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变厚度矩形薄板的线性和非线性理论的弹性平衡问题的Navier解

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摘　要

本文研究了四边简支变厚度矩形薄板的线性和非线性理论的弹性平衡问题．文中采用了Navier法，探索了求解的一般途径，并以示例说明求解的具体方法．最后，除提及解的收敛性外，还指出扩大解的应用范围的问题．

变厚度矩形薄板，一方面由于它在近代工程技术领域中有很大的实用价值；另一方面也由于表达薄板问题的微分方程本身的复杂性，所以近半个世纪以来，一直是力学工作者研究的重要课题之一．我国叶开沅教授对任意分布荷载下的非均匀变厚度矩形板的弯曲问题作过深入研究，并获得了统一的解式[1]．他还早在1953年冬就已提出了大挠度变厚度圆薄板方程[2]．最近，范家让同志通过积分变换引进特殊函数，处理了多种荷载下的四边简支和四边固定的线性变厚度矩形板的弯曲问题[3]．国外的R. G. Olsson，E. Reissner和H. Favre，B. Gilg曾先后探讨过板的弯曲刚度D和厚度h分别是y的线性函数的小挠度理论问题[4]．变厚度薄板的分析解本来就很困难，再加上板元微分方程"非线性"这一因素，难度就更大．因此，关于这方面的研究甚少．

C. L. Navier是简支矩形薄板弯曲问题的第一个解和利用双重三角级数来求这个解的创始人[5]．他解了均布荷载及在板中点作用一集中荷载这两种情形．后来，Mariotte（1886年），S. Woinowsky Kriegre（1932年），S. Levy（1942年），W. Nowacki（1954年），张福范（1955年）等人进一步发展了Navier的方法[4],[5],[6]．他们解决了多种荷载下的多种边界条件的等厚度矩形板的弹性平衡、稳定与振动问题．为了扩大Navier法的应用，本文试图通过方程的变形，对变厚度薄板给出一种统一的Navier解式；而对变厚度板属大挠度的情形，用Navier法亦可迎刃而解，从而使经典的Navier法成为板壳力学中一种更广泛的分析解题工具．

一、变厚度薄板的基本微分方程

设变厚度板的厚度没有突变，等厚度板的有关表达式可足够精确地应用于变厚度板的问

题[4]。若以 $D(x, y)$ 表示板的弯曲刚度，则弯矩和扭矩的表达式为：

$$\left.\begin{aligned} M_x &= -D(x, y)\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right) \\ M_y &= -D(x, y)\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right) \\ M_{xy} &= -(1-\mu)D(x, y)\frac{\partial^2 w}{\partial x\partial y} \end{aligned}\right\} \tag{1.1}$$

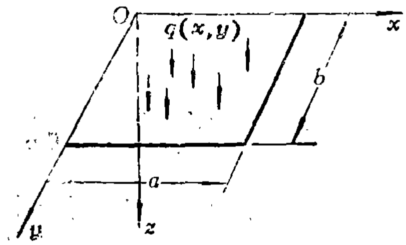

图 1

式中：$D(x, y)=\dfrac{Eh^3(x,y)}{12(1-\mu^2)}$，$E$为弹性模量，$\mu$为泊桑比，$h(x, y)$为板厚，$w$为板的挠度。

图 1 示一矩形薄板，$q(x, y)$为垂直板面的分布荷载，则板元的平衡微分方程为：

$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}=-q(x, y) \tag{1.2}$$

把（1.1）代入（1.2）后，我们得到：

$$\frac{\partial^2}{\partial x^2}\left[D(x, y)\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right)\right]+2(1-\mu)\frac{\partial^2}{\partial x\partial y}\left[D(x, y)\frac{\partial^2 w}{\partial x\partial y}\right]$$
$$+\frac{\partial^2}{\partial y^2}\left[D(x, y)\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)\right]=q(x,y) \tag{1.3}$$

或为：

$$D\nabla^2\nabla^2 w+2\frac{\partial D}{\partial x}\frac{\partial}{\partial x}\nabla^2 w+2\frac{\partial D}{\partial y}\frac{\partial}{\partial y}\nabla^2 w+\nabla^2 D\nabla^2 w$$
$$-(1-\mu)\left(\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2 D}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right)=q(x,y) \tag{1.4}$$

式中，$\nabla^2=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}$ 为 Laplace 算子。

此即小挠度（即线性理论）的变厚度弹性薄板的弯曲问题的平衡微分方程。

其次，我们来建立大挠度（即非线性理论）的变厚度弹性薄板的弹性平衡问题的基本微分方程。这时必须计及中面内各点由挠度引起的纵向位移，因此，也就必须考虑此项中面位移引起的中面应变和中面应力。

假设 x，y 平面内没有体力，而仅有垂直于板面的横向荷载 $q(x, y)$，则板元在中面内 x，y 轴向的平衡微分方程为：

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0,\quad \frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y^2}=0 \tag{1.5}$$

而板元在 z 轴向的平衡方程为：

$$\frac{\partial^2}{\partial x^2}\left[D\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right)\right]+2(1-\mu)\frac{\partial^2}{\partial x\partial y}\left[D\frac{\partial^2 w}{\partial x\partial y}\right]+\frac{\partial^2}{\partial y^2}\left[D\left(\frac{\partial^2 w}{\partial y^2}\right.\right.$$
$$\left.\left.+\mu\frac{\partial^2 w}{\partial x^2}\right)\right]-\left(h\sigma_x\frac{\partial^2 w}{\partial x^2}+h\sigma_y\frac{\partial^2 w}{\partial y^2}-2h\tau_{xy}\frac{\partial^2 w}{\partial x\partial y}\right)=q(x, y) \tag{1.6}$$

方程（1.5）和（1.6）里的σ_x，σ_y，τ_{xy}是由横向荷载所引起的中面应力。决定这三个变量，还需考虑板在弯曲时的中面应变。相应的应变分量为：

$$\varepsilon_x=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2,\quad \varepsilon_y=\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2,\quad \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{1.7}$$

取这些式子的二阶导数，便得形变协调方程：

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}-\frac{\partial^2\gamma_{xy}}{\partial x\partial y}=\left(\frac{\partial w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \tag{1.8}$$

用等价的式子：

$$\varepsilon_x=\frac{1}{E}(\sigma_x-\mu\sigma_y),\quad \varepsilon_y=\frac{1}{E}(\sigma_y-\mu\sigma_x),\quad \gamma_{xy}=\frac{2(1+\mu)}{E}\tau_{xy} \tag{1.9}$$

代入方程(1.8)，就得到用σ_x，σ_y，τ_{xy}表示的第三个方程。

为了简化微分方程，引入应力函数是可取的。不难看出，取[7]

$$\sigma_x=\frac{1}{h}\frac{\partial^2\varphi}{\partial y^2},\quad \sigma_y=\frac{1}{h}\frac{\partial^2\varphi}{\partial x^2},\quad \tau_{xy}=-\frac{1}{h}\frac{\partial^2\varphi}{\partial x\partial y} \tag{1.10}$$

方程（1.5）自然满足。而应变分量化为：

$$\varepsilon_x=\frac{1}{Eh}\left(\frac{\partial^2\varphi}{\partial y^2}-\mu\frac{\partial^2\varphi}{\partial x^2}\right),\quad \varepsilon_y=\frac{1}{Eh}\left(\frac{\partial^2\varphi}{\partial x^2}-\mu\frac{\partial^2\varphi}{\partial y^2}\right),\gamma_{xy}=-\frac{2(1+\mu)}{Eh}\frac{\partial^2\varphi}{\partial x\partial y} \tag{1.11}$$

则（1.6）和（1.8）两个微分方程变成为：

$$\left.\begin{aligned}
&\frac{\partial^2}{\partial x^2}\left[D\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right)\right]+2(1-\mu)\frac{\partial^2}{\partial x\partial y}\left[D\frac{\partial^2 w}{\partial x\partial y}\right]\\
&\quad+\frac{\partial^2}{\partial y^2}\left[D\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)\right]-\left(\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right.\\
&\quad\left.-2\frac{\partial^2\varphi}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}\right)=q(x,y)\\
&\frac{\partial^2}{\partial x^2}\left(\frac{1}{h}\frac{\partial^2\varphi}{\partial x^2}\right)+2\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{h}\frac{\partial^2\varphi}{\partial x\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{1}{h}\frac{\partial^2\varphi}{\partial y^2}\right)\\
&\quad-\mu\left[\frac{\partial^2}{\partial x^2}\left(\frac{1}{h}\frac{\partial^2\varphi}{\partial y^2}\right)-2\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{h}\frac{\partial^2\varphi}{\partial x\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{1}{h}\frac{\partial^2\varphi}{\partial x^2}\right)\right]\\
&=E\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]
\end{aligned}\right\} \tag{1.12}$$

此即为变厚度弹性薄板的大挠度问题的基本微分方程。它为变系数的高阶的非线性偏微分方程组。除了有些近似解法外，一般是无法从微分方程直接求解的。

二、边 界 条 件

薄板小挠度的线性理论的简支边界条件（图 1）为

$$x=0,\ a\text{时}:\quad w=0,\ M_x=\frac{\partial^2 w}{\partial x^2}=0 \tag{2.1a}$$

$$y=0,\ b\text{时：}\quad w=0,\ M_y=\frac{\partial^2 w}{\partial y^2}=0 \tag{2.1b}$$

对大挠度的非线性理论的薄板来说，由于方程（1.12）中的未知函数是挠度 w 和应力函数 φ。因此，边界条件除了 w 外还须用 φ 来表示。

当薄板问题属于有一对称轴（$x=a/2$ 或 $y=b/2$）的情形时[8]，一般简支情况的边界条件（图1）为

当 $x=0,\ a$ 时：

$$\left.\begin{aligned} &w=0,\quad M_x=-D(x,y)\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right)=0 \Rightarrow \frac{\partial^2 w}{\partial x^2}=0 \\ &N_x=\frac{\partial^2\varphi}{\partial y^2}=0,\ v=\frac{\partial^2\varphi}{\partial x^2}=0 \end{aligned}\right\} \tag{2.2a}$$

当 $y=0,b$ 时：

$$w=0,\quad M_y=\frac{\partial^2 w}{\partial y^2}=0,\quad N_y=\frac{\partial^2\varphi}{\partial x^2}=0,\ u=\frac{\partial^2\varphi}{\partial y^2}=0 \tag{2.2b}$$

在整个边界 Γ 上，上述边界条件(2.2a,b)也可统一表达为：

$$w=0,\ \nabla^2 w=0;\ \varphi=0,\ \nabla^2\varphi=0 \tag{2.2c}$$

三、统一的Navier解式

首先，我们来探求变厚度矩形薄板的非线性理论的弹性平衡问题的Navier解。方程组(1.12)，由于左端各项呈现变系数的缘故，尚无法直接使用Navier法的双重三角级数求解。为此，我们特将方程组（1.12）变形为：

$$\begin{aligned}\nabla^2\nabla^2 w=\frac{1}{D}\Big[&(1-\mu)\left(\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2 D}{\partial x\partial y}+\frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right)\\ &-2\frac{\partial D}{\partial x}\frac{\partial}{\partial x}\nabla^2 w-2\frac{\partial D}{\partial y}\frac{\partial}{\partial y}\nabla^2 w-\nabla^2 D\nabla^2 w\\ &+\left(\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}-2\frac{\partial^2\varphi}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}\right)+q(x,\ y)\Big]\end{aligned} \tag{3.1}$$

$$\begin{aligned}\nabla^2\nabla^2\varphi=&\frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial}{\partial x}\nabla^2\varphi+\frac{2}{h}\frac{\partial h}{\partial y}\nabla^2\varphi+\frac{1}{h}\left(\frac{\partial^2 h}{\partial x^2}\frac{\partial^2\varphi}{\partial x^2}+2\frac{\partial^2 h}{\partial x\partial y}\frac{\partial^2\varphi}{\partial x\partial y}\right.\\ &\left.+\frac{\partial^2 h}{\partial y^2}\frac{\partial^2\varphi}{\partial y^2}\right)-\frac{2}{h^2}\left[\left(\frac{\partial h}{\partial x}\right)^2\frac{\partial^2\varphi}{\partial x^2}+2\frac{\partial h}{\partial x}\frac{\partial h}{\partial y}\frac{\partial^2\varphi}{\partial x\partial y}+\left(\frac{\partial h}{\partial y}\right)^2\frac{\partial^2\varphi}{\partial y^2}\right]\\ &+\frac{2\mu}{h^2}\left[\left(\frac{\partial h}{\partial x}\right)^2\frac{\partial^2\varphi}{\partial y^2}-2\frac{\partial h}{\partial x}\frac{\partial h}{\partial y}\frac{\partial^2\varphi}{\partial x\partial y}+\left(\frac{\partial h}{\partial y}\right)^2\frac{\partial^2\varphi}{\partial x^2}\right]-\frac{\mu}{h}\left[\frac{\partial^2 h}{\partial x^2}\frac{\partial^2\varphi}{\partial y^2}\right.\\ &\left.-2\frac{\partial^2 h}{\partial x\partial y}\frac{\partial^2\varphi}{\partial x\partial y}+\frac{\partial^2 h}{\partial y^2}\frac{\partial^2\varphi}{\partial x^2}\right]+Eh\left[\left(\frac{\partial w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\end{aligned} \tag{3.2}$$

根据纳维叶（Navier）的方法，现设方程组(3.1)，(3.2)的解为如下的双重三角级数形式：

$$\left.\begin{aligned} w(x,\ y)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}w_{mn}\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y \\ \varphi(x,\ y)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\varphi_{mn}\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y \end{aligned}\right\} \tag{3.3}$$

显然，(3.3)式是满足边界条件 (2.2) 式的。

将方程 (3.1) 式和 (3.2) 式的右端分解为同样的双重三角级数形式：

$$\begin{aligned} &\frac{1}{D}\Big[(1-\mu)\Big(\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2 D}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\Big) \\ &\quad -2\frac{\partial D}{\partial x}\frac{\partial}{\partial x}\nabla^2 w-2\frac{\partial D}{\partial y}\frac{\partial}{\partial y}\nabla^2 w-\nabla^2 D\nabla^2 w \\ &\quad +\Big(\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}-2\frac{\partial^2\varphi}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}\Big)\Big] \\ &=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial}{\partial x}\nabla^2\varphi+\frac{2}{h}\frac{\partial h}{\partial y}\nabla^2\varphi+\frac{1}{h}\Big(\frac{\partial^2 h}{\partial x^2}\frac{\partial^2\varphi}{\partial x^2}+2\frac{\partial^2 h}{\partial x\partial y}\frac{\partial^2\varphi}{\partial x\partial y}+\frac{\partial^2 h}{\partial y^2}\frac{\partial^2\varphi}{\partial y^2}\Big) \\ &\quad -\frac{2}{h^2}\Big[\Big(\frac{\partial h}{\partial x}\Big)^2\frac{\partial^2\varphi}{\partial x^2}+2\frac{\partial h}{\partial x}\frac{\partial h}{\partial y}\frac{\partial^2\varphi}{\partial x\partial y}+\Big(\frac{\partial h}{\partial y}\Big)^2\frac{\partial^2\varphi}{\partial y^2}\Big] \\ &\quad +\frac{2\mu}{h^2}\Big[\Big(\frac{\partial h}{\partial x}\Big)^2\frac{\partial^2\varphi}{\partial y^2}-2\frac{\partial h}{\partial x}\frac{\partial h}{\partial y}\frac{\partial^2\varphi}{\partial x\partial y}+\Big(\frac{\partial h}{\partial y}\Big)^2\frac{\partial^2\varphi}{\partial x^2}\Big] \\ &\quad -\frac{\mu}{h}\Big[\frac{\partial^2 h}{\partial x^2}\frac{\partial^2\varphi}{\partial y^2}-2\frac{\partial^2 h}{\partial x\partial y}\frac{\partial^2\varphi}{\partial x\partial y}+\frac{\partial^2 h}{\partial y^2}\frac{\partial^2\varphi}{\partial x^2}\Big]+Eh\Big[\Big(\frac{\partial w}{\partial x\partial y}\Big)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\Big] \\ &=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y \end{aligned} \tag{3.5}$$

$$\frac{1}{D}q(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q_{mn}\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y \tag{3.6}$$

式中，A_{mn}，B_{mn}，q_{mn} 为双重三角级数的 Fourier 系数，它们由下列各式确定：

$$\begin{aligned} A_{mn}&=\frac{4}{ab}\int_0^a\int_0^b\frac{1}{D}\Big[(1-\mu)\Big(\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2 D}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\Big) \\ &\quad -2\frac{\partial D}{\partial x}\frac{\partial}{\partial x}\nabla^2 w-2\frac{\partial D}{\partial y}\frac{\partial}{\partial y}\nabla^2 w-\nabla^2 D\nabla^2 w+\Big(\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \\ &\quad +\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}-2\frac{\partial^2\varphi}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}\Big)\Big]\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\,dx\,dy \\ &=\psi_1^{(mn)}w_{mn}+\psi_2^{(mn)}\varphi_{mn}w_{mn} \end{aligned} \tag{3.7}$$

$$B_{mn}=\frac{4}{ab}\int_0^a\int_0^b\Big\{\frac{2}{h}\frac{\partial h}{\partial x}\frac{\partial}{\partial x}\nabla^2\varphi+\frac{2}{h}\frac{\partial h}{\partial y}\frac{\partial}{\partial y}\nabla^2\varphi+\frac{1}{h}\Big(\frac{\partial^2 h}{\partial x^2}\frac{\partial^2\varphi}{\partial x^2}$$

$$+2\frac{\partial^2 h}{\partial x\partial y}\frac{\partial^2\varphi}{\partial x\partial y}+\frac{\partial^2 h}{\partial y^2}\frac{\partial^2\varphi}{\partial y^2}\Big)-\frac{2}{h^2}\Big[\Big(\frac{\partial h}{\partial x}\Big)^2\frac{\partial^2\varphi}{\partial x^2}+2\frac{\partial h}{\partial x}\frac{\partial h}{\partial y}\frac{\partial^2\varphi}{\partial x\partial y}$$

$$+\Big(\frac{\partial h}{\partial y}\Big)^2\frac{\partial^2\varphi}{\partial y^2}\Big]+\frac{2\mu}{h^2}\Big[\Big(\frac{\partial h}{\partial x}\Big)^2\frac{\partial^2\varphi}{\partial y^2}-2\frac{\partial h}{\partial x}\frac{\partial h}{\partial y}\frac{\partial^2\varphi}{\partial x\partial y}+\Big(\frac{\partial h}{\partial y}\Big)^2\frac{\partial^2\varphi}{\partial x^2}\Big]$$

$$-\frac{\mu}{h}\Big[\frac{\partial^2 h}{\partial x^2}\frac{\partial^2\varphi}{\partial y^2}-2\frac{\partial^2 h}{\partial x\partial y}\frac{\partial^2\varphi}{\partial x\partial y}+\frac{\partial^2 h}{\partial y^2}\frac{\partial^2\varphi}{\partial x^2}\Big]+Eh\Big[\Big(\frac{\partial^2 w}{\partial x\partial y}\Big)^2$$

$$-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\Big]\Big\}\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\,dx\,dy$$

$$=\psi_3^{(mn)}\varphi_{mn}+\psi_4^{(mn)}w_{mn}^2 \tag{3.8}$$

$$q_{mn}=\frac{4}{ab}\int_0^a\int_0^b\frac{1}{D}q(x,\ y)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\,dxdy \tag{3.9}$$

式中

$$\psi_1^{(mn)}=4(\mu-1)\frac{\pi^2}{ab}\int_0^a\int_0^b\frac{1}{D}\Big[\Big(\frac{n^2}{b^2}\frac{\partial^2 D}{\partial x^2}+\frac{m^2}{a^2}\frac{\partial^2 D}{\partial y^2}\Big)\Big]\sin^2\frac{m\pi}{a}x\sin^2\frac{n\pi}{b}y\,dxdy$$

$$+2(\mu-1)\frac{mn\pi^2}{a^2b^2}\int_0^a\int_0^b\frac{1}{D}\frac{\partial^2 D}{\partial x\partial y}\sin\frac{2m\pi}{a}x\sin\frac{2n\pi}{b}y\,dxdy$$

$$+\frac{4m\pi^3}{a^2b}\Big(\frac{m^2}{a^2}+\frac{n^2}{b^2}\Big)\int_0^a\int_0^b\frac{1}{D}\frac{\partial D}{\partial x}\sin\frac{2m\pi}{a}x\sin^2\frac{n\pi}{b}y\,dxdy$$

$$+\frac{4n\pi^3}{ab^2}\Big(\frac{m^2}{a^2}+\frac{n^2}{b^2}\Big)\int_0^a\int_0^b\frac{1}{D}\frac{\partial D}{\partial y}\sin^2\frac{m\pi}{a}x\sin\frac{2n\pi}{b}y\,dxdy$$

$$+\frac{4\pi^2}{ab}\Big(\frac{m^2}{a^2}+\frac{n^2}{b^2}\Big)\int_0^a\int_0^b\frac{1}{D}\nabla^2 D\sin^2\frac{m\pi}{a}x\sin^2\frac{n\pi}{b}y\,dxdy \tag{3.10}$$

$$\psi_2^{(mn)}=\frac{8m^2n^2\pi^4}{a^3b^3}\int_0^a\int_0^b\frac{1}{D}\sin^3\frac{m\pi}{a}x\sin^3\frac{n\pi}{b}y\,dxdy$$

$$-\frac{8m^2n^2\pi^4}{a^3b^3}\int_0^a\int_0^b\frac{1}{D}\sin\frac{m\pi}{a}x\cos^2\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\cos^2\frac{n\pi}{b}y\,dxdy \tag{3.11}$$

$$\psi_3^{(mn)}=-\frac{4m\pi^3}{a^2b}\Big(\frac{m^2}{a^2}+\frac{n^2}{b^2}\Big)\int_0^a\int_0^b\frac{\partial h}{\partial x}\frac{1}{h}\sin\frac{2m\pi}{a}x\sin^2\frac{n\pi}{b}y\,dxdy$$

$$-\frac{4n\pi^3}{ab^2}\Big(\frac{m^2}{a^2}+\frac{n^2}{b^2}\Big)\int_0^a\int_0^b\frac{1}{h}\frac{\partial h}{\partial y}\sin^2\frac{m\pi}{a}x\sin\frac{2n\pi}{b}y\,dxdy$$

$$+\frac{4\pi^2}{ab}\Big(\mu\frac{n^2}{b^2}-\frac{m^2}{a^2}\Big)\int_0^a\int_0^b\frac{1}{h}\frac{\partial^2 h}{\partial x^2}\sin^2\frac{m\pi}{a}x\sin^2\frac{n\pi}{b}y\,dxdy$$

$$+\frac{2mn\pi^2}{a^2b^2}(1+\mu)\int_0^a\int_0^b\frac{1}{h}\frac{\partial^2 h}{\partial x\partial y}\sin\frac{2m\pi}{a}x\sin\frac{2n\pi}{b}y\,dxdy$$

$$+\frac{4\pi^2}{ab}\Big(\mu\frac{m^2}{a^2}-\frac{n^2}{b^2}\Big)\int_0^a\int_0^b\frac{1}{h}\frac{\partial^2 h}{\partial y^2}\sin^2\frac{m\pi}{a}x\sin^2\frac{n\pi}{b}y\,dxdy$$

$$+\frac{8\pi^2}{ab}\left(\frac{m^2}{a^2}-\mu\frac{n^2}{b^2}\right)\int_0^a\int_0^b\frac{1}{h^2}\left(\frac{\partial h}{\partial x}\right)^2\sin^2\frac{m\pi}{a}x\sin^2\frac{n\pi}{b}y\,dxdy$$

$$+\frac{4mn\pi^2}{a^2b^2}(\mu-1)\int_0^a\int_0^b\frac{1}{h^2}\frac{\partial h}{\partial x}\frac{\partial h}{\partial y}\sin\frac{2m\pi}{a}x\sin\frac{2n\pi}{b}y\,dxdy$$

$$+\frac{8\pi^2}{ab}\left(\frac{n^2}{b^2}-\mu\frac{m^2}{a^2}\right)\int_0^a\int_0^b\frac{1}{h^2}\left(\frac{\partial h}{\partial y}\right)^2\sin^2\frac{m\pi}{a}x\sin^2\frac{n\pi}{b}y\,dxdy \tag{3.12}$$

$$\psi_4^{(mn)}=\frac{4m^2n^2\pi^4E}{a^3b^3}\int_0^a\int_0^b h\sin\frac{m\pi}{a}x\cos^2\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\cos^2\frac{n\pi}{b}y\,dxdy$$

$$-\frac{4m^2n^2\pi^4E}{a^3b^3}\int_0^a\int_0^b h\sin^3\frac{m\pi}{a}x\sin^3\frac{n\pi}{b}y\,dxdy \tag{3.13}$$

将式（3.3）中的第一式，（3.4）、（3.6）式代入方程（3.1）得：

$$\pi^4\left[\frac{m^4}{a^4}+2\frac{m^2n^2}{a^2b^2}+\frac{n^4}{b^4}\right]w_{mn}\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y$$

$$=(A_{mn}+q_{mn})\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y \tag{3.14}$$

由此 $$w_{mn}=\psi_5^{(mn)}(A_{mn}+q_{mn}) \tag{3.15}$$

式中 $$\psi_5^{(mn)}=\frac{1}{\pi^4\left[\frac{m^4}{a^4}+2\frac{m^2n^2}{a^2b^2}+\frac{n^4}{b^4}\right]} \tag{3.16}$$

将式（3.7）代入式（3.15）得：

$$w_{mn}=\psi_5^{(mn)}(\psi_1^{(mn)}w_{mn}+\psi_2^{(mn)}\varphi_{mn}w_{mn}+q_{mn}) \tag{3.17}$$

再把式（3.3）中的第二式、（3.5）式代入方程（3.2）得：

$$\pi^4\left[\frac{m^4}{a^4}+\frac{m^2n^2}{a^2b^2}+\frac{n^4}{b^4}\right]\varphi_{mn}\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y=B_{mn}\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y \tag{3.18}$$

由此 $$\varphi_{mn}=\psi_5^{(mn)}B_{mn} \tag{3.19}$$

式中 $\psi_5^{(mn)}$ 仍如前式（3.16）。

将式（3.8）代入式（3.19）得：

$$\varphi_{mn}=\psi_5^{(mn)}(\psi_3^{(mn)}\varphi_{mn}+\psi_4^{(mn)}w_{mn}^2) \tag{3.20}$$

或 $$\varphi_{mn}=\psi_6^{(mn)}w_{mn}^2 \tag{3.21}$$

式中 $$\psi_6^{(mn)}=\frac{\psi_4^{(mn)}\psi_5^{(mn)}}{1-\psi_3^{(mn)}\psi_5^{(mn)}} \tag{3.22}$$

将式（3.21）代入式（3.17）得：

$$\psi_7^{(mn)}w_{mn}^3+\psi_8^{(mn)}w_{mn}+\psi_9^{(mn)}=0\qquad(m,n=1,\ 3,\ 5,\cdots) \tag{3.23}$$

式中：

$$\psi_7^{(mn)}=\psi_2^{(mn)}\psi_5^{(mn)}\psi_6^{(mn)},\quad \psi_8^{(mn)}=\psi_1^{(mn)}\psi_5^{(mn)}-1,\quad \psi_9^{(mn)}=\psi_5^{(mn)}q_{mn} \tag{3.24}$$

在方程式（3.23）中，给脚标 m, n 以不同的自然数奇数值，便得到无穷个关于 w_{mn} 的三次方程组。解此非线性方程组，可得量 w_{mn}。其次，把 w_{mn} 代入式（3.21）得量 φ_{mn}，进而，由（3.3）式求得解函数 $w(x,\ y)$，$\varphi(x,\ y)$，最后，将 $w(x,\ y)$，$\varphi(x,\ y)$ 代入（1.1）式和（1.10）式，可得弯曲内力和中面应力。还可由下式求得横剪力：

$$Q_x=\frac{\partial M_x}{\partial x}+\frac{\partial M_{yx}}{\partial y}\quad,\quad Q_y=\frac{\partial M_y}{\partial y}+\frac{\partial M_{xy}}{\partial x} \tag{3.25}$$

其次，关于小挠度的线性理论的变厚度薄板的Navier解式，仍可仿上推得。其解函数仍取为式（3.3）中第一式。今简介量 w_{mn} 的结果如下：

$$w_{mn}=\frac{q_{mn}}{\psi_{1\,0}^{(mn)}-\psi_{1\,1}^{(mn)}} \tag{3.26}$$

式中：

$$\psi_{1\,0}^{(mn)}=\pi^4\left(\frac{m^4}{a^4}+2\frac{m^2n^2}{a^2b^2}+\frac{n^4}{b^4}\right) \tag{3.27}$$

$$\begin{aligned}\psi_{1\,1}^{(mn)}=&4(\mu-1)\frac{\pi^2}{ab}\int_0^a\int_0^b\frac{1}{D}\left[\left(\frac{n^2}{b^2}\frac{\partial^2 D}{\partial x^2}+\frac{m^2}{a^2}\frac{\partial^2 D}{\partial y^2}\right)\right]\\&\cdot\sin^2\frac{m\pi}{a}x\sin^2\frac{n\pi}{b}y\,dxdy\\&+2(\mu-1)\frac{mn\pi^2}{a^2b^2}\int_0^a\int_0^b\frac{1}{D}\frac{\partial^2 D}{\partial x\partial y}\sin\frac{2m\pi}{a}x\sin\frac{2n\pi}{b}ydxdy\\&+\frac{4m\pi^3}{a^2b}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\int_0^a\int_0^b\frac{1}{D}\frac{\partial D}{\partial x}\sin\frac{2m\pi}{a}x\sin^2\frac{n\pi}{b}ydxdy\\&+\frac{4n\pi^3}{ab^2}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\int_0^a\int_0^b\frac{1}{D}\frac{\partial D}{\partial y}\sin^2\frac{m\pi}{a}x\sin\frac{2n\pi}{b}ydxdy\\&+\frac{4\pi^2}{ab}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\int_0^a\int_0^b\frac{1}{D}\nabla^2 D\sin^2\frac{m\pi}{a}x\sin^2\frac{n\pi}{b}ydxdy\end{aligned} \tag{3.28}$$

$$q_{mn}=\frac{4}{ab}\int_0^a\int_0^b\frac{q(x,y)}{D}\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}ydxdy \tag{3.29}$$

至此，问题遂告全部解决。

四、计 算 示 例

为说明本文方法的应用，并验证其可靠性，我们特沿用文献[4]p.186中的例，用本文方法求解，以资比较。

例1 按小挠度的线性理论求变厚度薄板的Navier解。

设变厚度薄板为方板，$a=b$，受静水压力的作用，$q=(q_0/a)y$。板的厚度只是y的线性函数，板在直线 $y=a/2$ 上的厚度用 h_0 表示，于是，板在任意一点有：$h=[1+\lambda(2y/a-1)]h_0$，其中 λ 是常数。我们考虑只求沿板中线 $x=a/2$ 处的挠度和内力，并取 $\mu=0.25$，$\lambda=0.2$。利用式(3.26)～(3.29)，给 m，n 一系列奇数值，将求得的 w_{mn} 回代（3.3）式第一式，求得 $w(x,y)$。再利用内力公式进而求得 M_x，M_y。其计算结果见表1，与文献[4]p.186中例题的比较见图2。

表 1

项目 / m,n / $\frac{y}{a}$	$w\left(\frac{a}{2}, y\right)$			$M_x\left(\frac{a}{2}, y\right)$			$M_y\left(\frac{a}{2}, y\right)$		
	$m=1$ $n=1$	$m=1,3$ $n=1,3$	$m=1,3,5$ $n=1,3,5$	$m=1$ $n=1$	$m=1,3$ $n=1,3$	$m=1,3,5$ $n=1,3,5$	$m=1$ $n=1$	$m=1,3$ $n=1,3$	$m=1,3,5$ $n=1,3,5$
0.25	0.001445	0.001438	0.001439	0.012992	0.011966	0.012255	0.012992	0.013313	0.013248
0.50	0.002043	0.002001	0.002004	0.025203	0.022458	0.023038	0.025203	0.022915	0.023370
0.75	0.001445	0.001438	0.001439	0.023720	0.021847	0.022375	0.023720	0.024308	0.024188

*) 表中数值分别乘 $\frac{q_0 a^4}{D_0}$; $q_0 a^2$; $q_0 a^2$.

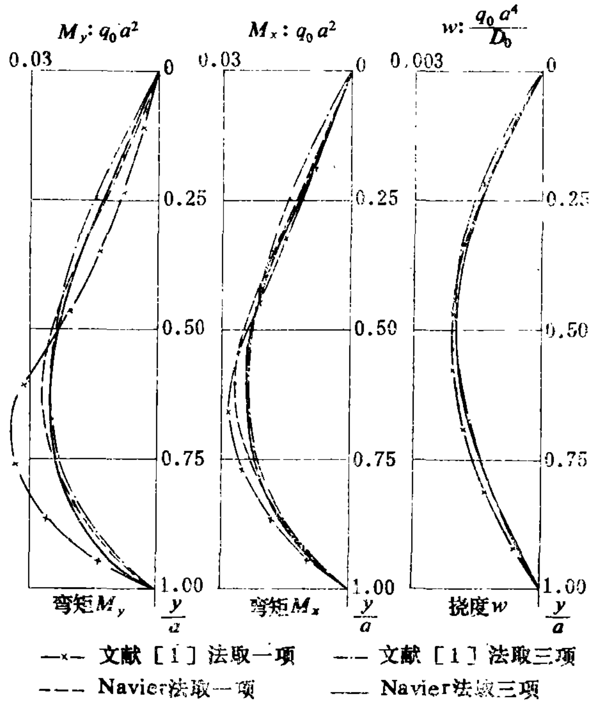

图 2

例 2 资料与例 1 同，按大挠度的非线性理论求变厚度薄板的 Navier 解。其挠度和内力的计算结果见表 2，与例 1 按小挠度的 Navier 解的比较见图3($q_0a^4/D_0h_0=400$时)。

表 2

项目 / m,n / $\frac{y}{a}$	$w\left(\frac{a}{2}, y\right)$			弯矩 $M_x\left(\frac{a}{2}, y\right)$			中面力 $N_x\left(\frac{a}{2}, y\right)$		
	$m=1$ $n=1$	$m=1,3$ $n=1,3$	$m=1,3,5$ $n=1,3,5$	$m=1$ $n=1$	$m=1,3$ $n=1,3$	$m=1,3,5$ $n=1,3,5$	$m=1$ $n=1$	$m=1,3$ $n=1,3$	$m=1,3,5$ $n=1,3,5$
0.25	0.481143	0.478431	0.478784	4.327240	3.916958	4.032755	0.435618	0.435660	0.435659
0.50	0.680438	0.663525	0.664858	8.394570	7.296361	7.527691	0.616057	0.615964	0.615964
0.75	0.481143	0.478431	0.478784	7.900627	7.151538	7.362959	0.435618	0.435660	0.435659

*) 表中挠度和内力值为取$\frac{q_0a^4}{D_0h_0}=400$时算得的挠度和内力值.表中数值 分 别 乘$\frac{q_0a^4}{D_0}$, q_0a^2, $\frac{q_0^2a^6Eh_0}{D_0^2}$.

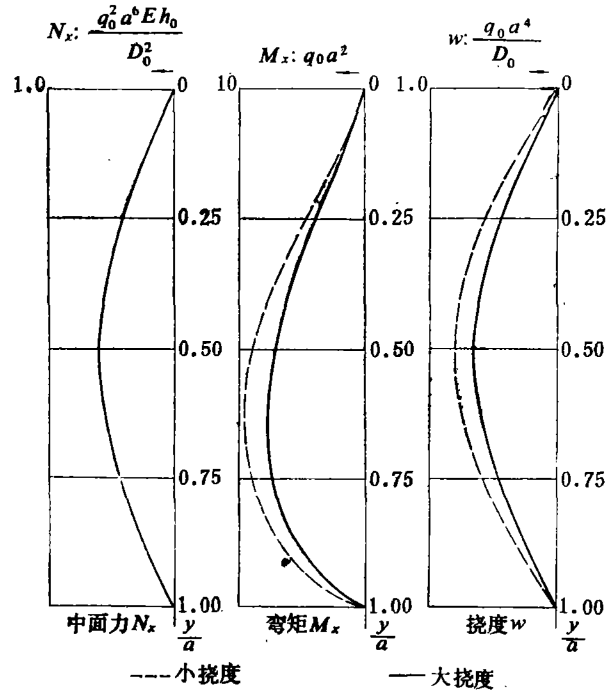

图 3

五、后 记

通过本文分析，有几点粗浅体会：

1. 从上述示例计算结果和荷载挠度曲线（图4，$x=a/2$，$y=a/2$）表明：

(1)小挠度理论假定横向荷载完全由板的弯曲承担，不考虑板的中面伸缩，即不计中面内力，这种假设只有当板的挠度远小于其厚度时才可成立。然而在大位移情况下就不合实际了。

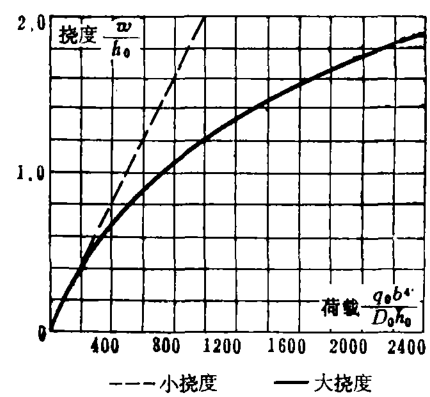

图 4

(2)大挠度理论认定横向荷载由板的弯曲和中面伸缩共同承担。不仅弯曲内力而且中面内力也随荷载的增加而增大，所以中面内力不可忽视。

(3)由图不难看出，小挠度理论夸大了挠度也夸大了内力。因此，按小挠度理论进行设计计算，显然是偏于安全和浪费的。

(4)当 $w_{max}<0.3h_0$ 时，荷载与挠度近似线性关系，故线性理论成立；当 $w_{max}>0.3h_0$时荷载与挠度成非线性关系，故力的线性迭加原理不再适用。

(5)用 Navier 法解变厚度矩形薄板的弹性平衡问题时，挠度和内力的收敛是相当快的。试以例 1 来说明其收敛情况。我们假定展开式第一项 $m=n=1$ 的数值取为 100%，中间截面

$y=0.5b$的挠度 w 的级数收敛情形见图 5，同一截面的内力 M_x 的收敛情形见图 6。从图形上看出，当仅取展开式第一项 $m=n=1$ 时，薄板挠度的收敛情形是比较快的，内力的收敛稍差一些，若计及展开式以后几项，则薄板级数解的收敛情形显然可以获得进一步的改善。同时，本法与文献[4]的解法比较，其精度不但十分吻合，而且收敛得更快。

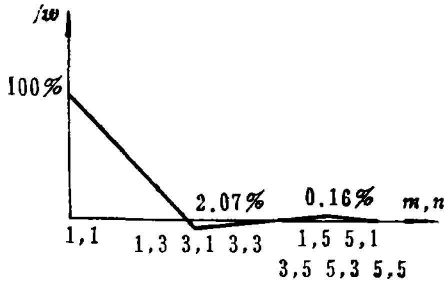

图 5

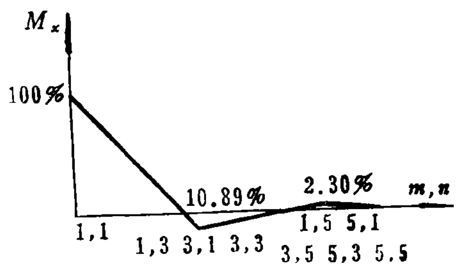

图 6

2. 在解决问题的过程中，给我们一个很大的启示，即：在混合边界条件（2.1）或（2.2）式的情形下，用 Navier 法的双重三角级数去处理线性的或非线性的变系数的高阶的偏微分方程组时，关键在于将方程适当变形，使方程左端留下的偶阶导数项摆脱变系数的困难，原则上皆可解决。

3. 本文提供的分析方法，若与结构力学中解静不定结构的“力法”相结合，则能解较复杂边界条件的小挠度变厚度板壳问题。同时，本法显然亦能解决四边简支的变厚度双曲扁薄壳的线性和非线性理论的弹性平衡问题以及线性理论的固有振动问题。

附　录

——变截面双曲扁壳的非线性理论的基本微分方程[9]

为了解变截面双曲扁壳的非线性理论的弹性平衡问题和线性理论的固有振动问题，现给出该问题的基本微分方程如下：

$$\left.\begin{aligned}&\frac{\partial^2}{\partial x^2}\left[D\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right)\right]+2(1-\mu)\frac{\partial^2}{\partial x\partial y}\left[\frac{\partial^2 w}{\partial x\partial y}\right]\\&\quad+\frac{\partial^2}{\partial y^2}\left[D\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)\right]=\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}-2\frac{\partial^2\varphi}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}\\&\quad+\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\nabla_k^2\varphi+q^{1)}\\&\frac{\partial^2}{\partial x^2}\left(\frac{1}{h}\frac{\partial^2\varphi}{\partial x^2}\right)+2\frac{\partial^2\varphi}{\partial x\partial y}\left(\frac{1}{h}\frac{\partial^2}{\partial x\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{1}{h}\frac{\partial^2\varphi}{\partial y^2}\right)\\&\quad-\mu\left[\frac{\partial^2}{\partial x^2}\left(\frac{1}{h}\frac{\partial^2\varphi}{\partial y^2}\right)-2\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{h}\frac{\partial^2\varphi}{\partial x\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{1}{h}\frac{\partial^2\varphi}{\partial x^2}\right)\right]\\&\quad=E\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\nabla_k^2 w\right]\end{aligned}\right\}. \tag{A.1}$$

其中引入了如下的微分算子：

$$\nabla_k^2=k_x\frac{\partial^2}{\partial y^2}-2k_{xy}\frac{\partial^2}{\partial x\partial y}+k_y\frac{\partial^2}{\partial x^2} \tag{A.2}$$

1) 当表述扁壳线性理论的固有振动问题时，此项q可改为$\frac{\gamma h(x,y)}{g}\omega^2 w$[10]，自然要去掉方程中的非线性项。

式中，k_x，k_y和k_{xy}分别为壳体中面的曲率和扭曲率；γ为扁壳材料的单位容重；g为重力加速度；ω为固有振动圆频率。在式(A.1)中，设$k_x=k_y=k_{xy}=0$，则式(A.1)退化为变厚度薄板的基本微分方程。

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Navier Solution for the Elastic Equilibrium Problems of Rectangular Thin Plates with Variable Thickness in Linear and Nonlinear Theories

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Abstract

This paper discusses the elastic equilibrium problems of rectangular thin plates of variable thickness and simply supported on all four sides in linear and nonlinear theories, using the Navier method to seek an approach to the problem, and to illustrate the solution with two examples. In conclusion, mention is made of the scope of application and the convergency of the solution.