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轴对称塑性问题摄动解 I ——圆杆颈缩*

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摘　　要

本文从文[11]中导得的轴对称塑性问题的一般方程出发，运用摄动技术，对圆杆颈缩问题进行渐近分析．所得的结果将使我们对拉伸试件的颈缩现象有进一步的认识，并使我们能更好地解释拉伸试件形成杯—锥状断口的原因．

一、引　　言

单向拉伸试件颈缩处应力与应变分布是相当复杂的，并且至今还没有得到满意的解决．但是，由于对于了解在破坏前一刹那的流动应力数值的重要性，这一问题早就引起了研究者的注意．

Bridgman(1944)[1]基于实验所提供的数据，假设在最小颈缩截面上变形是均匀的，即有径向速度 u 与 r 成比例，因而径向应变率 $\dot{\varepsilon}_r$ 与周向应变率 $\dot{\varepsilon}_\theta$ 相等，并与 r 无关．据此，立即得到 $\sigma_r=\sigma_\theta$．分析这一问题的困难还在于，颈部的形状是未知的，并且只能跟随它的逐渐发展来确定．因而，Bridgman 假定颈部外廓线为部分圆弧，即假设主应力面的曲率半径

$$\rho=\frac{a^2+2aR-r^2}{2r} \tag{1.1}$$

其中，a 为最小颈缩截面的半径，R 为最小颈缩截面处的曲率半径．

这样一来，就使问题得到很大的简化，由此，导得了最小颈缩截面上的应力分布近似公式．

Давиденков 和 Спиридонова(1945)[2]作了和 Bridgman 类似的分析，同样导得了最小颈缩截面上的应力分布近似公式．所不同的是，他们根据实测结果，未作进一步地说明，就假定主应力面的曲率

$$\frac{1}{\rho}=\frac{r}{aR} \tag{1.2}$$

Parker 等(1946)[3]采用半实验方法，计算了最小颈缩截面上的应力分布．结果表明，除对称轴外，σ_r 和 σ_θ 并不相等．

Yamashita(1966)[4]的实验研究表明，拉伸试件颈缩处的应变分布是不均匀的，他假设

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* 钱伟长推荐．

在最小颈缩截面上，轴向应变为 r 的二次函数，即

$$\dot{\varepsilon}_z=B-Ar^2 \tag{1.3}$$

其中 A, B 为待定常数。

其实验结果还表明，主应力面曲率与 Давиденков 公式(1.2)很好地相符合。据此，他计算了最小颈缩截面上的应力分布，得到的曲线高于 Bridgman 和 Давиденков 的结果。

Kaplan(1973)[5]基于均匀变形的假设，研究了最小颈缩截面邻近区域的应力与应变分布。得出，颈部塑性区范围为 $0\leqslant z/a\leqslant 0.87$，颈部外廓线为抛物线。

陈篪(1978)[6]基于实验数据，认为颈部外廓线为共焦双曲线。在均匀变形的假设下，导得了最小颈缩截面上应力分布的另一种近似解。

Chen(1971)[7], Needleman(1972)[8], Norris 等(1978)[9], Saje(1979)[10], Chen Liguo(1983)[12]，嵇醒等(1983)[13]，借助于计算机程序，对颈缩处的应力与应变进行数值分析，由于采用了不同的计算模型，得到一些有或多或少差别的结果。

为了获得对颈缩处整个塑性区应力与应变分布规律性的进一步认识，本文将从文[11]中导得的轴对称塑性问题的一般方程出发，运用摄动技术，来构造颈缩问题的渐近解。

二、摄 动 方 程

在文[11]中，我们导得了轴对称塑性问题的一般方程，并可以表为如下无量纲形式：

$$\left(\frac{1}{r}\frac{\partial\psi}{\partial z}-\frac{\partial^2\psi}{\partial r\partial z}\right)^2+\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)\left(\frac{\partial^2\psi}{\partial r\partial z}\right)+\frac{1}{4}\left(\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}-\frac{\partial^2\psi}{\partial z^2}\right)^2=\lambda^2r^2 \tag{2.1}$$

$$\frac{\partial^2}{\partial r\partial z}\left[\frac{1}{\lambda r}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}-2\frac{\partial^2\psi}{\partial r\partial z}\right)\right]+\frac{1}{r}\frac{\partial}{\partial z}\left[\frac{1}{\lambda r}\left(\frac{2}{r}\frac{\partial\psi}{\partial z}-\frac{\partial^2\psi}{\partial r\partial z}\right)\right]$$
$$-\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}-\frac{\partial^2}{\partial z^2}\right)\left[\frac{1}{2\lambda r}\left(\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}-\frac{\partial^2\psi}{\partial z^2}\right)\right]=0 \tag{2.2}$$

其中 $r=r/a,\ z=z/a,\ \psi=\psi/a^3,\ \lambda=k\lambda$ (2.3)

设方程的解 $\psi=\psi_0+\varepsilon\psi_1+\varepsilon^2\psi_2+o(\varepsilon^3),\ \lambda=\lambda_0+\varepsilon\lambda_1+\varepsilon^2\lambda_2+o(\varepsilon^3)$ (2.4)

其中 ε 为小参数。

将式(2.4)代入式(2.1)和(2.2)，我们可以得到

0 级摄动方程

$$\left(\frac{1}{r}\frac{\partial\psi_0}{\partial z}-\frac{\partial^2\psi_0}{\partial r\partial z}\right)^2+\left(\frac{1}{r}\frac{\partial\psi_0}{\partial z}\right)\left(\frac{\partial^2\psi_0}{\partial r\partial z}\right)+\frac{1}{4}\left(\frac{\partial^2\psi_0}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_0}{\partial r}-\frac{\partial^2\psi_0}{\partial z^2}\right)^2=\lambda_0^2r^2 \tag{2.5}$$

$$\frac{\partial^2}{\partial r\partial z}\left[\frac{1}{\lambda_0 r}\left(\frac{1}{r}\frac{\partial\psi_0}{\partial z}-2\frac{\partial^2\psi_0}{\partial r\partial z}\right)\right]+\frac{1}{r}\frac{\partial}{\partial z}\left[\frac{1}{\lambda_0 r}\left(\frac{2}{r}\frac{\partial\psi_0}{\partial z}-\frac{\partial^2\psi_0}{\partial r\partial z}\right)\right]$$
$$-\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}-\frac{\partial^2}{\partial z^2}\right)\left[\frac{1}{2\lambda_0 r}\left(\frac{\partial^2\psi_0}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_0}{\partial r}-\frac{\partial^2\psi_0}{\partial z^2}\right)\right]=0 \tag{2.6}$$

1 级摄动方程

$$2\left(\frac{1}{r}\frac{\partial\psi_0}{\partial z}-\frac{\partial^2\psi_0}{\partial r\partial z}\right)\left(\frac{1}{r}\frac{\partial\psi_1}{\partial z}-\frac{\partial^2\psi_1}{\partial r\partial z}\right)+\left(\frac{1}{r}\frac{\partial\psi_0}{\partial z}\right)\left(\frac{\partial^2\psi_1}{\partial r\partial z}\right)+\left(\frac{1}{r}\frac{\partial\psi_1}{\partial z}\right)\left(\frac{\partial^2\psi_0}{\partial r\partial z}\right)$$
$$+\frac{1}{2}\left(\frac{\partial^2\psi_0}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_0}{\partial r}-\frac{\partial^2\psi_0}{\partial z^2}\right)\left(\frac{\partial^2\psi_1}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_1}{\partial r}-\frac{\partial^2\psi_1}{\partial z^2}\right)\Big]=2\lambda_0\lambda_1r^2 \tag{2.7}$$

$$\frac{\partial^2}{\partial r\partial z}\left[\frac{1}{\lambda_0 r}\left(\frac{1}{r}\frac{\partial\psi_1}{\partial z}-2\frac{\partial^2\psi_1}{\partial r\partial z}\right)\right]+\frac{1}{r}\frac{\partial}{\partial z}\left[\frac{1}{\lambda_0 r}\left(\frac{2}{r}\frac{\partial\psi_1}{\partial z}-\frac{\partial^2\psi_1}{\partial r\partial z}\right)\right]$$

$$-\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}-\frac{\partial^2}{\partial z^2}\right)\left[\frac{1}{2\lambda_0 r}\left(\frac{\partial^2\psi_1}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_1}{\partial r}-\frac{\partial^2\psi_1}{\partial z^2}\right)\right]$$

$$=\frac{\partial^2}{\partial r\partial z}\left[\frac{1}{\lambda_0 r}\left(\frac{1}{r}\frac{\partial\psi_0}{\partial z}-2\frac{\partial^2\psi_0}{\partial r\partial z}\right)\frac{\lambda_1}{\lambda_0}\right]+\frac{1}{r}\frac{\partial}{\partial z}\left[\frac{1}{\lambda_0 r}\left(\frac{2}{r}\frac{\partial\psi_0}{\partial z}-\frac{\partial^2\psi_0}{\partial r\partial z}\right)\frac{\lambda_1}{\lambda_0}\right]$$

$$-\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}-\frac{\partial^2}{\partial z^2}\right)\left[\frac{1}{2\lambda_0 r}\left(\frac{\partial^2\psi_0}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_0}{\partial r}-\frac{\partial^2\psi_0}{\partial z^2}\right)\frac{\lambda_1}{\lambda_0}\right] \tag{2.8}$$

2 级摄动方程

$$2\left(\frac{1}{r}\frac{\partial\psi_0}{\partial z}-\frac{\partial^2\psi_0}{\partial r\partial z}\right)\left(\frac{1}{r}\frac{\partial\psi_2}{\partial z}-\frac{\partial^2\psi_2}{\partial r\partial z}\right)+\left(\frac{1}{r}\frac{\partial\psi_0}{\partial z}\right)\left(\frac{\partial^2\psi_2}{\partial r\partial z}\right)+\left(\frac{1}{r}\frac{\partial\psi_2}{\partial z}\right)\left(\frac{\partial^2\psi_0}{\partial r\partial z}\right)$$

$$+\frac{1}{2}\left(\frac{\partial^2\psi_0}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_0}{\partial r}-\frac{\partial^2\psi_0}{\partial z^2}\right)\left(\frac{\partial^2\psi_2}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_2}{\partial r}-\frac{\partial^2\psi_2}{\partial z^2}\right)=(2\lambda_0\lambda_2+\lambda_1^2)r^2$$

$$-\left[\left(\frac{1}{r}\frac{\partial\psi_1}{\partial z}-\frac{\partial^2\psi_1}{\partial r\partial z}\right)^2+\left(\frac{1}{r}\frac{\partial\psi_1}{\partial z}\right)\left(\frac{\partial^2\psi_1}{\partial r\partial z}\right)+\frac{1}{4}\left(\frac{\partial^2\psi_1}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_1}{\partial r}-\frac{\partial^2\psi_1}{\partial z^2}\right)^2\right] \tag{2.9}$$

$$\frac{\partial^2}{\partial r\partial z}\left[\frac{1}{\lambda_0 r}\left(\frac{1}{r}\frac{\partial\psi_2}{\partial z}-2\frac{\partial^2\psi_2}{\partial r\partial z}\right)\right]+\frac{1}{r}\frac{\partial}{\partial z}\left[\frac{1}{\lambda_0 r}\left(\frac{2}{r}\frac{\partial\psi_2}{\partial z}-\frac{\partial^2\psi_2}{\partial r\partial z}\right)\right]$$

$$-\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}-\frac{\partial^2}{\partial z^2}\right)\left[\frac{1}{2\lambda_0 r}\left(\frac{\partial^2\psi_2}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_2}{\partial r}-\frac{\partial^2\psi_2}{\partial z^2}\right)\right]$$

$$=\frac{\partial^2}{\partial r\partial z}\left[\frac{1}{\lambda_0 r}\left(\frac{1}{r}\frac{\partial\psi_0}{\partial z}-2\frac{\partial^2\psi_0}{\partial r\partial z}\right)\left(\frac{\lambda_2}{\lambda_0}-\frac{\lambda_1^2}{\lambda_0^2}\right)\right]+\frac{1}{r}\frac{\partial}{\partial z}\left[\frac{1}{\lambda_0 r}\left(\frac{2}{r}\frac{\partial\psi_0}{\partial z}-\frac{\partial^2\psi_0}{\partial r\partial z}\right)\left(\frac{\lambda_2}{\lambda_0}-\frac{\lambda_1^2}{\lambda_0^2}\right)\right]$$

$$-\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}-\frac{\partial^2}{\partial z^2}\right)\left[\frac{1}{2\lambda_0 r}\left(\frac{\partial^2\psi_0}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_0}{\partial r}-\frac{\partial^2\psi_0}{\partial z^2}\right)\left(\frac{\lambda_2}{\lambda_0}-\frac{\lambda_1^2}{\lambda_0^2}\right)\right]$$

$$+\frac{\partial^2}{\partial r\partial z}\left[\frac{1}{\lambda_0 r}\left(\frac{1}{r}\frac{\partial\psi_1}{\partial z}-2\frac{\partial^2\psi_1}{\partial r\partial z}\right)\frac{\lambda_1}{\lambda_0}\right]+\frac{1}{r}\frac{\partial}{\partial z}\left[\frac{1}{\lambda_0 r}\left(\frac{2}{r}\frac{\partial\psi_1}{\partial z}-\frac{\partial^2\psi_1}{\partial r\partial z}\right)\frac{\lambda_1}{\lambda_0}\right]$$

$$-\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}-\frac{\partial^2}{\partial z^2}\right)\left[\frac{1}{2\lambda_0 r}\left(\frac{\partial^2\psi_1}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_1}{\partial r}-\frac{\partial^2\psi_1}{\partial z^2}\right)\frac{\lambda_1}{\lambda_0}\right] \tag{2.10}$$

所有的方程，除了式(2.5)外，已线性化。适当地选取 0 级解，我们就能得到全部线性化方程，加之边界条件，就可以构造轴对称塑性问题摄动解。

三、颈缩问题摄动解

对于圆杆颈缩问题，我们有对称条件

$$\left.\begin{array}{ll}(1) & r=0,\ u=0\\ (2) & z=0,\ w=0\end{array}\right\} \tag{3.1}$$

其中，$u=u/a$，$w=w/a$ 为无量纲速度分量。

和应力边界条件，

$$z=0,\ r=1;\ \sigma_r=0 \tag{3.2}$$

其中，$\sigma_r=\sigma_r/k$ 为无量纲应力分量。

根据文[11]，我们有

$$u=\frac{1}{r}\frac{\partial\psi}{\partial z},\quad w=-\frac{1}{r}\frac{\partial\psi}{\partial r} \tag{3.3}$$

将式(2.4)代入式(3.3)，得

$$\left.\begin{aligned}u&=\left(\frac{1}{r}\frac{\partial\psi_0}{\partial z}\right)+\varepsilon\left(\frac{1}{r}\frac{\partial\psi_1}{\partial z}\right)+\varepsilon^2\left(\frac{1}{r}\frac{\partial\psi_2}{\partial z}\right)+o(\varepsilon^3)\\ w&=-\left(\frac{1}{r}\frac{\partial\psi_0}{\partial r}\right)-\varepsilon\left(\frac{1}{r}\frac{\partial\psi_1}{\partial r}\right)-\varepsilon^2\left(\frac{1}{r}\frac{\partial\psi_2}{\partial r}\right)+o(\varepsilon^3)\end{aligned}\right\} \tag{3.4}$$

因此，相应于各级摄动方程，有各级对称条件

0 级： $$r=0,\ \frac{1}{r}\frac{\partial\psi_0}{\partial z}=0;\quad z=0,\ \frac{1}{r}\frac{\partial\psi_0}{\partial r}=0 \tag{3.5}$$

1 级： $$r=0,\ \frac{1}{r}\frac{\partial\psi_1}{\partial z}=0;\quad z=0,\ \frac{1}{r}\frac{\partial\psi_1}{\partial r}=0 \tag{3.6}$$

2 级： $$r=0,\ \frac{1}{r}\frac{\partial\psi_2}{\partial z}=0;\quad z=0,\ \frac{1}{r}\frac{\partial\psi_2}{\partial r}=0 \tag{3.7}$$

由于应力分量σ_r可表为

$$\sigma_r=\sigma_r^{(0)}+\varepsilon\sigma_r^{(1)}+\varepsilon^2\sigma_r^{(2)}+o(\varepsilon^3) \tag{3.8}$$

同样，我们有各级边界条件

0 级： $$z=0,\ r=1;\ \sigma_r^{(0)}=0 \tag{3.9}$$

1 级： $$z=0,\ r=1;\ \sigma_r^{(1)}=0 \tag{3.10}$$

2 级： $$z=0,\ r=1;\ \sigma_r^{(2)}=0 \tag{3.11}$$

计及对称条件(3.5)，我们取 0 级摄动方程的解为

$$\psi_0=-a_0r^2z,\quad \lambda_0=\text{const} \tag{3.12}$$

则式(2.6)自动满足.

将式(3.12)代入式(2.5)，得

$$\lambda_0=\sqrt{3}\,a_0 \tag{3.13}$$

由式(3.12)，(3.13)，1 级摄动方程(2.7)、(2.8)变为

$$\frac{1}{r}\frac{\partial^2\psi_1}{\partial r\partial z}=-\frac{2}{\sqrt{3}}\lambda_1 \tag{3.14}$$

$$\begin{aligned}&\frac{\partial^2}{\partial r\partial z}\left[\frac{1}{r}\left(\frac{1}{r}\frac{\partial\psi_1}{\partial z}-\frac{1}{2}\frac{\partial^2\psi_1}{\partial r\partial z}\right)\right]+\frac{1}{r}\frac{\partial}{\partial z}\left[\frac{1}{r}\left(\frac{2}{r}\frac{\partial\psi_1}{\partial z}-\frac{\partial^2\psi_1}{\partial r\partial z}\right)\right]\\&\quad-\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}-\frac{\partial^2}{\partial z^2}\right)\left[\frac{1}{2r}\left(\frac{\partial^2\psi_1}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_1}{\partial r}-\frac{\partial^2\psi_1}{\partial z^2}\right)\right]=0\end{aligned} \tag{3.15}$$

计及对称条件(3.6)，我们取 1 级摄动方程的解为

$$\psi_1=r^2z(a_1r^2+a_2z^2),\quad \lambda_1=\sqrt{3}(c_1r^2+c_2z^2) \tag{3.16}$$

则式(3.15)自动满足.

将式(3.16)代入式(3.14)，得

$$c_1=-2a_1,\quad c_2=-3a_2 \tag{3.17}$$

那么

$$\lambda_1=-\sqrt{3}(2a_1r^2+3a_2z^2) \tag{3.18}$$

由式(3.12)、(3.13)、(3.16)、(3.18)，2 级摄动方程(2.9)、(2.10)变为

$$\frac{1}{r}\frac{\partial^2\psi_2}{\partial r\partial z}=-\frac{2}{\sqrt{3}}\lambda_2+\frac{1}{3a_0}[a_1^2r^4+(4a_1-3a_2)^2r^2z^2] \tag{3.19}$$

$$\begin{aligned}&\frac{\partial^2}{\partial r\partial z}\left[\frac{1}{r}\left(\frac{1}{r}\frac{\partial\psi_2}{\partial z}-\frac{1}{2}\frac{\partial^2\psi_2}{\partial r\partial z}\right)\right]+\frac{1}{r}\frac{\partial}{\partial z}\left[\frac{1}{r}\left(\frac{2}{r}\frac{\partial\psi_2}{\partial z}-\frac{\partial^2\psi_2}{\partial r\partial z}\right)\right]\\&\quad-\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}-\frac{\partial^2}{\partial z^2}\right)\left[\frac{1}{2r}\left(\frac{\partial^2\psi_2}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_2}{\partial r}-\frac{\partial^2\psi_2}{\partial z^2}\right)\right]\\&=\frac{2\psi}{a_0}(4a_1^2-6a_1a_2+3a_2^2)rz\end{aligned} \tag{3.20}$$

计及对称条件(3.7)，我们取 2 级摄动方程的解为

$$\left.\begin{aligned}\psi_2&=-\frac{1}{a_0}r^2z\left(2a_1^2r^4+4a_1a_2r^2z^2+\frac{159}{160}a_2^2z^4\right)\\\lambda_2&=\frac{\sqrt{3}}{a_0}(c_3r^4+c_4r^2z^2+c_6z^4)\end{aligned}\right\} \tag{3.21}$$

将式(3.21)代入式(3.20)，得

$$\frac{16}{3}a_1=a_2 \tag{3.22}$$

将式(3.21)代入式(3.19)，得

$$c_3=\frac{37}{6}a_1^2,\quad c_4=\frac{57}{2}a_1a_2=152a_1^2,\quad c_6=\frac{159}{32}a_2^2=\frac{424}{3}a_1^2 \tag{3.23}$$

因此，我们有

$$\psi_1=r^2z\left(r^2+\frac{16}{3}z^2\right)a_1,\quad \lambda_1=-\sqrt{3}(2r^2+16z^2)a_1 \tag{3.24}$$

和

$$\left.\begin{aligned}\psi_2&=-r^2z\left(2r^4+\frac{64}{3}r^2z^2+\frac{424}{15}z^4\right)\frac{a_1^2}{a_0}\\\lambda_2&=\sqrt{3}\left(\frac{37}{6}r^4+152r^2z^2+\frac{424}{3}z^4\right)\frac{a_1^2}{a_0}\end{aligned}\right\} \tag{3.25}$$

故

$$\left.\begin{aligned}\psi&=-a_0r^2z\left[1-\left(r^2+\frac{16}{3}z^2\right)\frac{a_1}{a_0}\varepsilon+\left(2r^4+\frac{64}{3}r^2z^2+\frac{424}{15}z^4\right)\frac{a_1^2}{a_0^2}\varepsilon^2\right]+o(\varepsilon^3)\\\lambda&=\sqrt{3}\,a_0\left[1-(2r^2+16z^2)\frac{a_1}{a_0}\varepsilon+\left(\frac{37}{6}r^4+152r^2z^2+\frac{424}{3}z^4\right)\frac{a_1^2}{a_0^2}\varepsilon^2\right]+o(\varepsilon^3)\end{aligned}\right\} \tag{3.26}$$

二次摄动，令

$$\frac{a_1}{a_0}\varepsilon=e \tag{3.27}$$

我们得到

$$\left.\begin{aligned}\psi&=-a_0r^2z\left[1-\left(r^2+\frac{16}{3}z^2\right)e+\left(2r^4+\frac{64}{3}r^2z^2+\frac{424}{15}z^4\right)e^2\right]+o(e^3)\\ \lambda&=\sqrt{3}\,a_0\left[1-(2r^2+16z^2)e+\left(\frac{37}{6}r^4+152r^2z^2+\frac{424}{3}z^4\right)e^2\right]+o(e^3)\end{aligned}\right\}\tag{3.28}$$

至此，仅系数 a_0 与小参数 e 尚未确定.

四、应 力 场

根据文[11]，我们有

$$\tau_{rz}=-\frac{1}{2\lambda r}\left(\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}-\frac{\partial^2\psi}{\partial z^2}\right)\tag{4.1}$$

$$\sigma_r-\sigma_\theta=\frac{1}{\lambda r}\left[\left(-\frac{2}{r}\frac{\partial\psi}{\partial z}+\frac{\partial^2\psi}{\partial r\partial z}\right)\right]\tag{4.2}$$

$$\sigma_z-\sigma_r=\frac{1}{\lambda r}\left[\left(\frac{1}{r}\frac{\partial\psi}{\partial z}-2\frac{\partial^2\psi}{\partial r\partial z}\right)\right]\tag{4.3}$$

其中 σ_z, σ_r, σ_θ, τ_{rz} 皆为无量纲应力分量.

将式(2.4)代入(4.1)、(4.2)、(4.3)，得

$$\begin{aligned}\tau_{rz}=&-\frac{1}{2\lambda_0 r}\left(\frac{\partial^2\psi_0}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_0}{\partial r}-\frac{\partial^2\psi_0}{\partial z^2}\right)-\frac{1}{2\lambda_0 r}\left[\left(\frac{\partial^2\psi_1}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_1}{\partial r}-\frac{\partial^2\psi_1}{\partial z^2}\right)\right.\\&\left.-\left(\frac{\partial^2\psi_0}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_0}{\partial r}-\frac{\partial^2\psi_0}{\partial z^2}\right)\frac{\lambda_1}{\lambda_0}\right]\varepsilon-\frac{1}{2\lambda_0 r}\left[\left(\frac{\partial^2\psi_2}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_2}{\partial r}-\frac{\partial^2\psi_2}{\partial z^2}\right)\right.\\&\left.-\left(\frac{\partial^2\psi_0}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_0}{\partial r}-\frac{\partial^2\psi_0}{\partial z^2}\right)\left(\frac{\lambda_2}{\lambda_0}-\frac{\lambda_1^2}{\lambda_0^2}\right)-\left(\frac{\partial^2\psi_1}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_1}{\partial r}-\frac{\partial^2\psi_1}{\partial z^2}\right)\frac{\lambda_1}{\lambda_0}\right]\varepsilon^2+o(\varepsilon^3)\end{aligned}\tag{4.4}$$

$$\begin{aligned}\sigma_r-\sigma_\theta=&\frac{1}{\lambda_0 r}\left(-\frac{2}{r}\frac{\partial\psi_0}{\partial z}+\frac{\partial^2\psi_0}{\partial r\partial z}\right)+\frac{1}{\lambda_0 r}\left[\left(-\frac{2}{r}\frac{\partial\psi_1}{\partial z}+\frac{\partial^2\psi_1}{\partial r\partial z}\right)-\left(-\frac{2}{r}\frac{\partial\psi_0}{\partial z}+\frac{\partial^2\psi_0}{\partial r\partial z}\right)\frac{\lambda_1}{\lambda_0}\right]\varepsilon\\&+\frac{1}{\lambda_0 r}\left[\left(-\frac{2}{r}\frac{\partial\psi_2}{\partial z}+\frac{\partial^2\psi_2}{\partial r\partial z}\right)-\left(-\frac{2}{r}\frac{\partial\psi_0}{\partial z}+\frac{\partial^2\psi_0}{\partial r\partial z}\right)\left(\frac{\lambda_2}{\lambda_0}-\frac{\lambda_1^2}{\lambda_0^2}\right)\right.\\&\left.-\left(-\frac{2}{r}\frac{\partial\psi_1}{\partial z}+\frac{\partial^2\psi_1}{\partial r\partial z}\right)\frac{\lambda_1}{\lambda_0}\right]\varepsilon^2+o(\varepsilon^3)\end{aligned}\tag{4.5}$$

$$\begin{aligned}\sigma_z-\sigma_r=&\frac{1}{\lambda_0 r}\left(\frac{1}{r}\frac{\partial\psi_0}{\partial z}-2\frac{\partial^2\psi_0}{\partial r\partial z}\right)+\frac{1}{\lambda_0 r}\left[\left(\frac{1}{r}\frac{\partial\psi_1}{\partial z}-2\frac{\partial^2\psi_1}{\partial r\partial z}\right)-\left(\frac{1}{r}\frac{\partial\psi_0}{\partial z}-2\frac{\partial^2\psi_0}{\partial r\partial z}\right)\frac{\lambda_1}{\lambda_0}\right]\varepsilon\\&+\frac{1}{\lambda_0 r}\left[\left(\frac{1}{r}\frac{\partial\psi_2}{\partial z}-2\frac{\partial^2\psi_2}{\partial r\partial z}\right)-\left(\frac{1}{r}\frac{\partial\psi_0}{\partial z}-2\frac{\partial^2\psi_0}{\partial r\partial z}\right)\left(\frac{\lambda_2}{\lambda_0}-\frac{\lambda_1^2}{\lambda_0^2}\right)\right.\\&\left.-\left(\frac{1}{r}\frac{\partial\psi_1}{\partial z}-2\frac{\partial^2\psi_1}{\partial r\partial z}\right)\frac{\lambda_1}{\lambda_0}\right]\varepsilon^2+o(\varepsilon^3)\end{aligned}\tag{4.6}$$

将式(3.26)、(3.27)代入，得

$$\tau_{rz}=\left(\frac{12}{\sqrt{3}}rz\right)e-\frac{1}{\sqrt{3}}\left(16r^3z+\frac{16}{3}rz^3\right)e^2+o(e^3)\tag{4.7}$$

$$\sigma_r-\sigma_\theta=\left(\frac{2}{\sqrt{3}}r^2\right)e-\frac{1}{\sqrt{3}}(4r^4+96r^2z^2)e^2+o(e^3) \tag{4.8}$$

$$\sigma_z-\sigma_r=\sqrt{3}-\left(\frac{1}{\sqrt{3}}r^2\right)e+\frac{1}{\sqrt{3}}\left(\frac{3}{2}r^4-24r^2z^2\right)e^2+o(e^3) \tag{4.9}$$

由式(4.8)、(4.9)，得

$$\tau_m=\frac{1}{2}(\sigma_z-\sigma_\theta)=\frac{\sqrt{3}}{2}\left[1+\left(\frac{r^2}{3}\right)e-\frac{1}{3}\left(\frac{5}{2}r^4+120r^2z^2\right)e^2\right]+o(e^3) \tag{4.10}$$

根据平衡方程，我们有

$$\frac{\partial\sigma_r}{\partial r}=-\left[\frac{\partial\tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}\right] \tag{4.11}$$

$$\frac{\partial\sigma_z}{\partial z}=-\left[\frac{\partial\tau_{rz}}{\partial r}+\frac{\tau_{rz}}{r}\right] \tag{4.12}$$

将式(4.7)、(4.8)代入(4.11)，积分得

$$\sigma_r=d^{(0)}+h^{(0)}(z)+\left[d^{(1)}+h^{(1)}(z)-\frac{7}{\sqrt{3}}r^2\right]e+\left[d^{(2)}+h^{(2)}(z)+\frac{1}{\sqrt{3}}(5r^4+56r^2z^2)\right]e^2+o(e^3)$$

其中 $d^{(i)}$ $(i=0,1,2)$为待定常数，$h^{(i)}(z)$仅为 z 的函数。

计及边界条件(3.9)、(3.10)、(3.11)，得

$$\sigma_r=h^{(0)}(z)+\left[h^{(1)}(z)+\frac{1}{\sqrt{3}}(7-7r^2)\right]e+\left[h^{(2)}(z)+\frac{1}{\sqrt{3}}(5r^4+56r^2z^2-5)\right]e^2+o(e^3) \tag{4.13}$$

且

$$h^{(i)}(0)=0 \qquad (i=0,1,2) \tag{4.14}$$

由式(4.9)、(4.13)，我们得

$$\sigma_z=\sqrt{3}+h^{(0)}(z)+\left[h^{(1)}(z)+\frac{1}{\sqrt{3}}(7-8r^2)\right]e+\left[h^{(2)}(z)+\frac{1}{\sqrt{3}}\left(\frac{13}{2}r^4+32r^2z^2-5\right)\right]e^2+o(e^3) \tag{4.15}$$

将式(4.7)代入式(4.12)，积分得

$$\sigma_z=b^{(0)}+g^{(0)}(r)+\left[b^{(1)}+g^{(1)}(r)-\frac{12}{\sqrt{3}}z^2\right]e+\left[b^{(2)}+g^{(2)}(r)+\frac{1}{\sqrt{3}}\left(32r^2z^2+\frac{8}{3}z^4\right)\right]e^2+o(e^3) \tag{4.16}$$

其中 $b^{(i)}$ $(i=0,1,2)$为待定常数，$g^{(i)}(r)$仅为 r 的函数。

式(4.15)、(4.16)必须匹配，因此，我们得

$$\left.\begin{array}{l} b^{(0)}=\sqrt{3},\ b^{(1)}=\dfrac{7}{\sqrt{3}},\ b^{(2)}=-\dfrac{5}{\sqrt{3}} \\ h^{(0)}(z)=0,\ h^{(1)}(z)=-\dfrac{12}{\sqrt{3}}z^2,\ h^{(2)}(z)=\dfrac{1}{\sqrt{3}}\left(\dfrac{8}{3}z^4\right) \\ g^{(0)}(r)=0,\ g^{(1)}(r)=-\dfrac{8}{\sqrt{3}}r^2,\ g^{(2)}(r)=\dfrac{1}{\sqrt{3}}\left(\dfrac{13}{2}r^4\right) \end{array}\right\} \tag{4.17}$$

那么

$$\sigma_z=\sqrt{3}+\frac{1}{\sqrt{3}}(7-8r^2-12z^2)e+\frac{1}{\sqrt{3}}\left(\frac{13}{2}r^4+32r^2z^2+\frac{8}{3}z^4-5\right)e^2+o(e^3) \tag{4.18}$$

$$\sigma_r=\frac{1}{\sqrt{3}}(7-7r^2-12z^2)e+\frac{1}{\sqrt{3}}\left(5r^4+56r^2z^2+\frac{8}{3}z^4-5\right)e^2+o(e^3) \tag{4.19}$$

$$\sigma_\theta=\frac{1}{\sqrt{3}}(7-9r^2-12z^2)e+\frac{1}{\sqrt{3}}\left(9r^4+152r^2z^2+\frac{8}{3}z^4-5\right)e^2+o(e^3) \tag{4.20}$$

因此，在最小颈缩截面($z=0$)上，我们有

$$\sigma_z=\sqrt{3}+\frac{1}{\sqrt{3}}(7-8r^2)e+\frac{1}{\sqrt{3}}\left(\frac{13}{2}r^4-5\right)e^2+o(e^3) \tag{4.21}$$

$$\sigma_r=\frac{1}{\sqrt{3}}(7-7r^2)e+\frac{1}{\sqrt{3}}(5r^4-5)e^2+o(e^3) \tag{4.22}$$

$$\sigma_\theta=\frac{1}{\sqrt{3}}(7-9r^2)e+\frac{1}{\sqrt{3}}(9r^4-5)e^2+o(e^3) \tag{4.23}$$

$$\tau_{rz}=0 \tag{4.24}$$

$$\tau_m=\frac{\sqrt{3}}{2}\left[1+\frac{r^2}{3}e-\frac{5}{6}r^4e^2\right]+o(e^3) \tag{4.25}$$

有了应力分布我们就可以来确定颈部外廓线方程。颈部的形状可以根据平行于最小颈缩截面上的轴向载荷相等的条件来得到。即根据条件

$$P=2\pi a^2k\int_0^1\sigma_z|_{z=0}rdr=2\pi a^2k\int_0^r\sigma_z|_{z=z}rdr \tag{4.26}$$

将式(4.18)、(4.21)代入条件(4.26)中，得颈部外廓线方程

$$\left(\frac{4}{9}r^2e^2\right)z^4-\left(2r^2e-\frac{8}{3}r^4e^2\right)z^2+\left(\frac{1}{2}+\frac{7}{6}e-\frac{5}{6}e^2\right)r^2-\frac{2}{3}er^4$$
$$+\frac{13}{36}e^2r^6-\left(\frac{1}{2}+\frac{1}{2}e-\frac{17}{36}e^2\right)=0 \tag{4.27a}$$

或

$$\left(\frac{13}{36}e^2\right)r^6-\left(\frac{2}{3}e-\frac{8}{3}z^2e^2\right)r^4+\left(\frac{1}{2}+\frac{7}{6}e-\frac{5}{6}e^2-2z^2e+\frac{4}{9}z^4e^2\right)r^2$$
$$-\left(\frac{1}{2}+\frac{1}{2}e-\frac{17}{36}e^2\right)=0 \tag{4.27b}$$

由式(4.27a)，我们有

$$z'=\frac{1+\frac{1}{3}(7-8r^2-12z^2)e+\frac{1}{3}\left(\frac{13}{2}r^4+32r^2z^2+\frac{8}{3}z^4-5\right)e^2+o(e^3)}{(4rz)e-\frac{1}{3}\left(16r^3z+\frac{16}{3}rz^3\right)e^2+o(e^3)} \tag{4.28}$$

和

$$z''=-\frac{(4r)e+\frac{1}{3}(56r-80r^3+32rz^3)e^2+o(e^3)}{\left[(4rz)e-\frac{1}{3}\left(16r^3z+\frac{16}{3}rz^3\right)e^2+o(e^3)\right]^3} \tag{4.29}$$

则外廓线曲率

$$\frac{a}{\rho}=\frac{|z''|}{(1+z'z')^{\frac{3}{2}}}=(4r)e-\frac{1}{3}(28r-16r^3-176rz^2)e^2+o(e^3) \tag{4.30}$$

当 $z=0$，$r=1$ 时，有 $\rho=R$，因而得

$$\frac{a}{R}=4e-4e^2+o(e^3) \tag{4.31}$$

如若给定 a/R，那么小参数 e 将由下式确定

$$e=\frac{1}{2}\left[1-\sqrt{1-\frac{a}{R}}\right] \tag{4.32}$$

因为 a 与 R 可由实验测定，一旦小参数 e 确定后，应力场与颈部外廓线方程也随之完全确定。

五、速度场与应变场

将式(3.28)代入式(3.3)，得速度分量无量纲表达式

$$u=-a_0\left[r-(r^3+16rz^2)e+\left(2r^5+64r^3z^2+\frac{424}{3}rz^4\right)e^2\right]+o(e^3) \tag{5.1}$$

$$w=a_0\left[2z-\left(4r^2z+\frac{32}{3}z^3\right)e+\left(12r^4z+\frac{256}{3}r^2z^3+\frac{848}{15}z^5\right)e^2\right]+o(e^3) \tag{5.2}$$

那么应变场为

$$\dot{\varepsilon}_r=\frac{\partial u}{\partial r}=-a_0\left[1-(3r^2+16z^2)e+\left(10r^4+192r^2z^2+\frac{424}{3}z^4\right)e^2\right]+o(e^3) \tag{5.3}$$

$$\dot{\varepsilon}_\theta=\frac{u}{r}=-a_0\left[1-(r^2+16z^2)e+\left(2r^4+64r^2z^2+\frac{424}{3}z^4\right)e^2\right]+o(e^3) \tag{5.4}$$

$$\dot{\varepsilon}_z=\frac{\partial w}{\partial z}=a_0\left[2-(4r^2+32z^2)e+\left(12r^4+256r^2z^2+\frac{848}{3}z^4\right)e^2\right]+o(e^3) \tag{5.5}$$

$$\dot{\gamma}_{rz}=\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)=a_0\left[(12rz)e-\left(40r^3z+\frac{592}{3}rz^3\right)e^2\right]+o(e^3) \tag{5.6}$$

因此，在最小颈缩截面($z=0$)上，有

$$u=-a_0[r-r^3e+2r^5e^2]+o(e^3) \tag{5.7}$$

$$w=0 \tag{5.8}$$

$$\dot{\varepsilon}_r=-a_0[1-3r^2e+10r^4e^2]+o(e^3) \tag{5.9}$$

$$\dot{\varepsilon}_\theta=-a_0[1-r^2e+2r^4e^2]+o(e^3) \tag{5.10}$$

$$\dot{\varepsilon}_z=a_0[2-4r^2e+12r^4e^2]+o(e^3) \tag{5.11}$$

$$\dot{\gamma}_{rz}=0 \tag{5.12}$$

显见，速度场与应变场的最终确定依赖于系数 a_0 的确定。

如果我们给定颈部塑性区 $z=z_n$ 截面上的平均轴向速度 w^*，那么系数 a_0 可由下列条件确定

$$\frac{2\pi\int_0^{a_n}w|_{z=z_n}rdr}{\pi a_n^2}=w^* \tag{5.13}$$

由此导出

$$a_0=\frac{w^*}{2z_n}\cdot\frac{1}{\left[1-\left(\frac{a_n^2}{a^2}+\frac{16}{3}\frac{z_n^2}{a^2}\right)e+\left(2\frac{a_n^4}{a^4}+\frac{64}{3}\frac{a_n^2z_n^2}{a^4}+\frac{424}{15}\frac{z_n^4}{a^4}\right)e^2+o(e^3)\right]} \tag{5.14}$$

其中 a_n/a 可由 z_n/a 代入外廓线方程(4.27b)得到，小参数 e 仍由式(4.32)按不同变形程度 a/R 得出。

可见，对于不同变形程度 a/R，a_0 取不同的值。

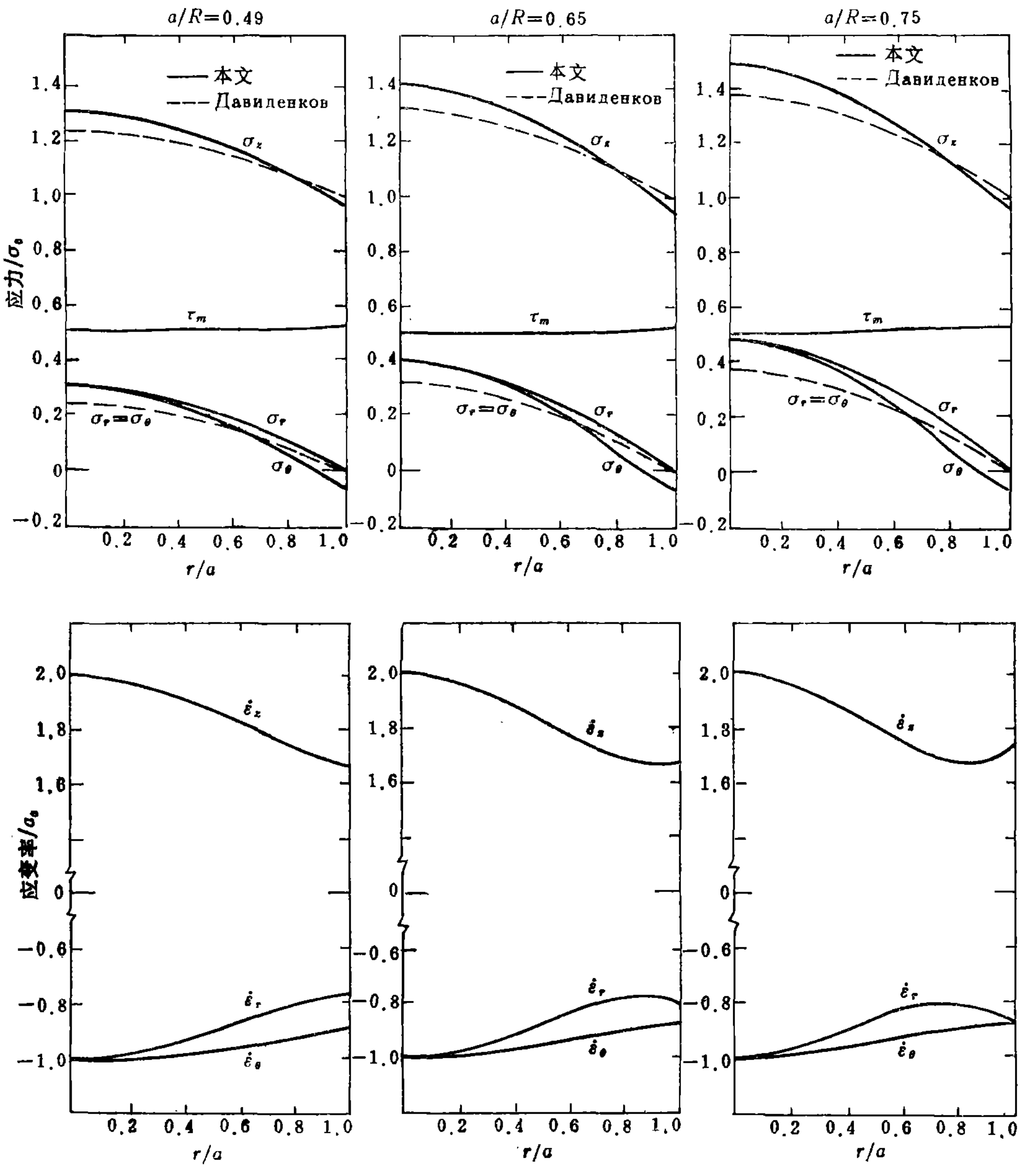

图1 最小颈缩截面上应力与应变率分布(a/R=0.49, 0.65, 0.75)

六、结论与讨论

(1) 根据式(4.21)～(4.25)及式(5.9)～(5.11)，图1分别绘出 $a/R=0.49$，0.65，0.75 不同变形状态时，最小颈缩截面上的应力与应变分布曲线，并与Давиденков的结果进行了比较。显见，随着 a/R 逐渐增大，轴向应力分布曲线呈双曲率型。

图2绘出 $a/R=0.75$ 时，颈缩处不同截面 ($z/a=0$，0.8)上的轴向应力分布曲线。这和Norris等[9]计算机模拟结果很相似。

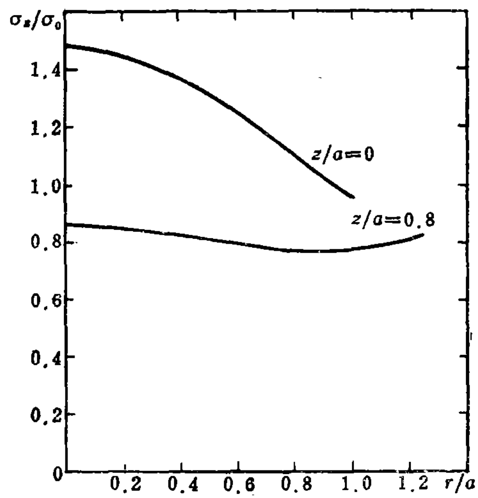

图2 不同截面轴向应力分布曲线($a/R=0.75$)

(2) 实验证明，即使在剪切断裂的情况下，断裂也开始于轴部。Parker 等在文 [3] 中对此有详细的论述。他们指出，要么 $\sigma_z-\sigma_r$ 在 $z=0$ 截面上并不是常数，而是在轴上有最大值；要么滑移面上的张应力对临界应力有影响。他们的实验结果支持前一个可能性。

由式(4.9)，在 $z=0$ 截面上

$$\sigma_z-\sigma_r=\sqrt{3}\left[1-\frac{1}{3}r^2e+\frac{1}{2}r^4e^2\right]+o(e^3) \tag{6.1}$$

不难看出，最大值在对称轴上($r=0$处)。渐近分析的结果同样证实前一个结论是正确的。

(3) 颈缩塑性区的应力与应变分布是远非均匀的。由式(5.7)、(5.9)、(5.10)可以看出，仅仅作为零级近似，才有径向速度 u 与 r 成比例，且有 $\dot{\varepsilon}_r=\dot{\varepsilon}_\theta$。因此，Bridgman 和 Давиденков 关于均匀变形的假设是相当粗糙的。

(4) 由式(4.30)、(4.31)，取一级近似，我们容易得到（有量纲形式）

$$\frac{1}{\rho}=\frac{r}{aR} \tag{6.2}$$

表1 颈部外廓线计算值与实测值比较

实测值[6]		计算值 $a/R\doteq0.432$ $r^2=(12.445)^2+0.492684505z^2$	实测值[6]		计算值 $a/R\doteq0.672$ $r^2=(10.365)^2+0.854574315z^2$
z	r	r	z	r	r
0.0	12.445	12.445	0.0	10.365	10.365
3.7	12.715	12.713	1.6	10.470	10.470
4.3	12.815	12.806	2.5	10.627	10.620
5.7	13.065	13.072	3.5	10.863	10.858
6.3	13.185	13.207	4.5	11.160	11.169
7.7	13.565	13.568	5.5	11.545	11.545
			6.6	12.030	12.027

这就是 Давиденков 假设。作为一级近似 Давиденков 假设无疑是正确的。

(5) 由式(4.27a)，取一级近似，我们可以导得（有量纲形式）
当 $0 \leqslant z/a \leqslant 0.7$ 时，有

$$r^2 = a^2 + 4z^2 e \tag{6.3}$$

表 1 给出按式(6.3)的计算值与文[6]中实测值的比较，可见，在拐点之内，用双曲线来描述颈部的外廓线具有相当好的精度。

(6) 文[11]中的一般方程是按理想塑性体导出的，考虑到实际问题中的强化效应，采用应变硬化假设，在计算应力与轴向载荷时，我们可让$\sqrt{3}k=\sigma_0$，σ_0 取应力-应变曲线上的瞬时值。

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Perturbation Solution of Axisymmetric Plastic Problem I ——Necking of a Bar

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Abstract

In this paper, basing on the general equations of axisymmetric plastic problems deduced in ref. [11], and employing perturbation technique, the asymptotic analysis for the necking problem is given. The result will provide knowledge of distribution of stress and strain in the whole plastic region, thus, it will lead to a better understanding of the necking phenomena in a tension specimen, such as cup-and-cone fracture.