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旋转薄壳的轴对称边缘弯曲问题*

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摘 要

本文给出了旋转薄壳轴对称边缘弯曲问题的一致有效渐近解．

符 号 说 明

$\tilde{A}_1$ 和 $\tilde{A}_2$，$\tilde{C}_1$ 至 $\tilde{C}_4$　复常数
B_1 至 B_4，ψ 和 ψ_1　实常数
E　弹性模量
H，V　径向和轴向内力
h　壁厚
M_φ，M_θ　经向和环向弯矩
N_φ，N_θ　经向和环向内力
Q_φ　横剪力
r　$r=r_2\sin\varphi$
r_1　经向主曲率半径
r_2　环向主曲率半径
s　从经线上某一基准计算起的经线长度
U　$U=r_2Q_\varphi$
Δ　径向位移
ϑ　经线切线方向的角位移
μ　泊松比
φ　壳面法线与旋转轴之间夹角
r^*，V^*，s^*　分别是 r，V，s 在上边界处的值
r_{2*}，s_*　分别是 r_2，s 在下边界处的值
$(\cdots)_0$　在某一边界处该值

一、前　言

在一般情况下，工程上的旋转薄壳是一个轴对称问题．旋转薄壳轴对称问题力矩理论的齐次方程是：

$$\left.\begin{aligned}
&r_2\frac{d^2\vartheta}{ds^2}+\left(\cot\varphi+3\frac{r_2}{h}\frac{dh}{ds}\right)\frac{d\vartheta}{ds}-\left(\frac{\cot^2\varphi}{r_2}+\frac{\mu}{r_1}\right.\\
&\qquad\left.-3\frac{\mu}{h}\frac{dh}{ds}\cot\varphi\right)\vartheta=-\frac{12(1-\mu^2)}{Eh^3}U\\
&r_2\frac{d^2U}{ds^2}+\left(\cot\varphi-\frac{r_2}{h}\frac{dh}{ds}\right)\frac{dU}{ds}-\left(\frac{\cot^2\varphi}{r_2}-\frac{\mu}{r_1}\right.\\
&\qquad\left.-\frac{\mu}{h}\frac{dh}{ds}\cot\varphi\right)U=Eh\vartheta
\end{aligned}\right\}\tag{1.1}$$

在旋转薄壳轴对称边缘弯曲问题中，一般是略去式(1.1)左边中 U 和 ϑ 及其一阶导数，

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即得：

$$r_2\frac{d^2\vartheta}{ds^2}=-\frac{12(1-\mu^2)}{Eh^3}U,\qquad r_2\frac{d^2U}{ds^2}=Eh\vartheta \tag{1.2}$$

由式(1.2)得：

$$\frac{d^4U}{ds^4}+4\delta^4U=0 \tag{1.3}$$

式中： $\delta=\sqrt[4]{3(1-\mu^2)}/\sqrt{r_2h}$

对一般旋转薄壳来讲，δ 不是一个常数。但在边缘弯曲问题中，在某一边缘及其附近，可将 δ 视作常数。那么，式(1.3)的解是：

$$U=B_1\exp[-\delta s]\cos\delta s+B_2\exp[-\delta s]\sin\delta s+B_3\exp[\delta s]\cos\delta s+B_4\exp[\delta s]\sin\delta s \tag{1.4}$$

通常称式(1.4)为一次近似解。它只适用于壳顶封闭或一个边缘对另一个边缘影响可以略去不计的薄壳。当有二个边缘时，二个边缘相矩又近，一个边缘对另一个边缘影响不能略去不计时，用式(1.4)将会引起很大的误差。本文给出了旋转薄壳轴对称弯曲问题的高次近似的一致有效渐近解和实用计算公式，提高了计算精度，并给出了二边缘相距较近时的计算公式。本文给出的一次渐近解也比上述一次近似解的精度高。对等厚圆球形薄壳来讲，本文给出的一次渐近解与S. Timoshenko[1]给出的相应的二次近似解是完全一致的。本文给出了旋转薄壳轴对称边缘弯曲问题的一次渐近解的一般式，重要的是本文给出了高次近似的一致有效渐近解。

二、基 本 方 程

旋转薄壳轴对称问题的载荷、内力和角位移、径向位移如图1。

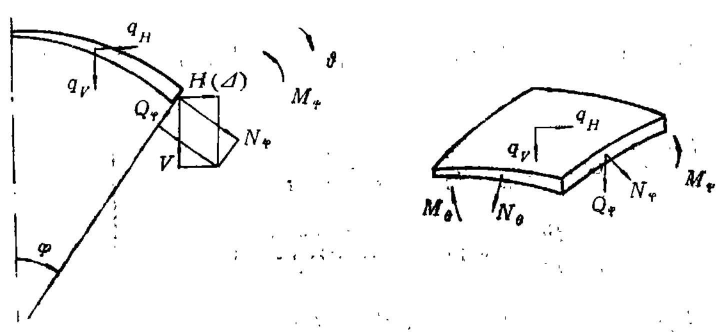


图 1

由图1可知，径向力和轴向力是：

$$H=N_\varphi\cos\varphi-Q_\varphi\sin\varphi,\qquad V=N_\varphi\sin\varphi+Q_\varphi\cos\varphi \tag{2.1}$$

在式(1.1)中没有考虑载荷 q_H 和 q_V 的作用，现我们考虑了 q_H 和 q_V 作用，并通过变换式(2.1)，在式(1.1)右边出现了载荷项，那末式(1.1)变成：

$$\left.\begin{aligned}&r_2\frac{d^2\vartheta}{ds^2}+\left(\cot\varphi+3\frac{r_2}{h}\frac{dh}{ds}\right)\frac{d\vartheta}{ds}-\left(\frac{\cot^2\varphi}{r_2}+\frac{\mu}{r_1}\right.\\&\qquad\left.-3\frac{\mu}{h}\frac{dh}{ds}\cot\varphi\right)\vartheta=\frac{12(1-\mu^2)}{Eh^3}(rH-rV\cot\varphi)\\&r_2\frac{d^2(rH)}{ds^2}+\left(\cot\varphi-\frac{r_2}{h}\frac{dh}{ds}\right)\frac{d(rH)}{ds}-\left(\frac{\cot^2\varphi}{r_2}-\frac{\mu}{r_1}\right.\\&\qquad\left.-\frac{\mu}{h}\frac{dh}{ds}\cot\varphi\right)(rH)=-Eh\vartheta+F(s)\end{aligned}\right\}\tag{2.2}$$

式中：
$$F(s)=-\left(2\cos\varphi+\mu\cos\varphi-\frac{r}{h}\frac{dh}{ds}\right)r_2q_H-r_2^2\frac{dq_H}{ds}\sin\varphi-\mu r_2q_V\sin\varphi$$
$$-\left(\frac{1}{r_2}+\frac{\mu}{r_1}-\mu\frac{\tan\varphi}{h}\frac{dh}{ds}\right)\left(\int_{s^*}^{s}rq_Vds-r^*V^*\right)\cot\varphi$$

作以下变换：

$$\left.\begin{aligned}&\Theta=\sqrt[4]{r_2h^5\sin^2\varphi}\ \vartheta\\&\Pi=\sqrt[4]{\frac{r_2\sin^2\varphi}{h^3}}\left[rH+\left(\int_{s^*}^{s}rq_Vds-r^*V^*\right)\cot\varphi\right]\\&\frac{d\xi}{ds}=\frac{1}{\lambda_0\sqrt{r_2h}}\end{aligned}\right\}\tag{2.3}$$

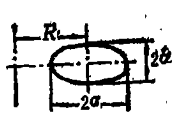
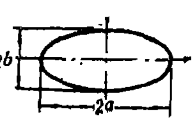
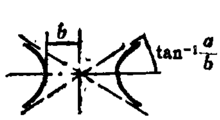
式中：对 φ 变化的薄壳，有：

$$\left.\begin{aligned}&r_1=R_1f_1(\varphi),\ r_2=R_2f_2(\varphi),\ h=tf_3(\varphi)\\&\lambda_0^2=\frac{R_1}{R_2t},\ \frac{d\xi}{ds}=\frac{1}{r_2f(\xi)},\ f(\xi)=\frac{R_1}{R_2}\sqrt{\frac{f_3(\varphi)}{f_2(\varphi)}}\end{aligned}\right\}\tag{2.4}$$

R_1，R_2 和 t 都是常数。对一些重要的 φ 可变化的旋转薄壳的 R_1，R_2 和 $f_1(\varphi)$，$f_2(\varphi)$ 的值如表 1。对圆柱薄壳有：

$$\left.\begin{aligned}&r_1=\infty,\ r_2=\text{const},\ h=tf_3(s)\\&\lambda_0^2=\frac{r_2}{t},\ \frac{d\xi}{ds}=\frac{1}{r_2f(\xi)},\ f(\xi)=\sqrt{f_3(s)}\end{aligned}\right\}\tag{2.5}$$

表 1　　R_1，R_2 和 $f_1(\varphi)$，$f_2(\varphi)$ 的值

薄壳类型	横　截　面	$f_1(\varphi)$	$f_2(\varphi)$	R_1	R_2
椭圆环和圆环 ($a=b$)		$\left[1+\left(\frac{a^2}{b^2}-1\right)\sin^2\varphi\right]^{-\frac{3}{2}}$	$\frac{1}{\sin\varphi}+\frac{a^2}{Rb}\left[1+\left(\frac{a^2}{b^2}-1\right)\cdot\sin^2\varphi\right]^{-\frac{1}{2}}$	$\frac{a^2}{b}$	R
椭球和圆球 ($a=b$)		$\left[1+\left(\frac{a^2}{b^2}-1\right)\sin^2\varphi\right]^{-\frac{3}{2}}$	$\left[1+\left(\frac{a^2}{b^2}-1\right)\sin^2\varphi\right]^{-\frac{1}{2}}$	$\frac{a^2}{b}$	$\frac{a^2}{b}$
双曲面		$\left[\left(\frac{b^2}{a^2}+1\right)\sin^2\varphi-1\right]^{-\frac{3}{2}}$	$\left[\left(\frac{b^2}{a^2}+1\right)\sin^2\varphi-1\right]^{-\frac{1}{2}}$	$\frac{b^2}{a}$	$\frac{b^2}{a}$

对圆锥薄壳有：

$$\left.\begin{aligned}&r_1=\infty,\ r_2=r_{2*}f_2(s),\ h=tf_3(s)\\&\lambda_0^2=\frac{r_{2*}}{t},\ \frac{d\xi}{ds}=\frac{1}{r_2 f(\xi)},\ f(\xi)=\sqrt{\frac{f_3(s)}{f_2(s)}}\end{aligned}\right\}\tag{2.6}$$

由式 (2.2) 得：

$$\frac{d^2\Theta}{d\xi^2}+g_1(\xi)\Theta=\frac{4\beta^4}{E}\lambda_0^2\Pi,\qquad \frac{d^2\Pi}{d\xi^2}+g_2(\xi)\Pi=-E\lambda_0^2(\Theta-\Theta_m)\tag{2.7}$$

式中：

$$\left.\begin{aligned}&\beta=\sqrt[4]{3(1-\mu^2)},\quad g_1(\xi)=\Omega+\Omega_1,\quad g_2(\xi)=\Omega+\Omega_2\\&\Omega=[f(\xi)]^2\Big[\Big(-\frac{15}{16}-\frac{r_2}{8r_1}+\frac{5r_2^2}{16r_1^2}\Big)\cot^2\varphi\\&\qquad+\frac{r_2}{4r_1}\Big(1+\frac{r_2}{r_1}+\frac{r_2}{r_1}\frac{dr_1}{ds}\cot\varphi\Big)\Big]\\&\Omega_1=-[f(\xi)]^2\Big[\frac{\mu r_2}{r_1}+\Big(\frac{11}{8}-3\mu+\frac{r_2}{8r_1}\Big)\frac{r_2}{h}\frac{dh}{ds}\cot\varphi\\&\qquad+\frac{15}{16}\frac{r_2^2}{h^2}\Big(\frac{dh}{ds}\Big)^2+\frac{5}{4}\frac{r_2^2}{h}\frac{d^2h}{ds^2}\Big]\\&\Omega_2=[f(\xi)]^2\Big[\frac{\mu r_2}{r_1}+\Big(\frac{5}{8}+\mu-\frac{r_2}{8r_1}\Big)\frac{r_2}{h}\frac{dh}{ds}\cot\varphi\\&\qquad-\frac{15}{16}\frac{r_2^2}{h^2}\Big(\frac{dh}{ds}\Big)^2+\frac{3}{4}\frac{r_2^2}{h}\frac{d^2h}{ds^2}\Big]\\&\Theta_m=\frac{\sqrt[4]{\ }\,r_2h\sin^2\varphi}{E}\Big\{-\Big(2\cos\varphi+\mu\cos\varphi-\frac{r}{h}\frac{dh}{ds}\Big)r_2q_H\\&\qquad-r_2^2\frac{dq_H}{ds}\sin\varphi+r_2^2\frac{dq_V}{ds}\cos\varphi+\Big(2\frac{\cos^2\varphi}{\sin\varphi}-2\frac{r_2}{r_1\sin\varphi}\\&\qquad-\mu\sin\varphi-\frac{r_2\cos\varphi}{h}\frac{dh}{ds}\Big)r_2q_V-\frac{\cos\varphi}{\sin^3\varphi}\Big[\frac{1}{r_2}+\frac{1}{r_1}-\frac{2r_2}{r_1^2}-\frac{r_2}{r_1^2}\\&\qquad\cdot\frac{dr_1}{ds}\tan\varphi-\Big(\frac{r_2}{r_1}+\mu\Big)\frac{\tan\varphi}{h}\frac{dh}{ds}\Big]\Big(\int_{s*}^{s}rq_Vds-r^*V^*\Big)\Big\}\end{aligned}\right\}\tag{2.8}$$

我们假定 $g_1(\xi)$和 $g_2(\xi)$ 相对于 λ_0^2 都是很小的，就必须要求 φ 角不接近 0° 或 180°，曲率半径 r_1 变化不快和壁厚变化较慢，并且不在壳顶点和壁厚顶点以及它们的附近。也就是说有：

$$\cot^2\varphi=O(1),\qquad \frac{dr_1}{ds}=O(1),\qquad \frac{1}{f_s}\frac{dh}{ds}=O(\lambda_0^{-2}),\qquad \frac{r_2}{f_3}\frac{d^2h}{ds^2}=O(\lambda_0^{-2})\tag{2.9}$$

将式(2.9)代入式(2.8)中，得：

$$g_1(\xi)=O(1),\qquad g_2(\xi)=O(1)\tag{2.10}$$

三、一致有效渐近解

3.1　齐次解

式(2.7)的齐次方程是：

$$\frac{d^2\bar{\Theta}}{d\xi^2}+g_1(\xi)\bar{\Theta}=\frac{4\beta^4}{E}\lambda_0^2\bar{\Pi},\qquad \frac{d^2\bar{\Pi}}{d\xi^2}+g_2(\xi)\bar{\Pi}=-E\lambda_0^2\bar{\Theta} \tag{3.1}$$

因为 λ_0 是一个大参数且 $g_1(\xi)=O(1)$ 和 $g_2(\xi)=O(1)$，对式(3.1)有比较方程：

$$\frac{d^2\theta}{d\xi^2}=\frac{4\beta^4}{E}\lambda_0^2 I,\qquad \frac{d^2 I}{d\xi^2}=-E\lambda_0^2\theta \tag{3.2}$$

方程组(3.2)可以合并成一个方程，即：

$$\frac{d^2X}{d\xi^2}+\lambda^2X=0 \tag{3.3}$$

式中： $X=I+i\dfrac{E}{2\beta^2}\theta,\qquad \lambda^2=2i\beta^2\lambda_0^2,\qquad i=\sqrt{-1}$

式(3.3)有解：

$$X=\widetilde{C}_1\exp[i\lambda\xi]+\widetilde{C}_2\exp[-i\lambda\xi]=\widetilde{C}_3\exp[(1+i)\beta y]+\widetilde{C}_4\exp[-(1+i)\beta y] \tag{3.4}$$

式中 $y=\displaystyle\int_{s*}^{s}\frac{ds}{\sqrt{r_2h}}$

那么，式(3.1)的一次渐近解是：

$$\Pi^{\mathrm{I}}+i\frac{E}{2\beta^2}\Theta^{\mathrm{I}}=\widetilde{A}_1\exp[-(1+i)\beta y]+\widetilde{A}_2\exp[-(1+i)\beta y_1] \tag{3.5}$$

式中 $y_1=\displaystyle\int_{s}^{s*}\frac{ds}{\sqrt{r_2h}}$

在本文中，式(3.1)的一致有效渐近解是：

$$\bar{\Theta}=\frac{2\beta^2}{E}\mathrm{Im}\left(\alpha_1X+\gamma_1\frac{dX}{d\xi}\right),\qquad \bar{\Pi}=\mathrm{Re}\left(\alpha_2X+\gamma_2\frac{dX}{d\xi}\right) \tag{3.6}$$

式中：

$$\left.\begin{aligned}&\alpha_1=\sum_{n=0}^{\infty}a_n(\xi)\lambda^{-n},\qquad \gamma_1=\sum_{n=0}^{\infty}b_n(\xi)\lambda^{-n}\\ &\alpha_2=\sum_{n=0}^{\infty}c_n(\xi)\lambda^{-n},\qquad \gamma_2=\sum_{n=0}^{\infty}e_n(\xi)\lambda^{-n}\end{aligned}\right\} \tag{3.7}$$

将式(3.3)、(3.6)代入式(3.1)中，得：

$$\left.\begin{aligned}&\left[\frac{d^2\alpha_1}{d\xi^2}-\lambda^2(\alpha_1-\alpha_2)+g_1(\xi)\alpha_1-2\lambda^2\frac{d\gamma_1}{d\xi}\right]X\\ &\qquad+\left[2\frac{d\alpha_1}{d\xi}+\frac{d^2\gamma_1}{d\xi^2}-\lambda^2(\gamma_1-\gamma_2)+g_1(\xi)\gamma_1\right]\frac{dX}{d\xi}=0\\ &\left[\frac{d^2\alpha_2}{d\xi^2}-\lambda^2(\alpha_2-\alpha_1)+g_2(\xi)\alpha_2-2\lambda^2\frac{d\gamma_2}{d\xi}\right]X\\ &\qquad+\left[2\frac{d\alpha_2}{d\xi}+\frac{d^2\gamma_2}{d\xi^2}-\lambda^2(\gamma_2-\gamma_1)+g_2(\xi)\gamma_2\right]\frac{dX}{d\xi}=0\end{aligned}\right\} \tag{3.8}$$

令 X 和 $dX/d\xi$ 的系数分别为零，得：

$$\frac{d^2\alpha_1}{d\xi^2}-\lambda^2(\alpha_1-\alpha_2)+g_1(\xi)\alpha_1-2\lambda^2\frac{d\gamma_1}{d\xi}=0 \tag{3.9a}$$

$$\frac{d^2\alpha_2}{d\xi^2}-\lambda^2(\alpha_2-\alpha_1)+g_2(\xi)\alpha_2-2\lambda^2\frac{d\gamma_2}{d\xi}=0 \tag{3.9b}$$

$$2\frac{d\alpha_1}{d\xi}+\frac{d^2\gamma_1}{d\xi^2}-\lambda^2(\gamma_1-\gamma_2)+g_1(\xi)\gamma_1=0 \tag{3.9c}$$

$$2\frac{d\alpha_2}{d\xi}+\frac{d^2\gamma_2}{d\xi^2}-\lambda^2(\gamma_2-\gamma_1)+g_2(\xi)\gamma_2=0 \tag{3.9d}$$

由式(3.9)得：

$$\left.\begin{aligned}&\frac{d^2(\alpha_1+\alpha_2)}{d\xi^2}+g_1(\xi)\alpha_1+g_2(\xi)\alpha_2-2\lambda^2\frac{d(\gamma_1+\gamma_2)}{d\xi}=0\\&2\frac{d(\alpha_1+\alpha_2)}{d\xi}+\frac{d^2(\gamma_1+\gamma_2)}{d\xi^2}+g_1(\xi)\gamma_1+g_2(\xi)\gamma_2=0\end{aligned}\right\} \tag{3.10}$$

将式(3.7)代入式(3.9a)、(3.9c)、(3.10)中，由 λ 的同次幂的系数为零，得：

$$\left.\begin{aligned}&\frac{d^2}{d\xi^2}[a_n(\xi)+c_n(\xi)]+g_1(\xi)a_n(\xi)+g_2(\xi)c_n(\xi)\\&\qquad-2\frac{d}{d\xi}[b_{n+2}(\xi)+e_{n+2}(\xi)]=0\\&2\frac{da_n(\xi)}{d\xi}+\frac{d^2b_n(\xi)}{d\xi^2}-[b_{n+2}(\xi)-e_{n+2}(\xi)]+g_1(\xi)b_n(\xi)=0\\&2\frac{d}{d\xi}[a_n(\xi)+c_n(\xi)]+\frac{d^2}{d\xi^2}[b_n(\xi)+e_n(\xi)]+g_1(\xi)b_n(\xi)\\&\qquad+g_2(\xi)e_n(\xi)=0\\&\frac{d^2a_n(\xi)}{d\xi^2}-[a_{n+2}(\xi)-c_{n+2}(\xi)]+g_1(\xi)a_n(\xi)-2\frac{db_{n+2}(\xi)}{d\xi}=0\end{aligned}\right\} \tag{3.11}$$

由式(3.11)得：

$$\left.\begin{aligned}&a_{2n+1}(\xi)=b_{2n+1}(\xi)=c_{2n+1}(\xi)=e_{2n+1}(\xi)=0\\&a_0(\xi)=c_0(\xi)=1\\&b_0(\xi)=e_0(\xi)=0\\&a_2(\xi)=\frac{1}{8}g_1(\xi)-\frac{3}{8}g_2(\xi)-\frac{1}{4}\int^{\xi}[g_1(\xi)+g_2(\xi)]b_2(\xi)d\xi\\&b_2(\xi)=\frac{1}{4}\int^{\xi}[g_1(\xi)+g_2(\xi)]d\xi\\&c_2(\xi)=-\frac{3}{8}g_1(\xi)+\frac{1}{8}g_2(\xi)-\frac{1}{4}\int^{\xi}[g_1(\xi)+g_2(\xi)]b_2(\xi)d\xi\\&e_2(\xi)=\frac{1}{4}\int^{\xi}[g_1(\xi)+g_2(\xi)]d\xi\\&a_{2n+2}(\xi)=-\frac{1}{4}\left\{\frac{d}{d\xi}[b_{2n+2}(\xi)+e_{2n+2}(\xi)]+\int^{\xi}[g_1(\xi)b_{2n+2}(\xi)\right.\\&\qquad\left.+g_2(\xi)e_{2n+2}(\xi)]d\xi\right\}+\frac{1}{2}\frac{d^2a_{2n}(\xi)}{d\xi^2}+\frac{1}{2}g_1(\xi)a_{2n}(\xi)-\frac{db_{2n+2}(\xi)}{d\xi}\\&b_{2n+2}(\xi)=\frac{1}{4}\left\{\frac{d}{d\xi}[a_{2n}(\xi)+c_{2n}(\xi)]+\int^{\xi}[g_1(\xi)a_{2n}(\xi)\right.\\&\qquad\left.+g_2(\xi)c_{2n}(\xi)]d\xi\right\}+\frac{da_{2n}(\xi)}{d\xi}+\frac{1}{2}\frac{d^2b_{2n}(\xi)}{d\xi^2}+\frac{1}{2}g_1(\xi)b_{2n}(\xi)\\&c_{2n+2}(\xi)=a_{2n+2}(\xi)-\frac{d^2a_{2n}(\xi)}{d\xi^2}-g_1(\xi)a_{2n}(\xi)+2\frac{db_{2n+2}(\xi)}{d\xi}\\&e_{2n+2}(\xi)=b_{2n+2}(\xi)-2\frac{da_{2n}(\xi)}{d\xi}-\frac{d^2b_{2n}(\xi)}{d\xi^2}-g_1(\xi)b_{2n}(\xi)\end{aligned}\right\} \tag{3.12}$$

我们求得位移 ϑ 和 $\varDelta$，内力和弯矩如下式：

$$
\left.\begin{aligned}
\bar{H}=&\frac{1}{r_2\sin\varphi}\sqrt[4]{\frac{h^3}{r_2\sin^2\varphi}}\{B_1\exp[-\beta y_1][f_{11}\sin(\beta y_1+\psi_1)-f_{12}\cos(\beta y_1\\
&+\psi_1)]+B_2\exp[-\beta y][f_{13}\sin(\beta y+\psi)-f_{14}\cos(\beta y+\psi)]\}+H_m\\
\bar{\vartheta}=&\frac{2\beta^2}{E\sqrt[4]{r_2h^5\sin^2\varphi}}\{B_1\exp[-\beta y_1][f_{21}\cos(\beta y_1+\psi_1)+f_{22}\sin(\beta y_1+\psi_1)]\\
&+B_2\exp[-\beta y][f_{23}\cos(\beta y+\psi)+f_{24}\sin(\beta y+\psi)]\}\\
\bar{M}_\varphi=&-\frac{h^{\frac{1}{4}}}{2\beta^2\sqrt[4]{r_2\sin^2\varphi}}\{B_1\exp[-\beta y_1][f_{31}\cos(\beta y_1+\psi_1)\\
&+f_{32}\sin(\beta y_1+\psi_1)]+B_2\exp[-\beta y][f_{33}\cos(\beta y+\psi)\\
&+f_{34}\sin(\beta y+\psi)]\}\\
\bar{M}_\theta=&-\frac{h^{\frac{1}{4}}}{2\beta^2\sqrt[4]{r_2\sin^2\varphi}}\{B_1\exp[-\beta y_1][f_{41}\cos(\beta y_1+\psi_1)\\
&+f_{42}\sin(\beta y_1+\varphi_1)]+B_2\exp[-\beta y][f_{43}\cos(\beta y+\psi)\\
&+f_{44}\sin(\beta y+\psi)]\}\\
\bar{N}_\varphi=&\bar{H}\cos\varphi\\
\bar{Q}_\varphi=&-\bar{H}\sin\varphi\\
\bar{N}_\theta=&\sqrt[4]{\frac{h^3}{r_2\sin^2\varphi}}\{B_1\exp[-\beta y_1][f_{51}\cos(\beta y_1+\psi_1)+f_{52}\sin(\beta y_1+\psi_1)]\\
&+B_2\exp[-\beta y][f_{53}\cos(\beta y+\psi)+f_{54}\sin(\beta y+\psi)]\}+\frac{d(rH_m)}{ds}\\
\bar{\varDelta}=&\frac{1}{E}\sqrt[4]{\frac{r_2^3\sin^2\varphi}{h}}\{B_1\exp[-\beta y_1][f_{61}\cos(\beta y_1+\psi_1)\\
&+f_{62}\sin(\beta y_1+\psi_1)]+B_2\exp[-\beta y][f_{63}\cos(\beta y+\psi)\\
&+f_{64}\sin(\beta y+\psi)]\}+\frac{r}{Eh}\left[\frac{d(rH_m)}{ds}-\mu H_m\cos\varphi\right]
\end{aligned}\right\}\quad(3.13)
$$

式中：

$$rH_m=-\left(\int_{s^*}^{s}rq_Vds-r^*V^*\right)\cot\varphi$$

$$f_{11}+if_{12}=\alpha_2+\beta(1+i)\lambda_0\gamma_2$$

$$f_{13}+if_{14}=\alpha_2-\beta(1+i)\lambda_0\gamma_2$$

$$f_{21}+if_{22}=\alpha_1+\beta(1+i)\lambda_0\gamma_1$$

$$f_{23}+if_{24}=\alpha_1-\beta(1+i)\lambda_0\gamma_1$$

$$f_{31}=\beta\frac{f_{21}-f_{22}}{\sqrt{r_2h}}+\frac{df_{21}}{ds}-f_{21}\frac{\cot\varphi}{4}\left(\frac{1-4\mu}{r_2}+\frac{1}{r_1}\right)-\frac{5}{4}\frac{f_{21}}{h}\frac{dh}{ds}$$

$$f_{32}=\beta\frac{f_{21}+f_{22}}{\sqrt{r_2h}}+\frac{df_{22}}{ds}-f_{22}\frac{\cot\varphi}{4}\left(\frac{1-4\mu}{r_2}+\frac{1}{r_1}\right)-\frac{5}{4}\frac{f_{22}}{h}\frac{dh}{ds}$$

$$f_{33}=-\beta\frac{f_{23}-f_{24}}{\sqrt{r_2h}}+\frac{df_{23}}{ds}-f_{23}\frac{\cot\varphi}{4}\left(\frac{1-4\mu}{r_2}+\frac{1}{r_1}\right)-\frac{5}{4}\frac{f_{23}}{h}\frac{dh}{ds}$$

$$f_{34}=-\beta\frac{f_{23}+f_{24}}{\sqrt{r_2h}}+\frac{df_{24}}{ds}-f_{24}\frac{\cot\varphi}{4}\left(\frac{1-4\mu}{r_2}+\frac{1}{r_1}\right)-\frac{5}{4}\frac{f_{24}}{h}\frac{dh}{ds}$$

$$f_{41}=\mu\beta\frac{f_{21}-f_{22}}{\sqrt{r_2h}}+\mu\frac{df_{21}}{ds}-f_{21}\frac{\cot\varphi}{4}\left(\frac{\mu-4}{r_2}+\frac{\mu}{r_1}\right)-\frac{5}{4}\frac{\mu f_{21}}{h}\frac{dh}{ds}$$

$$f_{42}=\mu\beta\frac{f_{21}+f_{22}}{\sqrt{r_2h}}+\mu\frac{df_{22}}{ds}-f_{22}\frac{\cot\varphi}{4}\left(\frac{\mu-4}{r_2}+\frac{\mu}{r_1}\right)-\frac{5}{4}\frac{\mu f_{22}}{h}\frac{dh}{ds}$$

$$f_{43}=-\mu\beta\frac{f_{23}-f_{24}}{\sqrt{r_2h}}+\mu\frac{df_{23}}{ds}-f_{23}\frac{\cot\varphi}{4}\left(\frac{\mu-4}{r_2}+\frac{\mu}{r_1}\right)-\frac{5}{4}\frac{\mu f_{23}}{h}\frac{dh}{ds}$$

$$f_{44}=-\mu\beta\frac{f_{23}+f_{24}}{\sqrt{r_2h}}+\mu\frac{df_{24}}{ds}-f_{24}\frac{\cot\varphi}{4}\left(\frac{\mu-4}{r_2}+\frac{\mu}{r_1}\right)-\frac{5}{4}\frac{\mu f_{24}}{h}\frac{dh}{ds}$$

$$f_{51}=-\beta\frac{f_{11}+f_{12}}{\sqrt{r_2h}}-\frac{df_{12}}{ds}+f_{12}\frac{\cot\varphi}{4}\left(\frac{1}{r_2}+\frac{1}{r_1}\right)-\frac{3f_{12}}{4h}\frac{dh}{ds}$$

$$f_{52}=\beta\frac{f_{11}-f_{12}}{\sqrt{r_2h}}+\frac{df_{11}}{ds}-f_{11}\frac{\cot\varphi}{4}\left(\frac{1}{r_2}+\frac{1}{r_1}\right)+\frac{3f_{11}}{4h}\frac{dh}{ds}$$

$$f_{53}=\beta\frac{f_{13}+f_{14}}{\sqrt{r_2h}}-\frac{df_{14}}{ds}+f_{14}\frac{\cot\varphi}{4}\left(\frac{1}{r_2}+\frac{1}{r_1}\right)-\frac{3f_{14}}{4h}\frac{dh}{ds}$$

$$f_{54}=-\beta\frac{f_{13}-f_{14}}{\sqrt{r_2h}}+\frac{df_{13}}{ds}-f_{13}\frac{\cot\varphi}{4}\left(\frac{1}{r_2}+\frac{1}{r_1}\right)+\frac{3f_{13}}{4h}\frac{dh}{ds}$$

$$f_{61}=f_{51}+\mu f_{12}\cot\varphi$$
$$f_{62}=f_{52}-\mu f_{11}\cot\varphi$$
$$f_{63}=f_{53}+\mu f_{14}\cot\varphi$$
$$f_{64}=f_{54}-\mu f_{13}\cot\varphi$$

如果二边缘相距不太近时，例如 $\eta_2=\int_{s*}^{s*}\frac{ds}{\sqrt{r_2h}}\geqslant3$ 时，下边缘处由式(3.13)中的B_1项来决定，而上边缘处由 B_2 项来决定。下面我们给出了二个重要情况下的实用公式：

（1）在边界 $s=s_0$ 处仅有边缘力矩 $\overline{M}_{\varphi0}$ 作用：

$$\left.\begin{aligned}\bar{\Delta}_0&=-\frac{2\beta^2r_{20}\sin\varphi_0}{Eh_0^2}\left(\frac{f_{11}f_{61}+f_{12}f_{62}}{f_{11}f_{31}+f_{12}f_{32}}\right)_0\overline{M}_{\varphi0}\\ \bar{\vartheta}_0&=-\frac{4\beta^4}{Eh_0^3}\left(\frac{f_{11}f_{21}+f_{12}f_{22}}{f_{11}f_{31}+f_{12}f_{32}}\right)_0\overline{M}_{\varphi0}\end{aligned}\right\}\qquad(3.14)$$

（2）在边界 $s=s_0$ 处仅有边缘径向力 $\overline{H}_0$ 作用：

$$\left.\begin{aligned}\bar{\Delta}_0&=-\frac{r_{20}^2\sin^2\varphi_0}{Eh_0}\left(\frac{f_{32}f_{61}-f_{31}f_{62}}{f_{11}f_{31}+f_{12}f_{32}}\right)_0\overline{H}_0\\ \bar{\vartheta}_0&=-\frac{2\beta^2r_{20}\sin\varphi_0}{Eh_0^2}\left(\frac{f_{21}f_{32}-f_{22}f_{31}}{f_{11}f_{31}+f_{12}f_{32}}\right)_0\overline{H}_0\end{aligned}\right\}\qquad(3.15)$$

对一次渐近解，由式(3.14)、(3.15)得：

仅在边缘力矩 $\overline{M}_{\varphi0}$作用下，有：

$$\bar{\Delta}_0^1=\frac{2\beta^2r_{20}\sin\varphi_0}{K_1Eh_0^2}\overline{M}_{\varphi0},\qquad\bar{\vartheta}_0^1=-\frac{4\beta^3}{K_1E}\sqrt{\frac{r_{20}}{h_0^5}}\,\overline{M}_{\varphi0}\qquad(3.16)$$

式中：$K_1=1-\frac{\cot\varphi_0}{4}\left(1-4\mu+\frac{r_{20}}{r_{10}}\right)\sqrt{\frac{h_0}{r_{20}}}-\frac{5}{4\beta}\sqrt{\frac{r_{20}}{h_0}}\left(\frac{dh}{ds}\right)_0$

仅在边缘径向力 $\overline{H}_0$ 作用下，有：

$$\left.\begin{aligned}\varDelta_0^{\rm I}&=\frac{\beta}{E}\left(\frac{r_{20}}{h_0}\right)^{\frac{3}{2}}\left(\frac{1}{K_1}+K_2\right)\overline{H}_0\sin^2\varphi_0\\ \vartheta_0^{\rm I}&=-\frac{2\beta^2 r_{20}}{K_1 E h_0^2}\overline{H}_0\sin\varphi_0\end{aligned}\right\}\tag{3.17}$$

式中： $K_2=1-\frac{\cot\varphi_0}{4}\left(1+4\mu+\frac{r_{20}}{r_{10}}\right)\sqrt{\frac{h_0}{r_{20}}}+\frac{3}{4\beta}\sqrt{\frac{r_{20}}{h_0}}\left(\frac{dh}{ds}\right)_0$

当二个边缘相距较近时，一个边缘对另一个边缘的影响不能略去不计，由式 (3.5) 引出以下形式的一次渐近解：

$$\Pi^{\rm I}+i\frac{E}{2\beta^2}\Theta^{\rm I}=\left(B_1-i\frac{B_3}{2}\right)(Y_1+2iY_3)+\left(B_2-i\frac{B_4}{2}\right)(Y_2+2iY_4)\tag{3.18}$$

式中：Y_1 至 Y_4 是 Крылов 函数：

$$Y_1=\mathrm{ch}(\beta y)\cos(\beta y),\quad Y_2=\frac{1}{2}[\mathrm{ch}(\beta y)\sin(\beta y)+\mathrm{sh}(\beta y)\cos(\beta y)]$$

$$Y_3=\frac{1}{2}\mathrm{sh}(\beta y)\sin(\beta y),\quad Y_4=\frac{1}{4}[\mathrm{ch}(\beta y)\sin(\beta y)-\mathrm{sh}(\beta y)\cos(\beta y)]$$

我们求得位移 ϑ 和 $\varDelta$，内力和弯矩如下式：

$$\left.\begin{aligned}
\overline{H}^{\rm I}&=\frac{1}{r_2\sin\varphi}\sqrt[4]{\frac{h^3}{r_2\sin\varphi}}(B_1Y_1+B_2Y_2+B_3Y_3+B_4Y_4)+H_m\\
\overline{\vartheta}^{\rm I}&=\frac{\beta^2}{E\sqrt[4]{r_2h^5\sin^2\varphi}}(-4B_1Y_3-4B_2Y_4+B_3Y_1+B_4Y_2)\\
\overline{M}_\varphi^{\rm I}&=-\frac{h^{\frac{1}{4}}}{4\beta^2\sqrt[4]{r_2\sin^2\varphi}}\Big[-\left(\frac{1}{4r_1}+\frac{1-4\mu}{4r_2}\right)(-4B_1Y_3-4B_2Y_4+B_3Y_1\\
&\quad+B_4Y_2)\cot\varphi+\frac{\beta}{\sqrt{r_2h}}(-4B_1Y_2-4B_2Y_3-4B_3Y_4+B_4Y_1)\Big]\\
\overline{M}_\theta^{\rm I}&=-\frac{h^{\frac{1}{4}}}{4\beta^2\sqrt[4]{r_2\sin^2\varphi}}\Big[-\left(\frac{\mu}{4r_1}-\frac{4-\mu}{4r_2}\right)(-4B_1Y_3-4B_2Y_4+B_3Y_1\\
&\quad+B_4Y_2)\cot\varphi+\frac{\mu\beta}{\sqrt{r_2h}}(-4B_1Y_2-4B_2Y_3-4B_3Y_4+B_4Y_1)\Big]\\
\overline{N}_\varphi^{\rm I}&=\overline{H}^{\rm I}\cos\varphi\\
\overline{Q}_\varphi^{\rm I}&=-\overline{H}^{\rm I}\sin\varphi\\
\overline{N}_\theta^{\rm I}&=\sqrt[4]{\frac{h^3}{r_2\sin^2\varphi}}\Big[-\frac{\cot\varphi}{4}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)(B_1Y_1+B_2Y_2+B_3Y_3+B_4Y_4)\\
&\quad+\frac{\beta}{\sqrt{r_2h}}(-4B_1Y_4+B_2Y_1+B_3Y_2+B_4Y_3)\Big]+\frac{d(rH_m)}{ds}\\
\overline{\varDelta}^{\rm I}&=\frac{r_2\sin\varphi}{E\sqrt[4]{r_2h\sin^2\varphi}}\Big[-\frac{\cot\varphi}{4}\left(\frac{1}{r_1}+\frac{1+4\mu}{r_2}\right)(B_1Y_1+B_2Y_2+B_3Y_3\\
&\quad+B_4Y_4)+\frac{\beta}{\sqrt{r_2h}}(-4B_1Y_4+B_2Y_1+B_3Y_2+B_4Y_3)\Big]\\
&\quad+\frac{r_2\sin\varphi}{Eh}\left[\frac{d(rH_m)}{ds}-\mu H_m\cos\varphi\right]
\end{aligned}\right\}\tag{3.19}$$

因为Love-Kirchhoff薄壳理论假定，它的允许误差是$|\lambda^{-2}|$阶数量级的。而本文给出的二次渐近解，它的误差是在Love-Kirchhoff薄壳理论允许误差范围之内的，即$|\lambda^{-2}|$阶数量级的。本文给出的二次渐近解是：

$$\Pi^{\text{I}}+i\frac{E}{2\beta^2}\Theta^{\text{I}}=\widetilde{C}_1\left[1+\frac{1-i}{2\beta\lambda_0}b_2(\xi)\right]\exp[-(1+i)\beta y]$$

$$+\widetilde{C}_2\left[1-\frac{1-i}{2\beta\lambda_0}b_2(\xi)\right]\exp[-(1+i)\beta y_1] \tag{3.20}$$

3.2 特解

因为λ是一个大参数且有$g_1(\xi)=O(1)$，$g_2(\xi)=O(1)$，那么式(2.7)的一次渐近特解由下式决定：

$$\frac{d^2}{d\xi^2}\left(\Pi^{P\text{I}}+i\frac{E}{2\beta^2}\Theta^{P\text{I}}\right)+\lambda^2\left(\Pi^{P\text{I}}+i\frac{E}{2\beta^2}\Theta^{P\text{I}}\right)=E\lambda_0^2\Theta_m \tag{3.21}$$

上式有特解：

$$\Pi^{P\text{I}}+i\frac{E}{2\beta^2}\Theta^{P\text{I}}=\frac{E\lambda_0^2}{\lambda}\left(\sin\lambda\xi\int^{\xi}\Theta_m\cos\lambda\xi d\xi-\cos\lambda\xi\int^{\xi}\Theta_m\sin\lambda\xi d\xi\right) \tag{3.22}$$

式(2.7)的高次近似特解由常数变易法来求得：

$$\left.\begin{aligned}\Theta^P&=u_1\Theta_1+u_2\Theta_2+u_3\Theta_3+u_4\Theta_4\\ \Pi^P&=u_1\Pi_1-u_2\Pi_2+u_3\Pi_3-u_4\Pi_4\end{aligned}\right\} \tag{3.23}$$

式中：

$$\begin{aligned}\Theta_2+i\Theta_1&=[\alpha_1-\beta(1+i)\lambda_0\gamma_1]\exp[-(1+i)\beta y]\\ \Theta_4+i\Theta_3&=[\alpha_1+\beta(1+i)\lambda_0\gamma_1]\exp[-(1+i)\beta y_1]\\ \Pi_1+i\Pi_2&=[\alpha_2-\beta(1+i)\lambda_0\gamma_2]\exp[-(1+i)\beta y]\\ \Pi_3+i\Pi_4&=[\alpha_2+\beta(1+i)\lambda_0\gamma_2]\exp[-(1+i)\beta y_1]\end{aligned}$$

u_1至u_4由下式决定：

$$\left.\begin{aligned}&\Theta_1\frac{du_1}{d\xi}+\Theta_2\frac{du_2}{d\xi}+\Theta_3\frac{du_3}{d\xi}+\Theta_4\frac{du_4}{d\xi}=0\\ &\frac{d\Theta_1}{d\xi}\frac{du_1}{d\xi}+\frac{d\Theta_2}{d\xi}\frac{du_2}{d\xi}+\frac{d\Theta_3}{d\xi}\frac{du_3}{d\xi}+\frac{d\Theta_4}{d\xi}\frac{du_4}{d\xi}=0\\ &\Pi_1\frac{du_1}{d\xi}-\Pi_2\frac{du_2}{d\xi}+\Pi_3\frac{du_3}{d\xi}-\Pi_4\frac{du_4}{d\xi}=0\\ &\frac{d\Pi_1}{d\xi}\frac{du_1}{d\xi}-\frac{d\Pi_2}{d\xi}\frac{du_2}{d\xi}+\frac{d\Pi_3}{d\xi}\frac{du_3}{d\xi}-\frac{d\Pi_4}{d\xi}\frac{du_4}{d\xi}=E\lambda_0^2\Theta_m\end{aligned}\right\} \tag{3.24}$$

如果我们有$d^2(E\Theta_m)/d\xi^2=O(1)$，那末式(2.7)的第一次和第二次渐近特解都是：

$$\Pi^{P\text{I}}+i\frac{E}{2\beta^2}\Theta^{P\text{I}}=\Pi^{P\text{II}}+i\frac{E}{2\beta^2}\Theta^{P\text{II}}=i\frac{E}{2\beta^2}\Theta_m \tag{3.25}$$

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The Axial Symmetrical Edge Problems for Thin-Walled Shells of Revolution

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Abstract

In this paper, the uniformly valid asymptotic solutions for the axial symmetrical edge problems of thin-walled shells of revolution in bending are given.