0(\eta)$是待定的任意函数（参见(1.60)和(1.61)）。若把w_0和v_0代入方程(1.15)，则得关于ψ_0的微分方程：

$$\begin{aligned}u_r^3\frac{\partial^3\psi_0}{\partial\xi^3}=&-\frac{u_r^2}{2\eta}c_0^2(\eta)\exp(-2\xi)-\left\{\frac{u_r}{\eta}\left[\frac{q\eta}{2(a_1+a_3\eta^2)}\right.\right.\\&+\frac{q\delta^2}{8(a_1\eta+a_3\eta^3)}\left(\pi^2c\eta^2-\pi c\eta\sin 2\pi\eta+\frac{c}{2}(1-\cos 2\pi\eta)\right)\\&\left.\left.-\frac{q\delta^2(e_1\eta+e_3\eta^3+e_5\eta^5)}{2(a_1+a_3\eta^2)^2}\right]\right\}c_0(\eta)\exp(-\xi)\end{aligned} \tag{1.53}$$

由方程(1.53)可得关于ψ_0的边界层型的解：

$$\begin{aligned}\psi_0=&\frac{1}{16\eta u_r}c_0^2(\eta)\exp(-2\xi)\\&+\frac{1}{\eta u_r^2}\left\{\frac{q\eta}{2(a_1+a_3\eta^2)}+\frac{q\delta^2}{8(a_1\eta+a_3\eta^3)}[\pi^2c\eta^2\right.\\&-\pi c\eta\sin2\pi\eta+\frac{c}{2}(1-\cos2\pi\eta)\\&\left.-\frac{q\delta^2(e_1\eta+e_3\eta^3+e_5\eta^5)}{2(a_1+a_3\eta^2)^2}\right\}c_0(\eta)\exp(-\xi)\end{aligned} \tag{1.54}$$

我们再把w_0和$d\varphi_0/dr$代入方程(1.12)和(1.13)（分别取$n=1$），则得关于w_1和φ_1的微分方程，然后先消去dw_1/dr，并略去包含δ^2的项，得φ_1的微分方程：

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\varphi_1}{dr}\right)\right]-\frac{q^2}{4(a_1+a_3r^2)^2}\frac{d\varphi_1}{dr}=0 \tag{1.55}$$

由方程(1.55)和边界条件(1.26)及(1.27)（分别取$n=1$），类似地，利用幂级数解法可确定$d\varphi_1/dr$：

$$\frac{d\varphi_1}{dr}=b_1r+b_3r^3+\cdots \tag{1.56}$$

其中 $$b_1=-\frac{1}{(1-\mu)+q^2(3-\mu)/32a_1^3}\left\{\left[\frac{q}{2(a_1+a_3)}+\frac{qc\delta^2(\pi^2-1)}{8(a_1+a_3)}-\frac{q\delta^2(e_1+e_3+e_5)}{2(a_1+a_3)}\right]c_0(1)+\frac{u_r(1)c_0^2(1)}{4}\right\},$$

$$b_3=\frac{q^2}{32a_1^3}b_1$$

我们把(1.47)，(1.48)和(1.56)代入(1.12)，并利用边界条件(1.23)（取$n=1$），则得：

$$\begin{aligned}w_1=&\frac{b_3q}{4a_3^2}\ln\left(\frac{a_1+a_3r^2}{a_1+a_3}\right)\\&-\frac{q}{4a_3}\left(\frac{b_1+b_3r^2}{a_1+a_3r^2}-\frac{b_1+b_3}{a_1+a_3}\right)-c_0(1)\\&+\frac{q\delta^2}{16}\int_1^r\frac{(b_1+b_3r^2)\left(\pi^2cr^2-\pi cr\sin 2\pi r+\frac{c}{2}(1-\cos 2\pi r)\right)}{r(a_1+a_3r^2)^3}dr\\&+\frac{q\delta^2}{2}\int_1^r\frac{e_1r^2+e_3r^4+e_5r^6}{(a_1+a_3r^2)^2}dr\end{aligned}\tag{1.57}$$

由方程(1.16)（取$n=1$），得关于v_1的微分方程：

$$D_0v_1-\frac{u_r}{\eta}\frac{\partial\varphi_0}{\partial\eta}\frac{\partial v_1}{\partial\xi}=-D_1v_0+\frac{u_r}{\eta}\frac{\partial\varphi_1}{\partial\eta}\frac{\partial v_0}{\partial\xi}+\frac{1}{\eta}\frac{\partial\varphi_0}{\partial\eta}\frac{\partial v_0}{\partial\eta}\tag{1.58}$$

展开(1.58)的右端，并令exp(-ξ)的系数等于零，得关于$c_0(\eta)$的微分方程：

$$\begin{aligned}&2[(a_1+a_3\eta^2)+\delta^2(e_1+e_3\eta^2+e_5\eta^4)]\frac{\partial c_0(\eta)}{\partial\eta}+\frac{1}{\eta}\{(a_1+a_3\eta^2)\\&\quad+\delta^2(e_1+e_3\eta^2+e_5\eta^4)+3[a_3\eta^2+\delta^2(e_3\eta^2+2e_5\eta^4)]\\&\quad+[(a_1+a_3\eta^2)+\delta^2(e_1+e_3\eta^2+e_5\eta^4)]^{1/2}(b_1\eta+b_3\eta^3)\}c_0(\eta)=0\end{aligned}\tag{1.59}$$

由方程(1.59)和边界条件(1.19)则得：

$$\begin{aligned}c_0(\eta)=c_0(1)\exp\Big[\int_r^1&A^{-1}\{a_1+4a_3\eta^2+\delta^2(e_1+4e_3\eta^2+7e_5\eta^4)\\&+[(a_1+a_3\eta^2)+\delta^2(e_1+e_3\eta^2+e_5\eta^4)]^{1/2}(b_1\eta+b_3\eta^3)\}d\eta\Big]\end{aligned}\tag{1.60}$$

其中 $A=2\eta[(a_1+a_3\eta^2)+\delta^2(e_1+e_3\eta^2+e_5\eta^4)]$

$$c_0(1)=\frac{q\left[1+\frac{\pi^2c\delta^2}{4}-\frac{\delta^2(e_1+e_3+e_5)}{(a_1+a_3)}\right]}{2(a_1+a_3)[(a_1+a_3)+\delta^2(e_1+e_3+e_5)]^{1/2}}\tag{1.61}$$

把(1.60)代入(1.52)，并引入截断函数$\zeta(r)$：$\zeta(r)=1$，当$2/3\leqslant r\leqslant 1$时；$\zeta(r)=0$，当$0\leqslant r<2/3$时，则得$v_0$：

$$\begin{aligned}v_0=\zeta(r)c_0(1)\exp\Big[&-\varepsilon^{-1}\int_r^1[a_1+a_3\eta^2+\delta^2(e_1+e_3\eta^2+e_5\eta^4)]^{1/2}d\eta\\&+\int_r^1A^{-1}\{a_1+4a_3\eta^2+\delta^2(e_1+4e_3\eta^2+7e_5\eta^4)\\&+[a_1+a_3\eta^2+\delta^2(e_1+e_3\eta^2+e_5\eta^4)]^{1/2}(b_1\eta+b_3\eta^3)\}d\eta\Big]\end{aligned}\tag{1.62}$$

式中A同(1.60)式中的A。此时，关于v_1的微分方程(1.58)变为：

$$u_r^3\frac{\partial^3 v_1}{\partial\xi^3}-\frac{u_r}{\eta}\frac{\partial\varphi_0}{\partial\eta}\frac{\partial v_1}{\partial\xi}=0 \tag{1.63}$$

类似地，我们可以求得关于v_1的边界层型解：

$$v_1=c_1(\eta)\exp\left[-\varepsilon^{-1}\int_r^1[a_1+a_3\eta^2+\delta^2(e_1+e_3\eta^2+e_6\eta^4)]^{\frac{1}{2}}d\eta\right. \tag{1.64}$$

其中$c_1(\eta)$是待定的任意函数。

由方程(1.17)(取$n=1$)，则得关于ψ_1的微分方程：

$$D_0\psi_1=-D_1\psi_0-\frac{u_r^2}{\eta}\frac{\partial v_0}{\partial\xi}\frac{\partial v_1}{\partial\xi}-\frac{u_r}{\eta}\frac{\partial v_0}{\partial\xi}\frac{\partial v_0}{\partial\eta}$$

$$-\frac{u_r}{\eta}\frac{\partial w_0}{\partial\eta}\frac{\partial v_1}{\partial\xi}-\frac{u_r}{\eta}\frac{\partial w_1}{\partial\eta}\frac{\partial v_0}{\partial\xi}-\frac{1}{\eta}\frac{\partial w_0}{\partial\eta}\frac{\partial v_0}{\partial\eta}$$

$$+\frac{u_r}{\eta}\pi\delta\frac{\partial v_1}{\partial\xi}\sin\pi\eta+\frac{1}{\eta}\pi\delta\frac{\partial v_0}{\partial\eta}\sin\pi\eta \tag{1.65}$$

把w_0，w_1，v_0，v_1和ψ_0代入方程(1.65)；并积分，可得关于ψ_1的边界层型的解。

按以上步骤可逐次求得w_n，v_n，φ_n，$\psi_n(n=0,\ 1,\ 2,\ \cdots,\ N)$，然后把它们分别代入(1.9a)和(1.9b)式，将得到关于挠度函数W和应力函数Φ的N阶一致有效渐近解，限于一阶近似解，在此情况下，我们得到：

$$W=-\frac{q}{4a_3}\ln\left(\frac{1-(1-\mu)r^2/(3-\mu)}{1-(1-\mu)/(3-\mu)}\right)-\delta(1+\cos\pi r)$$

$$-\delta^2\left\{\frac{q\pi^2c}{16a_3}\ln\left(\frac{a_1+a_3r^2}{a_1+a_3}\right)+\frac{qc}{16}\int_1^r\frac{(1-\cos2\pi r)}{r(a_1+a_3r^2)}dr\right.$$

$$\left.-\frac{qc}{8}\int_1^r\frac{\sin2\pi r}{a_1+a_3r^2}dr-\frac{q}{2}\int_1^r\frac{(e_1r+e_3r^3+e_6r^5)}{(a_1+a_3r^2)^2}dr\right\}$$

$$+\varepsilon\left\{\frac{b_3q}{4a_3^2}\ln\left(\frac{a_1+a_3r^2}{a_1+a_3}\right)-\frac{q}{4a_3}\left(\frac{b_1+b_3r^2}{a_1+a_3r^2}-\frac{b_1+b_3}{a_1+a_3}\right)-c_0(1)\right.$$

$$+\frac{q\delta^2}{16}\int_1^r\frac{(b_1+b_3r^2)[\pi^2cr^2-\pi cr\sin2\pi r+(1-\cos2\pi r)c/2]}{r(a_1+a_3r^2)^3}dr$$

$$\left.+\frac{q\delta^2}{2}\int_1^r\frac{e_1r^2+e_3r^4+e_6r^6}{(a_1+a_3r^2)}dr\right\}$$

$$+\varepsilon\left\{\zeta(r)c_0(1)\exp\left[-\varepsilon^{-1}\int_r^1[(a_1+a_3\eta^2)+\delta^2(e_1+e_3\eta^2+e_6\eta^4)]^{1/2}d\eta\right.\right.$$

$$+\int_r^1A^{1-}\left\{a_1+4a_3\eta^2+\delta^2(e_1+4e_3\eta^2+7e_6\eta^4)\right.$$

$$+[(a_1+a_3\eta^2)+\delta^2(e_1+e_3\eta^2+e_6\eta^4)]^{1/2}$$

$$\left.\left.\left.\cdot(b_1\eta+b_3\eta^3)\right\}d\eta\right]\right\}+O(\varepsilon^2) \tag{1.66}$$

式中A同式(1.61)中A。

$$\frac{d\Phi}{dr}=\{(a_1r+a_3r^3)+\delta^2(e_1r+e_3r^3+e_6r^5)\}$$

$$+\varepsilon(b_1r+b_3r^3)-\varepsilon^2\zeta(r)\left\{\frac{1}{8r}c_0^2(r)\exp(-2\xi)\right.$$

$$+\frac{1}{ru_r}\left\{\left[\frac{qr}{2(a_1+a_3r^2)}+\frac{q\delta^2}{8(a_1r+a_3r^3)}(\pi^2cr^2\right.\right.$$

$$\left.-\pi cr\sin 2\pi r+\frac{c}{2}(1-\cos 2\pi r)\right]$$

$$\left.\left.-\frac{qr\delta^2}{2(a_1+a_3r^2)}(e_1r+e_3r^3+e_5r^5)\right\}c_0(r)\exp(-\xi)\right\}+O(\varepsilon^2) \tag{1.67}$$

二、结 论

1．由方程(1.66)和(1.67)式我们可以看出，当初始挠度为零时，具有初始挠度的圆薄板的解就变为圆薄板的非线性弯曲问题的解[6]：

$$W=-\frac{q}{4a_3}\ln\left(\frac{1-(1-\mu)r^2/(3-\mu)}{1-(1-\mu)/(3-\mu)}\right)+\varepsilon\left\{\frac{b_3q}{4a_3^2}\ln\left(\frac{a_1+a_3r^2}{a_1+a_3}\right)\right.$$

$$\left.-\frac{q}{4a_3}\left(\frac{b_1+b_3r^2}{a_1+a_3r^2}-\frac{b_1+b_3}{a_1+a_3}\right)-c_0(1)\right\}$$

$$+\varepsilon\eta(r)\left\{\frac{q}{2(a_1+a_3)^{3/2}}\exp\left[-\varepsilon^{-1}\int_r^1(a_1+a_3\eta^2)^{1/2}d\eta\right.\right.$$

$$\left.\left.+\int_r^1\frac{[a_1+4a_3\eta^2+(a_1+a_3\eta^2)^{1/2}(b_1\eta+b_3\eta^3)]}{2\eta(a_1+a_3\eta^2)}d\eta\right]\right\}+O(\varepsilon^2) \tag{2.1}$$

$$\frac{d\Phi}{dr}=(a_1r+a_3r^3)+\varepsilon(b_1r+b_3r^3)$$

$$-\varepsilon^2\zeta(r)\left\{\frac{c_0^2(\eta)}{8\eta}\exp(-2\xi)+\frac{qc_0(\eta)}{2u_r(a_1+a_3\eta^2)}\exp(-\xi)\right\}+O(\varepsilon^2) \tag{2.2}$$

因此，当初始挠很小时，具有初始挠度的圆薄板不会发生跳跃现象。

2．如果初始挠度较大，且初始挠度与横向载荷强度 q 的符号相反时，当横向载荷强度达到某一值时，它将产生跳跃现象，图1表明具有初始挠度的圆薄板的挠度-载荷曲线。

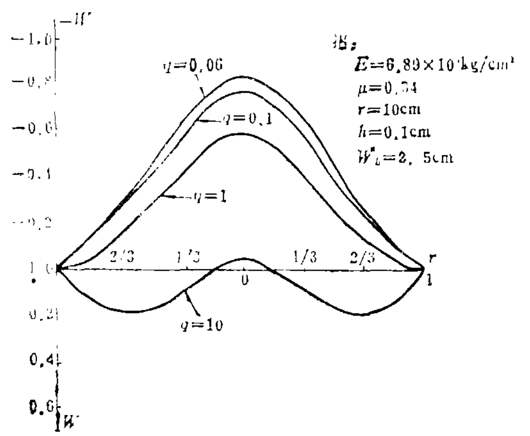

图1 具有初始挠度的圆薄板的挠度曲线

3．本文利用奇异摄动与正则摄动相结合的方法研究了含有两个独立小参数的非线性微分方程的边值问题，我们把这种方法称为混合摄动法。这种方法也适用于处理含有多个独立小参数的边值问题。

参考文献

[1] Chien Wei-zang, Asymptotic behavior of a thin clamped circular plate under uniform normal pressure at very large deflection, *Sci. Rep. Nat. Tsinghua Univ. Ser.*, A, 5 (1948), 71-94.

[2] 江福汝，环形和圆形薄板在各种支承条件下的非对称弯曲问题(Ⅰ)，应用数学和力学，**3**,5 (1982), 629-640.

[3] 江福汝，环形和圆形薄板在各种支承条件下的非对称弯曲问题(Ⅱ)，应用数学和力学，**5**,2 (1984), 191-202.

[4] 江福汝，摄动方法在薄板弯曲问题中的某些应用，应用数学和力学，**1**, 1 (1980), 37-53.

[5] 秦圣立、张爱淑，关于环形薄板的屈曲问题，应用数学和力学，**6**, 2 (1985), 175-190.

[6] 秦圣立、张爱淑、康连成，关于圆形薄板的轴对称非线性弯曲问题，第一届华东固体力学学术讨论会论文集，(1984).

[7] R. E. 奥写利，《奇异摄动引论》，科学出版社 (1983), 24-30.

[8] A. C. 沃耳密尔，《柔韧板与柔韧壳》，科学出版社 (1959), 162-193.

On The Jumping Problems of A Circular Thin Plate With Initial Deflection

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Abstract

In this paper, the jumping problems of a circular thin plate with initial deflection are studied by using "the method of two variables"[3][4] proposed by Jiang Fu-ru and the method of the normal perturbation (in this paper (1.1),(1.2)). we obtained Nth-order uniformly valid asymptotic expansion of the solution of this problem ((1.66), (1.67)). When the initial deflection vanishes the solution of a circular thin plate with inital deflection is reduced to the solution of the problems of the nonlinear bending of a circular thin plate[6]. If the initial deflection is largish and the signs of the initial deflection with the intensity of the transverse load are opposite, when the intensity of the transverse load reaches a certain value, the circular thin plate with initial deflection should produce the jumping phenomenon[8].