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用调和函数表示弹性理论方程组的一般解*

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摘要

本文用七个调和函数表示弹性理论方程组的一般解，但七个调和函数中只有三个是彼此独立的，并且每个调和函数具有一定的力学意义．文中提供了应用一般解求解弹性静力学中若干简单逆问题的例子．

一、用三个独立的调和函数表示弹性理论方程组的一般解**)

按位移表示的线性弹性理论方程组（不考虑体积力）如下[1]：

$$\frac{E}{1-2\nu}\frac{\partial e}{\partial x}+E\Delta u=0,\quad \frac{E}{1-2\nu}\frac{\partial e}{\partial y}+E\Delta v=0,\quad \frac{E}{1-2\nu}\frac{\partial e}{\partial z}+E\Delta w=0 \tag{1.1}$$

其中E是弹性模量，ν是泊松比，u，v，w分别是平行坐标轴x，y，z的位移分量，体积膨胀

$$e=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$$

调和算子，

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

对于方程组(1.1)的一般解，已经知道有多种表示方法[1]．1963年找到的一种表示如下：

$$\left.\begin{aligned}u&=\frac{3}{2E}P-\frac{x}{2E}\frac{\partial P}{\partial x}-\frac{1+\nu}{E}\phi\\ v&=\frac{3}{2E}Q-\frac{y}{2E}\frac{\partial Q}{\partial y}-\frac{1+\nu}{E}A\\ w&=\frac{1+\nu}{E}(\psi+B)-\frac{1+2\nu}{E}R+\frac{1}{2E}\left(x\frac{\partial R}{\partial x}+y\frac{\partial R}{\partial y}\right)\end{aligned}\right\} \tag{1.2}$$

其中P，Q，R，ϕ，ψ，A，B是调和函数，它们满足

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**) 本节结果由作者在1963年获得（参照《古典弹性理论方程的一般解》．湖南大学科技资料，总38期，数力8号，1963）．

$$\frac{\partial P}{\partial x}=\frac{\partial Q}{\partial y}=\frac{\partial R}{\partial z},\quad \frac{\partial \phi}{\partial x}=\frac{\partial \psi}{\partial z},\quad \frac{\partial A}{\partial y}=\frac{\partial B}{\partial z} \tag{1.3}$$

利用(1.3)与调和函数的性质，容易验证(1.2)满足(1.1)，因此(1.2)是(1.1)的解。

另一方面，假定(u,v,w)是(1.1)的一组已知解。将(1.1)的三个方程分别对x,y,z微分，我们发现

$$\Delta e=\Delta\left(\frac{1}{1-2\nu}\left(e-\frac{x}{2}\frac{\partial e}{\partial x}\right)-\frac{\partial u}{\partial x}\right)=\Delta\left(\frac{1}{1-2\nu}\left(e-\frac{y}{2}\frac{\partial e}{\partial y}\right)-\frac{\partial v}{\partial y}\right)=0$$

即 $e,\ \frac{1}{1-2\nu}\left(e-\frac{x}{2}\frac{\partial e}{\partial x}\right)-\frac{\partial u}{\partial x},\quad \frac{1}{1-2\nu}\left(e-\frac{y}{2}\frac{\partial e}{\partial y}\right)-\frac{\partial v}{\partial y}$

是调和函数。令

$$\left.\begin{aligned}&\frac{\partial P}{\partial x}=\frac{\partial Q}{\partial y}=\frac{\partial R}{\partial z}=\frac{E}{1-2\nu}e\\&\frac{\partial \phi}{\partial x}=\frac{\partial \psi}{\partial z}=\frac{E}{(1+\nu)(1-2\nu)}\left(e-\frac{x}{2}\frac{\partial e}{\partial x}\right)-\frac{E}{1+\nu}\frac{\partial u}{\partial x}\\&\frac{\partial A}{\partial y}=\frac{\partial B}{\partial z}=\frac{E}{(1+\nu)(1-2\nu)}\left(e-\frac{y}{2}\frac{\partial e}{\partial y}\right)-\frac{E}{1+\nu}\frac{\partial v}{\partial y}\end{aligned}\right\} \tag{1.4}$$

则调和函数

$$\frac{\partial P}{\partial x}=\frac{\partial Q}{\partial y}=\frac{\partial R}{\partial z},\quad \frac{\partial \phi}{\partial x}=\frac{\partial \psi}{\partial z},\quad \frac{\partial A}{\partial y}=\frac{\partial B}{\partial z}$$

完全由(1.4)确定。为了使由(1.4)确定的P,Q,R,ϕ,ψ,A,B满足(1.2)和(1.1)必须要求

$$\Delta\left(\frac{3}{2}P-(1+\nu)\phi\right)=0$$

$$\Delta\left(\frac{3}{2}Q-(1+\nu)A\right)=0$$

$$(1+\nu)\Delta(\psi+B)-2\nu\Delta R+\frac{1}{2}\left(x\Delta\frac{\partial R}{\partial x}+y\Delta\frac{\partial R}{\partial y}\right)=0$$

令P,Q,R,ϕ,ψ,A,B是调和函数，则上述诸等式均被满足。这就证明了，存在满足(1.3)的七个调和函数，使得(1.2)对(1.1)的任何已知解成立。

利用应变分量与位移之间的几何关系及虎克定律[1]，我们获得应力分量如下：

$$\left.\begin{aligned}&\sigma_x=\frac{\partial P}{\partial x}-\frac{x}{2(1+\nu)}\frac{\partial^2 P}{\partial x^2}-\frac{\partial \phi}{\partial x}\\&\sigma_y=\frac{\partial Q}{\partial y}-\frac{y}{2(1+\nu)}\frac{\partial^2 Q}{\partial y^2}-\frac{\partial A}{\partial y}\\&\sigma_z=\frac{\partial \phi}{\partial x}+\frac{\partial A}{\partial y}-\frac{\partial R}{\partial z}+\frac{1}{2(1+\nu)}\left(x\frac{\partial^2 P}{\partial x^2}+y\frac{\partial^2 Q}{\partial y^2}\right)\\&\tau_{xy}=\frac{3}{4(1+\nu)}\left(\frac{\partial P}{\partial y}+\frac{\partial Q}{\partial x}\right)-\frac{1}{4(1+\nu)}\left(x\frac{\partial^2 P}{\partial x\partial y}+y\frac{\partial^2 Q}{\partial x\partial y}\right)\\&\qquad-\frac{1}{2}\left(\frac{\partial \phi}{\partial y}+\frac{\partial A}{\partial x}\right)\end{aligned}\right\} \tag{1.5}$$

$$\left.\begin{aligned}\tau_{yz}=&\frac{3}{4(1+\nu)}\frac{\partial Q}{\partial z}-\frac{1+2\nu}{2(1+\nu)}\frac{\partial R}{\partial y}-\frac{y}{2(1+\nu)}\frac{\partial^2 Q}{\partial y\partial z}\\&+\frac{1}{4(1+\nu)}\frac{\partial}{\partial x}\left(x\frac{\partial R}{\partial y}-y\frac{\partial R}{\partial x}\right)-\frac{1}{2}\frac{\partial A}{\partial z}+\frac{1}{2}\frac{\partial(\psi+B)}{\partial y}\\\tau_{zx}=&\frac{3}{4(1+\nu)}\frac{\partial P}{\partial z}-\frac{1+2\nu}{2(1+\nu)}\frac{\partial R}{\partial x}-\frac{x}{2(1+\nu)}\frac{\partial^2 P}{\partial z\partial x}\\&-\frac{1}{4(1+\nu)}\frac{\partial}{\partial y}\left(x\frac{\partial R}{\partial y}-y\frac{\partial R}{\partial x}\right)-\frac{1}{2}\frac{\partial\phi}{\partial z}+\frac{1}{2}\frac{\partial(\psi+B)}{\partial x}\end{aligned}\right\}$$

其中 $$\frac{\partial P}{\partial x}=\frac{\partial Q}{\partial y}=\frac{\partial R}{\partial z}=\sigma_x+\sigma_y+\sigma_z=\theta \tag{1.6}$$

表示应力张量的第一不变量，而

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial z}=-\sigma_x,\quad \frac{\partial A}{\partial y}=\frac{\partial B}{\partial z}=-\sigma_y;\quad \forall\sigma_x+\sigma_y+\sigma_z=0 \tag{1.7}$$

(1.2)或(1.5)中每个调和函数的力学意义可以用(1.6)或(1.7)解释.

二、具有常应力分量的解

假定所有的应力分量为常数，分别以$\sigma_x^0,\sigma_y^0,\sigma_z^0,\tau_{xy}^0,\tau_{yz}^0,\tau_{zx}^0$表示，令

$$\theta_0=\sigma_x^0+\sigma_y^0+\sigma_z^0,\quad \theta_1=\sigma_y^0+\sigma_z^0,\quad \theta_2=\sigma_z^0+\sigma_x^0$$

由(1.6)我们立刻有

$$P=\theta_0x+P_1,\quad Q=\theta_0y+Q_2,\quad R=\theta_0z+R_3 \tag{2.1}$$

其中P_1,Q_2,R_3分别是不依赖于x,y,z的二元调和函数. 从(1.5)的前两个等式我们又得知

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial z}=\theta_1,\qquad \frac{\partial A}{\partial y}=\frac{\partial B}{\partial z}=\theta_2$$

因而

$$\phi=\theta_1x+\phi_1,\quad \psi=\theta_1z+\psi_3,\quad A=\theta_2y+A_2,\quad B=\theta_2z+B_3 \tag{2.2}$$

其中ϕ_1,ψ_3,A_2,B_3是二元调和函数，类似地，ϕ_1,A_2分别不依赖于x,y，ψ_3和B_3不依赖于z.

(1.5)的前三个等式因(2.1)和(2.2)而被满足. 为了满足(1.5)的其余等式，我们只需要在$P_1,Q_2,R_3,\phi_1,A_2,\psi_3,B_3$中选择三个二元调和函数. 注意到(1.2)和(1.5)，我们可以假定

$$P_1=Q_2=R_3=B_3=0 \tag{2.3}$$

而不失一般性. 利用(2.1)～(2.3)，我们从(1.5)得到

$$\frac{\partial\phi_1}{\partial y}+\frac{\partial A_2}{\partial x}=-2\tau_{xy}^0,\quad \frac{\partial\psi_3}{\partial y}-\frac{\partial A_2}{\partial z}=2\tau_{yz}^0,\quad \frac{\partial\psi_3}{\partial x}-\frac{\partial\phi_1}{\partial z}=2\tau_{zx}^0 \tag{2.4}$$

因为ϕ_1,A_2分别是不依赖于x,y的调和函数，所以从(2.4)的第一个方程易知

$$\frac{\partial\phi_1}{\partial y}=a_1z+a_2-\tau_{xy}^0,\quad \frac{\partial A_2}{\partial x}=a_1z-a_2-\tau_{xy}^0$$

其中a_1和a_2是常数. 类似地，从(2.4)的其余方程得知

$$\frac{\partial\psi_3}{\partial y}=b_1x+b_2+\tau_{yz}^0,\quad \frac{\partial A_2}{\partial z}=b_1x+b_2-\tau_{yz}^0$$

$$\frac{\partial\psi_3}{\partial x}=c_1y+c_2+\tau^0_{zx},\quad \frac{\partial\phi_1}{\partial z}=c_1y+c_2-\tau^0_{zx}$$

其中b_1,b_2,c_1,c_2是常数．比较并积分上述各方程，我们得到

$$\left.\begin{aligned}&a_1=b_1=c_1=0\\&\phi_1=(a_2-\tau^0_{xy})y+(c_2-\tau^0_{zx})z+a_3\\&A_2=(b_2-\tau^0_{yz})z-(a_2+\tau^0_{xy})x+b_3\\&\psi_3=(c_2+\tau^0_{zx})x+(b_2+\tau^0_{yz})y+c_3\end{aligned}\right\}\qquad(2.5)$$

其中a_3,b_3,c_3为常数．综合(1.2)、(2.1)、(2.2)、(2.3)和(2.5)，我们做出结论如下：

$$\left.\begin{aligned}&u=\frac{\sigma^0_x-\nu(\sigma^0_y+\sigma^0_z)}{E}x-\frac{1+\nu}{E}((a_2-\tau^0_{xy})y+(c_2-\tau^0_{zx})z+a_3)\\&v=\frac{\sigma^0_y-\nu(\sigma^0_z+\sigma^0_x)}{E}y-\frac{1+\nu}{E}((b_2-\tau^0_{yz})z-(a_2+\tau^0_{xy})x+b_3)\\&w=\frac{\sigma^0_z-\nu(\sigma^0_x+\sigma^0_y)}{E}z+\frac{1+\nu}{E}((c_2+\tau^0_{zx})x+(b_2+\tau^0_{yz})y+c_3)\end{aligned}\right\}\qquad(2.6)$$

对于零应力状态，即 $\sigma^0_x=\sigma^0_y=\sigma^0_z=\tau^0_{xy}=\tau^0_{yz}=\tau^0_{zx}=0$，(2.6)表示的位移是刚体的旋转和平移，这正是应该得到的结论．不考虑刚体位移，我们有

$$\left.\begin{aligned}&u=\frac{\sigma^0_x-\nu(\sigma^0_y+\sigma^0_z)}{E}x+\frac{1+\nu}{E}(\tau^0_{xy}y+\tau^0_{zx}z)\\&v=\frac{\sigma^0_y-\nu(\sigma^0_z+\sigma^0_x)}{E}y+\frac{1+\nu}{E}(\tau^0_{xy}x+\tau^0_{yz}z)\\&w=\frac{\sigma^0_z-\nu(\sigma^0_x+\sigma^0_y)}{E}z+\frac{1+\nu}{E}(\tau^0_{zx}x+\tau^0_{yz}y)\end{aligned}\right\}\qquad(2.7)$$

作为例子，令 $\tau^0_{xy}=\tau^0_{yz}=\tau^0_{zx}=0$，则(2.7)可以表示一块矩形厚板的位移，这块厚板被放置在没有摩擦的刚性基础上而表面承受均匀正压力．

三、具有常体积膨胀的厚板

假定一块矩形厚板刚好被放进一个刚性而无摩擦的盒内，板的上表面承受压力．我们提出如下问题：

什么样的压力使得该板具有常体积膨胀？

以$2a,2b,h$分别表示板的长、宽、厚．以下表面的中心点为坐标原点建立坐标系，使x，y，z轴分别平行于板的三条棱．我们有如下边界条件：

$$\left.\begin{aligned}&w=\tau_{yz}=\tau_{zx}=0\qquad(z=0)\\&u=\tau_{xy}=\tau_{zx}=0\qquad(x=\pm a)\\&v=\tau_{xy}=\tau_{yz}=0\qquad(y=\pm b)\\&\sigma_z=-f(x,y),\ \tau_{zx}=-g_1(x,y),\ \tau_{yz}=-g_2(x,y),\ (z=h)\end{aligned}\right\}\qquad(3.1)$$

此外$g_1(x,y)$，$g_2(x,y)$，$f(x,y)$是表面压力分量，它们被如此地确定，使得板内任何点的体积膨胀为常数e_0．

因为 $$\frac{\partial P}{\partial x}=\frac{\partial Q}{\partial y}=\frac{\partial R}{\partial z}=\theta_0=\frac{E}{1-2\nu}e_0$$

所以按(2.1)和(2.3)的观点我们可以写

$$P=\theta_0 x,\quad Q=\theta_0 y,\quad R=\theta_0 z \tag{3.2}$$

令

$$\mu_{mn}=\sqrt{\frac{m^2}{4a^2}+\frac{n^2}{4b^2}}$$

$$\left.\begin{aligned}
\phi&=\frac{\theta_0}{1+\nu}x+\sum_{m,n}\alpha_{mn}\mu_{mn}\pi\,\mathrm{ch}\mu_{mn}\pi z\sin\frac{m\pi}{2a}(x+a)\cos\frac{n\pi}{2b}(y+b)\\
A&=\frac{\theta_0}{1+\nu}y+\sum_{m,n}\beta_{mn}\mu_{mn}\pi\,\mathrm{ch}\mu_{mn}\pi z\cos\frac{m\pi}{2a}(x+a)\sin\frac{n\pi}{2b}(y+b)\\
\psi&=\frac{\theta_0}{1+\nu}z+\sum_{m,n}\alpha_{mn}\frac{m\pi}{2a}\,\mathrm{sh}\mu_{mn}\pi z\cos\frac{m\pi}{2a}(x+a)\cos\frac{n\pi}{2b}(y+b)\\
B&=\frac{\theta_0}{1+\nu}z+\sum_{m,n}\beta_{mn}\frac{n\pi}{2b}\,\mathrm{sh}\mu_{mn}\pi z\cos\frac{m\pi}{2a}(x+a)\cos\frac{n\pi}{2b}(y+b)
\end{aligned}\right\} \tag{3.3}$$

其中 m, n 是整数而 α_{mn}, β_{mn} 是实数。显然，(3.2)和(3.3)定义的调和函数满足(1.3)。因此，将(3.2)和(3.3)代入(1.2)和(1.5)，我们就可以得到满足(1.1)的位移和应力分量。容易验证这样确定的位移和应力满足(3.1)中$z=0$, $x=\pm a$ 及$y=\pm b$ 的边界条件。

为了满足(3.1)中$z=h$的边界条件，我们要求

$$\left.\begin{aligned}
f(x,y)&=-\frac{1-\nu}{1+\nu}\theta_0-\sum_{m,n}\mu_{mn}\pi\left(\frac{m\pi}{2a}\alpha_{mn}+\frac{n\pi}{2b}\beta_{mn}\right)\\
&\quad\cdot\mathrm{ch}\mu_{mn}\pi h\cos\frac{m\pi}{2a}(x+a)\cos\frac{n\pi}{2b}(y+b)\\
g_1(x,y)&=\sum_{m,n}\left(\left(\frac{\mu_{mn}^2\pi^2}{2}+\frac{m^2\pi^2}{8a^2}\right)\alpha_{mn}+\frac{mn\pi^2}{8ab}\beta_{mn}\right)\\
&\quad\cdot\mathrm{sh}\mu_{mn}\pi h\sin\frac{m\pi}{2a}(x+a)\cos\frac{n\pi}{2b}(y+b)\\
g_2(x,y)&=\sum_{m,n}\left(\left(\frac{mn\pi^2}{8ab}\alpha_{mn}+\left(\frac{\mu_{mn}^2\pi^2}{2}+\frac{n^2\pi^2}{8b^2}\right)\beta_{mn}\right)\right.\\
&\quad\cdot\mathrm{sh}\mu_{mn}\pi h\cos\frac{m\pi}{2a}(x+a)\sin\frac{n\pi}{2b}(y+b)
\end{aligned}\right\} \tag{3.4}$$

如果限制

$$g_2=(x,y)=0$$

则在(3.4)中置 $n\equiv 0$（比方说），我们得到

$$f(x,y)=-\frac{1-\nu}{1+\nu}\theta_0-\frac{\pi^2}{4a^2}\sum_m m^2\alpha_m\mathrm{ch}\frac{m\pi}{2a}h\cos\frac{m\pi}{2a}(x+a)$$

$$g_1(x,y)=\frac{\pi}{4a^2}\sum_m m^2\alpha_m\mathrm{sh}\frac{m\pi}{2a}h\sin\frac{m\pi}{2a}(x+a)$$

如果$g_1(x,y)=g_2(x,y)=0$，则在(3.4)中$\alpha_{mn}=\beta_{mn}=0$，而

$$f(x,y)=-\frac{1+\nu}{1-\nu}\theta_0$$

参 考 文 献

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Representing General Solution of Equations in Theory of Elasticity by Harmonic Functions

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Abstract

The general solution of the equations in the theory of elasticity is represented by seven harmonic functions, where there are only three harmonic functions independent of each other and every one of them has certain mechanics meaning. The examples applying the general solution to solve several simple inverse problems in elastostatics are presented