

严格求解夹层圆柱壳在轴压下的轴对称失稳问题*

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摘 要

本文在文献[1]的基础上, 用严格的方法求解两端简支的夹层圆柱壳在均匀轴压下的轴对称失稳问题。内、外表层很薄弹性模量又大, 按薄壳理论处理; 夹心较厚弹性模量又相当小, 横向剪切变形的影响必须考虑, 在研究夹层壳的整体失稳尤其是局部失稳时, 横向的拉伸和压缩变形也不可忽略, 用数学弹性力学的方法处理。本文导得了可求解轴对称整体失稳和局部失稳临界载荷的超越方程, 用数值计算的方法可算得临界载荷的最小值。对于整体失稳的情况, 给出算例, 与夹层壳理论的解作了比较。

一、引 言

夹层结构具有重量轻、刚度大等优点, 已得到广泛的应用。近二、三十年来, 在板壳理论的基础上, 采用各种简化假定, 发展了夹层板壳理论。对于表板, 都采用克希荷夫假定; 对于夹心, 变形前中面的法线, 变形后虽不再是法线, 仍假定为一一直线。此外, 还有各种假定。有的理论单独或同时假定 $E_z = \infty$, $E_x = E_y = G_{xy} = 0$, 有的不作这些假定。其中, z 为法线方向, x 和 y 在板壳的中面上。采用夹层板壳理论, 已得到可供设计应用的结果。但是夹层板壳理论的解与弹性力学的解相比具有多少误差, 这是理论工作者所关心的问题。用数学弹性力学的方法, 严格而又正确地求解夹层板壳的稳定问题, 这是一个复杂而又重要的课题, 至今还未见到这方面的文献。因此本文用严格的方法对这个问题进行研究, 用算例将夹层壳理论所得的结果与严格解作比较, 考察夹层壳理论的可靠性, 具有重要的理论意义和一定的实际意义。

本文对夹层壳的内、外表层采用克希荷夫假定, 按板壳理论处理。在文献[1]中用数学弹性力学的方法解决了单层圆柱壳在轴压下的轴对称失稳问题。算例表明, 采用板壳理论与按数学弹性力学方法算得的临界载荷相比, 误差小于0.3%。夹层圆柱壳内、外表层的厚度要比上述算例中的单层壳小得多, 因此对表层壳采用克希荷夫假定, 仅带来千分之一左右的误差, 已足够精确。对于夹心, 本文考虑了失稳前在夹心中的应力, 严格按照数学弹性力学的

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方法处理。考虑了夹心在沿厚度方向的压缩、拉伸和剪切变形,可以求解整体失稳和局部失稳的临界载荷。假定三层都是各向同性材料,泊桑比相同。导得了可求解不同材料不等厚度的表层与软、硬夹心的夹层圆柱壳在轴压下轴对称整体失稳和局部失稳临界载荷的超越方程。用数值计算的方法可算得临界载荷的最小值,与夹心采用横向不可压缩假定的夹层壳理论算得的整体失稳临界载荷作了比较。对于表层壳材料相同厚度相等的情况,通过算例与考虑夹心的横向拉伸、压缩和剪切变形的夹层壳理论算得的整体失稳临界载荷作了比较。

除特别说明外,本文所用的符号与文献[1]相同。

二、基本方程

在失稳前的平衡位置,三层壳的应力和位移分量中, $\sigma_r^0 = \sigma_\theta^0 = \tau_{r\theta}^0 = \tau_{\theta z}^0 = u_\theta^0 = 0$, 其余的量为:
内层壳:

$$\sigma_z^0 = -p_1, \quad u_r^0 = \frac{\nu p_1}{E_1} r, \quad u_z^0 = -\frac{p_1}{E_1} z \quad (2.1)$$

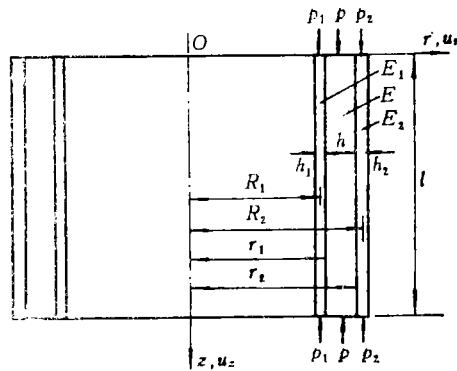


图 1

外层壳:

$$\sigma_z^0 = -p_2, \quad u_r^0 = \frac{\nu p_2}{E_2} r, \quad u_z^0 = -\frac{p_2}{E_2} z \quad (2.2)$$

夹心壳:

$$\sigma_z^0 = -p, \quad u_r^0 = \frac{\nu p}{E} r, \quad u_z^0 = -\frac{p}{E} z \quad (2.3)$$

由于轴向应变相同, 所以

$$\frac{p_1}{E_1} = \frac{p_2}{E_2} = \frac{p}{E} \quad (2.4)$$

设 u_{r1} , u_{z1} , u_{r2} , u_{z2} 分别为内外表层壳中面上从失稳前的平衡位置到失稳后的平衡位置的附加位移分量; σ_r , τ_{rz} 为附加应力分量; $(\sigma_r)_{r=r_1}$, $(\tau_{rz})_{r=r_1}$ 和 $(\sigma_r)_{r=r_2}$, $(\tau_{rz})_{r=r_2}$ 分别为 $r=r_i$ ($i=1,2$) 处的应力; R_1 , R_2 为内外表层壳中面的半径。由壳体微元的平衡条件, 可得下列求解稳定问题的平衡方程。

内层壳:

$$\left. \begin{aligned} & \frac{E_1 h_1}{1-\nu^2} \left(\frac{\nu}{R_1} \frac{\partial u_{r1}}{\partial z} + \frac{\partial^2 u_{z1}}{\partial z^2} \right) + (\tau_{rz})_{r=r_1} = 0 \\ & \frac{E_1 h_1}{1-\nu^2} \left[\left(\frac{h_1^2}{12} \frac{\partial^4 u_{r1}}{\partial z^4} + \frac{u_{r1}}{R_1^2} \right) + \frac{\nu}{R_1} \frac{\partial u_{z1}}{\partial z} \right] + p_1 h_1 \frac{\partial^2 u_{r1}}{\partial z^2} \\ & - (\sigma_r)_{r=r_1} - \frac{h_1}{2} \frac{\partial}{\partial z} (\tau_{rz})_{r=r_1} = 0 \end{aligned} \right\} \quad (2.5)$$

外层壳:

$$\left. \begin{aligned} & \frac{E_2 h_2}{1-\nu^2} \left(\frac{\nu}{R_2} \frac{\partial u_{r2}}{\partial z} + \frac{\partial^2 u_{z2}}{\partial z^2} \right) - (\tau_{rz})_{r=r_2} = 0 \\ & \frac{E_2 h_2}{1-\nu^2} \left[\left(\frac{h_2^2}{12} \frac{\partial^4 u_{r2}}{\partial z^4} + \frac{u_{r2}}{R_2^2} \right) + \frac{\nu}{R_2} \frac{\partial u_{z2}}{\partial z} \right] + p_2 h_2 \frac{\partial^2 u_{r2}}{\partial z^2} \\ & + (\sigma_r)_{r=r_2} - \frac{h_2}{2} \frac{\partial}{\partial z} (\tau_{rz})_{r=r_2} = 0 \end{aligned} \right\} \quad (2.6)$$

对于夹心壳, 直接引用文献[1]中的结果。由文献[1]中(2.51)和(2.52)式, 对于两端简支的夹层圆柱壳的轴对称失稳问题, $n=0$, 可得

$$\left. \begin{aligned} u_r &= [m_1 A_1 I_1(m_1 r) - m_1 B_1 K_1(m_1 r) + m_2 A_2 I_1(m_2 r) - m_2 B_2 K_1(m_2 r)] \sin \frac{m\pi z}{l} \\ u_z &= \frac{m\pi}{l} \left\{ \left(1 - \frac{p}{G}\right) [A_1 I_0(m_1 r) + B_1 K_0(m_1 r)] + [A_2 I_0(m_2 r) \right. \\ & \left. + B_2 K_0(m_2 r)] \right\} \cos \frac{m\pi z}{l} \end{aligned} \right\} \quad (2.7)$$

和

$$\left. \begin{aligned} \sigma_r &= 2G \left\{ \left(1 - \frac{p}{G}\right) \frac{m^2 \pi^2}{l^2} [A_1 I_0(m_1 r) + B_1 K_0(m_1 r)] \right. \\ & \left. + \left(1 - \frac{p}{2G}\right) \frac{m^2 \pi^2}{l^2} [A_2 I_0(m_2 r) + B_2 K_0(m_2 r)] - \frac{1}{r} [m_1 A_1 I_1(m_1 r) \right. \\ & \left. - m_1 B_1 K_1(m_1 r) + m_2 A_2 I_1(m_2 r) - m_2 B_2 K_1(m_2 r)] \right\} \sin \frac{m\pi z}{l} \\ \tau_{rz} &= 2G \frac{m\pi}{l} \left\{ \left(1 - \frac{p}{2G}\right) [m_1 A_1 I_1(m_1 r) - m_1 B_1 K_1(m_1 r)] \right. \\ & \left. + [m_2 A_2 I_1(m_2 r) - m_2 B_2 K_1(m_2 r)] \right\} \cos \frac{m\pi z}{l} \end{aligned} \right\} \quad (2.8)$$

其中

$$m_1 = \frac{m\pi}{l} \left(1 - \frac{p}{G}\right)^{\frac{1}{2}}, \quad m_2 = \frac{m\pi}{l} \left(1 - \frac{p}{\lambda + 2G}\right)^{\frac{1}{2}} \quad (2.9)$$

在轴对称失稳时, 设内外表层中面的位移分别为

$$u_{r1} = C_{11} \sin \frac{m\pi z}{l}, \quad u_{z1} = C_{12} \cos \frac{m\pi z}{l} \quad (2.10)$$

$$u_{r2} = C_{21} \sin \frac{m\pi z}{l}, \quad u_{z2} = C_{22} \cos \frac{m\pi z}{l} \quad (2.11)$$

则两端的简支条件已经满足, 即 $z=0$ 和 l 时,

$$(\mathbf{u}_r)_{z=0} = \left(\frac{\partial^2 \mathbf{u}_r}{\partial z^2} \right)_{z=0} = (\mathbf{u}_r)_{z=l} = \left(\frac{\partial^2 \mathbf{u}_r}{\partial z^2} \right)_{z=l} = 0 \quad (2.12)$$

将(2.10)式代入(2.5)式, 注意到(2.8)式, 可得

$$\begin{aligned} \frac{\nu}{R_1} C_{11} - \frac{m\pi}{l} C_{12} + \frac{1-\nu^2}{E_1 h_1} \cdot 2G \left\{ \left(1 - \frac{p}{2G}\right) [m_1 A_1 I_1(m_1 r_1) - m_1 B_1 K_1(m_1 r_1)] \right. \\ \left. + [m_2 A_2 I_1(m_2 r_1) - m_2 B_2 K_1(m_2 r_1)] \right\} = 0 \end{aligned} \quad (2.13)$$

$$\begin{aligned} \left(\frac{h_1^2}{12} \frac{m^4 \pi^4}{l^4} + \frac{1}{R_1^2} \right) C_{11} - \frac{\nu}{R_1} \frac{m\pi}{l} C_{12} - 2G \frac{1-\nu^2}{E_1 h_1} \left\{ \left(1 - \frac{p}{G}\right) \frac{m^2 \pi^2}{l^2} [A_1 I_0(m_1 r_1) \right. \\ \left. + B_1 K_0(m_1 r_1)] + \left(1 - \frac{p}{2G}\right) \frac{m^2 \pi^2}{l^2} [A_2 I_0(m_2 r_1) + B_2 K_0(m_2 r_1)] \right. \\ \left. - \frac{1}{r_1} [m_1 A_1 I_1(m_1 r_1) - m_1 B_1 K_1(m_1 r_1) + m_2 A_2 I_1(m_2 r_1) - m_2 B_2 K_1(m_2 r_1)] \right. \\ \left. - \frac{h_1}{2} \frac{m^2 \pi^2}{l^2} \left[\left(1 - \frac{p}{2G}\right) m_1 A_1 I_1(m_1 r_1) - \left(1 - \frac{p}{2G}\right) m_1 B_1 K_1(m_1 r_1) \right. \right. \\ \left. \left. + m_2 A_2 I_1(m_2 r_1) - m_2 B_2 K_1(m_2 r_1) \right] \right\} - \frac{p_1}{E_1} (1-\nu^2) \frac{m^2 \pi^2}{l^2} C_{11} = 0 \end{aligned} \quad (2.14)$$

在(2.13)和(2.14)式中消去 C_{12} 可得

$$\begin{aligned} \left(D_1 \frac{m^4 \pi^4}{l^4} - p_1 h_1 \frac{m^2 \pi^2}{l^2} + \frac{E_1 h_1}{R_1^2} \right) C_{11} - 2G \left\{ \left(1 - \frac{p}{G}\right) \frac{m^2 \pi^2}{l^2} [A_1 I_0(m_1 r_1) \right. \\ \left. + B_1 K_0(m_1 r_1)] + \left(1 - \frac{p}{2G}\right) \frac{m^2 \pi^2}{l^2} [A_2 I_0(m_2 r_1) + B_2 K_0(m_2 r_1)] \right. \\ \left. - \frac{1}{r_1} [m_1 A_1 I_1(m_1 r_1) - m_1 B_1 K_1(m_1 r_1) + m_2 A_2 I_1(m_2 r_1) - m_2 B_2 K_1(m_2 r_1)] \right. \\ \left. + \left(\frac{\nu}{R_1} - \frac{h_1}{2} \frac{m^2 \pi^2}{l^2} \right) \left[\left(1 - \frac{p}{2G}\right) m_1 A_1 I_1(m_1 r_1) - \left(1 - \frac{p}{2G}\right) m_1 B_1 K_1(m_1 r_1) \right. \right. \\ \left. \left. + m_2 A_2 I_1(m_2 r_1) - m_2 B_2 K_1(m_2 r_1) \right] \right\} = 0 \end{aligned} \quad (2.15)$$

将(2.11)式代入(2.6)式, 注意到(2.8)式, 可得

$$\begin{aligned} \frac{\nu}{R_2} C_{21} - \frac{m\pi}{l} C_{22} - \frac{1-\nu^2}{E_2 h_2} \cdot 2G \left\{ \left(1 - \frac{p}{2G}\right) [m_1 A_1 I_1(m_1 r_2) - m_1 B_1 K_1(m_1 r_2)] \right. \\ \left. + [m_2 A_2 I_1(m_2 r_2) - m_2 B_2 K_1(m_2 r_2)] \right\} = 0 \end{aligned} \quad (2.16)$$

$$\begin{aligned} \left(\frac{h_2^2}{12} \frac{m^4 \pi^4}{l^4} + \frac{1}{R_2^2} \right) C_{21} - \frac{\nu}{R_2} \frac{m\pi}{l} C_{22} + 2G \frac{1-\nu^2}{E_2 h_2} \left\{ \left(1 - \frac{p}{G}\right) \frac{m^2 \pi^2}{l^2} [A_1 I_0(m_1 r_2) \right. \\ \left. + B_1 K_0(m_1 r_2)] + \left(1 - \frac{p}{2G}\right) \frac{m^2 \pi^2}{l^2} [A_2 I_0(m_2 r_2) + B_2 K_0(m_2 r_2)] \right. \\ \left. - \frac{1}{r_2} [m_1 A_1 I_1(m_1 r_2) - m_1 B_1 K_1(m_1 r_2) + m_2 A_2 I_1(m_2 r_2) - m_2 B_2 K_1(m_2 r_2)] \right. \\ \left. + \frac{h_2}{2} \frac{m^2 \pi^2}{l^2} \left[\left(1 - \frac{p}{2G}\right) m_1 A_1 I_1(m_1 r_2) - \left(1 - \frac{p}{2G}\right) m_1 B_1 K_1(m_1 r_2) \right. \right. \end{aligned}$$

$$+m_2 A_2 I_1(m_2 r_2) - m_2 B_2 K_1(m_2 r_2)] \left. \right\} - \frac{p_2}{E_2} (1-\nu^2) \frac{m^2 \pi^2}{l^2} C_{21} = 0 \quad (2.17)$$

在(2.16)和(2.17)式中消去 C_{22} 可得

$$\begin{aligned} & \left(D_2 \frac{m^4 \pi^4}{l^4} - p_2 h_2 \frac{m^2 \pi^2}{l^2} + \frac{E_2 h_2}{R_2^2} \right) C_{21} + 2G \left\{ \left(1 - \frac{p}{G} \right) \frac{m^2 \pi^2}{l^2} [A_1 I_0(m_1 r_2) \right. \\ & \quad \left. + B_1 K_0(m_1 r_2)] + \left(1 - \frac{p}{2G} \right) \frac{m^2 \pi^2}{l^2} [A_2 I_0(m_2 r_2) + B_2 K_0(m_2 r_2)] \right. \\ & \quad \left. - \frac{1}{r_2} [m_1 A_1 I_1(m_1 r_2) - m_1 B_1 K_1(m_1 r_2) + m_2 A_2 I_1(m_2 r_2) - m_2 B_2 K_1(m_2 r_2)] \right. \\ & \quad \left. + \left(\frac{\nu}{R_2} + \frac{h_2 m^2 \pi^2}{2 l^2} \right) \left[\left(1 - \frac{p}{2G} \right) m_1 A_1 I_1(m_1 r_2) - \left(1 - \frac{p}{2G} \right) m_1 A_1 K_1(m_1 r_2) \right. \right. \\ & \quad \left. \left. + m_2 A_2 I_1(m_2 r_2) - m_2 B_2 K_1(m_2 r_2) \right] \right\} = 0 \end{aligned} \quad (2.18)$$

在(2.15)和(2.18)式中

$$D_1 = \frac{E_1 h_1^3}{12(1-\nu^2)}, \quad D_2 = \frac{E_2 h_2^3}{12(1-\nu^2)}, \quad R_1 = r_1 - \frac{h_1}{2}, \quad R_2 = r_2 + \frac{h_2}{2} \quad (2.19)$$

内外层壳在采用克希荷夫假定时, 可得

$$(u_r)_{r=r_1} = u_{r1}, \quad (u_z)_{r=r_1} = u_{z1} - \frac{h_1}{2} \frac{\partial u_{r1}}{\partial z} \quad (2.20)$$

$$(u_r)_{r=r_2} = u_{r2}, \quad (u_z)_{r=r_2} = u_{z2} + \frac{h_2}{2} \frac{\partial u_{r2}}{\partial z} \quad (2.21)$$

在夹心壳和内外层壳的联接处即 $r=r_1, r_2$ 处, 有相同的位移, 由(2.7)、(2.10)、(2.20)和(2.7)、(2.11)和(2.21)式分别可得

$$C_{11} = m_1 A_1 I_1(m_1 r_1) - m_1 B_1 K_1(m_1 r_1) + m_2 A_2 I_1(m_2 r_1) - m_2 B_2 K_1(m_2 r_1) \quad (2.22)$$

$$\begin{aligned} C_{12} = & \frac{h_1}{2} \frac{m\pi}{l} C_{11} + \frac{m\pi}{l} \left\{ \left(1 - \frac{p}{G} \right) [A_1 I_0(m_1 r_1) + B_1 K_0(m_1 r_1)] \right. \\ & \left. + [A_2 I_0(m_2 r_1) + B_2 K_0(m_2 r_1)] \right\} \end{aligned} \quad (2.23)$$

$$C_{21} = m_1 A_1 I_1(m_1 r_2) - m_1 B_1 K_1(m_1 r_2) + m_2 A_2 I_1(m_2 r_1) - m_2 B_2 K_1(m_2 r_2) \quad (2.24)$$

$$\begin{aligned} C_{22} = & -\frac{h_2}{2} \frac{m\pi}{l} C_{21} + \frac{m\pi}{l} \left\{ \left(1 - \frac{p}{G} \right) [A_1 I_0(m_1 r_2) + B_1 K_0(m_1 r_2)] \right. \\ & \left. + [A_2 I_0(m_2 r_2) + B_2 K_0(m_2 r_2)] \right\} \end{aligned} \quad (2.25)$$

将(2.22)代入(2.15)式得

$$\begin{aligned} & \left(D_1 \frac{m^4 \pi^4}{l^4} - p_1 h_1 \frac{m^2 \pi^2}{l^2} + \frac{E_1 h_1}{R_1^2} \right) [m_1 A_1 I_1(m_1 r_1) - m_1 B_1 K_1(m_1 r_1) \\ & \quad + m_2 A_2 I_1(m_2 r_1) - m_2 B_2 K_1(m_2 r_1)] - 2G \left\{ \left(1 - \frac{p}{G} \right) \frac{m^2 \pi^2}{l^2} [A_1 I_0(m_1 r_1) \right. \\ & \quad \left. + B_1 K_0(m_1 r_1)] + \left(1 - \frac{p}{2G} \right) \frac{m^2 \pi^2}{l^2} [A_2 I_0(m_2 r_1) + B_2 K_0(m_2 r_1)] \right. \\ & \quad \left. - \frac{1}{r_1} [m_1 A_1 I_1(m_1 r_1) - m_1 B_1 K_1(m_1 r_1) + m_2 A_2 I_1(m_2 r_1) - m_2 B_2 K_1(m_2 r_1)] \right\} \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{\nu}{R_1} - \frac{h_1}{2} \frac{m^2 \pi^2}{l^2} \right) \left[\left(1 - \frac{p}{2G} \right) m_1 A_1 I_1(m_1 r_1) - \left(1 - \frac{p}{2G} \right) m_1 B_1 K_1(m_1 r_1) \right. \\
& \left. + m_2 A_2 I_1(m_2 r_1) - m_2 B_2 K_1(m_2 r_1) \right] = 0
\end{aligned} \tag{2.26}$$

将(2.24)代入(2.18)式得

$$\begin{aligned}
& \left(D_2 \frac{m^4 \pi^4}{l^4} - p_2 h_2 \frac{m^2 \pi^2}{l^2} + \frac{E_2 h_2}{R_2^2} \right) [m_1 A_1 I_1(m_1 r_2) - m_1 B_1 K_1(m_1 r_2) \\
& + m_2 A_2 I_1(m_2 r_2) - m_2 B_2 K_1(m_2 r_2)] + 2G \left\{ \left(1 - \frac{p}{G} \right) \frac{m^2 \pi^2}{l^2} [A_1 I_0(m_1 r_2) \right. \\
& + B_1 K_0(m_1 r_2)] + \left(1 - \frac{p}{2G} \right) \frac{m^2 \pi^2}{l^2} [A_2 I_0(m_2 r_2) + B_2 K_0(m_2 r_2)] \right. \\
& - \frac{1}{r_2} [m_1 A_1 I_1(m_1 r_2) - m_1 B_1 K_1(m_1 r_2) + m_2 A_2 I_1(m_2 r_2) - m_2 B_2 K_1(m_2 r_2)] \\
& \left. + \left(\frac{\nu}{R_2} + \frac{h_2}{2} \frac{m^2 \pi^2}{l^2} \right) \left[\left(1 - \frac{p}{2G} \right) m_1 A_1 I_1(m_1 r_2) - \left(1 - \frac{p}{2G} \right) m_1 B_1 K_1(m_1 r_2) \right. \right. \\
& \left. \left. + m_2 A_2 I_1(m_2 r_2) - m_2 B_2 K_1(m_2 r_2) \right] \right\} = 0
\end{aligned} \tag{2.27}$$

由(2.13)、(2.22)和(2.23)式消去 C_{11} 和 C_{12} 后可得

$$\begin{aligned}
& \left(\frac{\nu}{R_1} - \frac{m^2 \pi^2}{l^2} \frac{h_1}{2} \right) [m_1 A_1 I_1(m_1 r_1) - m_1 B_1 K_1(m_1 r_1) + m_2 A_2 I_1(m_2 r_1) \\
& - m_2 B_2 K_1(m_2 r_1)] - \frac{m^2 \pi^2}{l^2} \left\{ \left(1 - \frac{p}{G} \right) [A_1 I_0(m_1 r_1) + B_1 K_0(m_1 r_1)] \right. \\
& \left. + [A_2 I_0(m_2 r_1) + B_2 K_0(m_2 r_1)] \right\} + \frac{1-\nu^2}{E_1 h_1} 2G \left\{ \left(1 - \frac{p}{2G} \right) [m_1 A_1 I_1(m_1 r_1) \right. \\
& \left. - m_1 B_1 K_1(m_1 r_1)] + [m_2 A_2 I_1(m_2 r_1) - m_2 B_2 K_1(m_2 r_1)] \right\} = 0
\end{aligned} \tag{2.28}$$

由(2.16)、(2.24)和(2.25)式消去 C_{21} 和 C_{22} 后可得

$$\begin{aligned}
& \left(\frac{\nu}{R_2} + \frac{m^2 \pi^2}{l^2} \frac{h_2}{2} \right) [m_1 A_1 I_1(m_1 r_2) - m_1 B_1 K_1(m_1 r_2) + m_2 A_2 I_1(m_2 r_2) \\
& - m_2 B_2 K_1(m_2 r_2)] - \frac{m^2 \pi^2}{l^2} \left\{ \left(1 - \frac{p}{G} \right) [A_1 I_0(m_1 r_2) + B_1 K_0(m_1 r_2)] \right. \\
& \left. + [A_2 I_0(m_2 r_2) + B_2 K_0(m_2 r_2)] \right\} - \frac{1-\nu^2}{E_2 h_2} 2G \left\{ \left(1 - \frac{p}{2G} \right) [m_1 A_1 I_1(m_1 r_2) \right. \\
& \left. - m_1 B_1 K_1(m_1 r_2)] + [m_2 A_2 I_1(m_2 r_2) - m_2 B_2 K_1(m_2 r_2)] \right\} = 0
\end{aligned} \tag{2.29}$$

将(2.28)、(2.29)、(2.26)和(2.27)整理后可得

$$\left. \begin{aligned}
F_{11} A_1 + F_{12} B_1 + F_{13} A_2 + F_{14} B_2 &= 0 \\
F_{21} A_1 + F_{22} B_1 + F_{23} A_2 + F_{24} B_2 &= 0 \\
F_{31} A_1 + F_{32} B_1 + F_{33} A_2 + F_{34} B_2 &= 0 \\
F_{41} A_1 + F_{42} B_1 + F_{43} A_2 + F_{44} B_2 &= 0
\end{aligned} \right\} \tag{2.30}$$

在(2.30)式中

$$\begin{aligned}
 F_{11} &= \left[\left(\frac{\nu h_1}{R_1} - \frac{m^2 \pi^2}{l^2} \frac{h_1^2}{2} \right) + (1-\nu^2) \frac{2G-p}{E_1} \right] I_1(m_1 r_1) \\
 &\quad - \frac{m^2 \pi^2 h_1}{l^2 m_1} \left(1 - \frac{p}{G} \right) I_0(m_1 r_1) \\
 F_{12} &= - \left[\left(\frac{\nu h_1}{R_1} - \frac{m^2 \pi^2}{l^2} \frac{h_1^2}{2} \right) + (1-\nu^2) \frac{2G-p}{E_1} \right] K_1(m_1 r_1) \\
 &\quad - \frac{m^2 \pi^2 h_1}{l^2 m_1} \left(1 - \frac{p}{G} \right) K_0(m_1 r_1) \\
 F_{13} &= \left[\left(\frac{\nu h_1}{R_1} - \frac{m^2 \pi^2}{l^2} \frac{h_1^2}{2} \right) + (1-\nu^2) \frac{2G}{E_1} \right] I_1(m_2 r_1) - \frac{m^2 \pi^2 h_1}{l^2 m_2} I_0(m_2 r_1) \\
 F_{14} &= - \left[\left(\frac{\nu h_1}{R_1} - \frac{m^2 \pi^2}{l^2} \frac{h_1^2}{2} \right) + (1-\nu^2) \frac{2G}{E_1} \right] K_1(m_2 r_1) - \frac{m^2 \pi^2 h_1}{l^2 m_2} K_0(m_2 r_1) \\
 F_{21} &= \left[\left(\frac{\nu h_2}{R_2} + \frac{m^2 \pi^2}{l^2} \frac{h_2^2}{2} \right) - (1-\nu^2) \frac{2G-p}{E_2} \right] I_1(m_1 r_2) \\
 &\quad - \frac{m^2 \pi^2 h_2}{l^2 m_1} \left(1 - \frac{p}{G} \right) I_0(m_1 r_2) \\
 F_{22} &= - \left[\left(\frac{\nu h_2}{R_2} + \frac{m^2 \pi^2}{l^2} \frac{h_2^2}{2} \right) - (1-\nu^2) \frac{2G-p}{E_2} \right] K_1(m_1 r_2) \\
 &\quad - \frac{m^2 \pi^2 h_2}{l^2 m_1} \left(1 - \frac{p}{G} \right) K_0(m_1 r_2) \\
 F_{23} &= \left[\left(\frac{\nu h_2}{R_2} + \frac{m^2 \pi^2}{l^2} \frac{h_2^2}{2} \right) - (1-\nu^2) \frac{2G}{E_2} \right] I_1(m_2 r_2) - \frac{m^2 \pi^2 h_2}{l^2 m_2} I_0(m_2 r_2) \\
 F_{24} &= - \left[\left(\frac{\nu h_2}{R_2} + \frac{m^2 \pi^2}{l^2} \frac{h_2^2}{2} \right) - (1-\nu^2) \frac{2G}{E_2} \right] K_1(m_2 r_2) - \frac{m^2 \pi^2 h_2}{l^2 m_2} K_0(m_2 r_2) \\
 F_{31} &= \left[\left(D_1 \frac{m^4 \pi^4}{l^4} - p_1 h_1 \frac{m^2 \pi^2}{l^2} + \frac{E_1 h_1}{R_1^2} \right) \frac{h_1}{2G} + \frac{h_1}{r_1} - \left(\frac{\nu h_1}{R_1} - \frac{m^2 \pi^2}{l^2} \frac{h_1^2}{2} \right) \right] I_1(m_1 r_1) \\
 &\quad - \left(1 - \frac{p}{G} \right) \frac{m^2 \pi^2 h_1}{l^2 m_1} I_0(m_1 r_1) \\
 F_{32} &= - \left[\left(D_1 \frac{m^4 \pi^4}{l^4} - p_1 h_1 \frac{m^2 \pi^2}{l^2} + \frac{E_1 h_1}{R_1^2} \right) \frac{h_1}{2G} + \frac{h_1}{r_1} \right. \\
 &\quad \left. - \left(\frac{\nu h_1}{R_1} - \frac{m^2 \pi^2}{l^2} \frac{h_1^2}{2} \right) \right] K_1(m_1 r_1) - \left(1 - \frac{p}{G} \right) \frac{m^2 \pi^2 h_1}{l^2 m_1} K_0(m_1 r_1) \\
 F_{33} &= \left[\left(D_1 \frac{m^4 \pi^4}{l^4} - p_1 h_1 \frac{m^2 \pi^2}{l^2} + \frac{E_1 h_1}{R_1^2} \right) \frac{h_1}{2G} + \frac{h_1}{r_1} - \left(\frac{\nu h_1}{R_1} - \frac{m^2 \pi^2}{l^2} \frac{h_1^2}{2} \right) \right] I_1(m_2 r_1) \\
 &\quad - \left(1 - \frac{p}{2G} \right) \frac{m^2 \pi^2 h_1}{l^2 m_2} I_0(m_2 r_1) \\
 F_{34} &= - \left[\left(D_1 \frac{m^4 \pi^4}{l^4} - p_1 h_1 \frac{m^2 \pi^2}{l^2} + \frac{E_1 h_1}{R_1^2} \right) \frac{h_1}{2G} + \frac{h_1}{r_1} \right.
 \end{aligned} \tag{2.31}$$

$$\begin{aligned}
& -\left(\frac{\nu h_1}{R_1} - \frac{m^2 \pi^2}{l^2} \frac{h_1^2}{2}\right) \Big] K_1(m_2 r_1) - \left(1 - \frac{p}{2G}\right) \frac{m^2 \pi^2 h_1}{l^2 m_2} K_0(m_2 r_1) \\
F_{41} = & \left[\left(D_2 \frac{m^4 \pi^4}{l^4} - p_2 h_2 \frac{m^2 \pi^2}{l^2} + \frac{E_2 h_2}{R_2^2} \right) \frac{h_2}{2G} - \frac{h_2}{r_2} + \left(\frac{\nu h_2}{R_2} + \frac{m^2 \pi^2}{l^2} \frac{h_2^2}{2} \right) \right] I_1(m_1 r_2) \\
& + \left(1 - \frac{p}{G}\right) \frac{m^2 \pi^2 h_2}{l^2 m_1} I_0(m_1 r_2) \\
F_{42} = & - \left[\left(D_2 \frac{m^4 \pi^4}{l^4} - p_2 h_2 \frac{m^2 \pi^2}{l^2} + \frac{E_2 h_2}{R_2^2} \right) \frac{h_2}{2G} - \frac{h_2}{r_2} + \left(\frac{\nu h_2}{R_2} \right. \right. \\
& \left. \left. + \frac{m^2 \pi^2}{l^2} \frac{h_2^2}{2} \right) \right] K_1(m_1 r_2) + \left(1 - \frac{p}{G}\right) \frac{m^2 \pi^2 h_2}{l^2 m_1} K_0(m_1 r_2) \\
F_{43} = & \left[\left(D_2 \frac{m^4 \pi^4}{l^4} - p_2 h_2 \frac{m^2 \pi^2}{l^2} + \frac{E_2 h_2}{R_2^2} \right) \frac{h_2}{2G} - \frac{h_2}{r_2} + \left(\frac{\nu h_2}{R_2} + \frac{m^2 \pi^2}{l^2} \frac{h_2^2}{2} \right) \right] I_1(m_2 r_2) \\
& + \left(1 - \frac{p}{2G}\right) \frac{m^2 \pi^2 h_2}{l^2 m_2} I_0(m_2 r_2) \\
F_{44} = & - \left[\left(D_2 \frac{m^4 \pi^4}{l^4} - p_2 h_2 \frac{m^2 \pi^2}{l^2} + \frac{E_2 h_2}{R_2^2} \right) \frac{h_2}{2G} - \frac{h_2}{r_2} + \left(\frac{\nu h_2}{R_2} + \frac{m^2 \pi^2}{l^2} \frac{h_2^2}{2} \right) \right] K_1(m_2 r_2) \\
& + \left(1 - \frac{p}{2G}\right) \frac{m^2 \pi^2 h_2}{l^2 m_2} K_0(m_2 r_2)
\end{aligned}$$

在(2.30)式中, 待定常数具有非零解的条件为

$$\begin{vmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{vmatrix} = 0 \quad (2.32)$$

由此式可通过试算求解临界应力 p_{1cr} , p_{2cr} 和 p_{cr} .

由于行列式中的每一个元素都相当复杂, 待求的临界应力 p 和失稳时在轴向形成的半波数 m 都包含在贝塞尔函数中, 因此(2.32)式是一个超越方程. 要从(2.32)式求得临界应力的显式非常困难. 为此需要用计算机进行数值计算, 求解临界应力的最小值. 用夹层壳理论可比较方便地求得夹层圆柱壳在均匀轴压下轴对称整体失稳时的临界载荷与相应的半波数. 由此近似解出发, 取一系列的试算值, 代入(2.32)式, 可求得整体失稳时临界应力的最小值. 当夹心的弹性模量、剪切模量较大而厚度不大时, 整体失稳的临界载荷总是小于局部失稳的临界载荷, 因而不必计算局部失稳的临界载荷. 当夹心的弹性模量、剪切模量较小而厚度较大时, 局部失稳的临界载荷可小于整体失稳的临界载荷. 由(2.32)式也可通过试算求得轴对称局部失稳时的临界载荷, 这在下一节中讨论. 有了临界应力的精确解和相应的半波数 m , 要判别究竟是整体失稳还是局部失稳的临界应力, 则可由(2.30)式算得 B_1/A_1 , A_2/A_1 , B_2/A_1 , 由(2.7)式, 当 $r=r_0=(r_1+r_2)/2$ 时, u_r 的最大值为

$$\bar{u}_r = m_1 A_1 I_1(m_1 r_0) - m_1 B_1 K_1(m_1 r_0) + m_2 A_2 I_1(m_2 r_0) - m_2 B_2 K_1(m_2 r_0) \quad (2.33)$$

再由(2.10)和(2.11)式, 当 $r=r_1$ 和 r_2 时, u_r 的最大值为 C_{11} 和 C_{21} . 如 \bar{u}_r 和 C_{11} (或 C_{21}) 属同一量级, 则求得的是整体失稳的临界应力; 如 \bar{u}_r 远小于 C_{11} (或 C_{21}), 这里指绝对值, 则求得的是局部失稳的临界应力.

三、夹层壳理论的解

(1) 对于两端简支不对称构造 (两表层材料不同, 厚度不等, 即 $E_1 \neq E_2$, $h_1 \neq h_2$) 的横观各向同性夹心 (厚度为 h) 的夹层圆柱壳在轴压下轴对称失稳时, 如夹心沿厚度方向不可压缩, 由文献[2](3.23)式可得整体失稳临界载荷 N_{cr} 的算式.

$$N_{cr} = C_4 M + \frac{C_1(1-\nu^2)}{R^2 M} + \frac{C_3 M \left(1 + \frac{D_3}{C_5} \frac{h_1 + h_2}{2h} M \right)}{1 + \frac{M}{C_5} \left[\frac{h}{2} (B_1 \beta_1 + B_2 \beta_2) + \frac{D_3 H}{h} \right]} \quad (3.1)$$

其中

$$\left. \begin{aligned} C_1 &= B_1 + B_2 + B_3, \quad C_3 = \frac{h+h_1}{2} B_1 \beta_1 + \frac{h+h_2}{2} B_2 \beta_2 + D_3 \frac{H}{h} \\ C_4 &= D_1 + D_2 - \frac{h_1+h_2}{2h} D_3, \quad C_5 = GH, \quad H = \frac{1}{2}(2h+h_1+h_2) \\ \beta_1 &= \frac{B_2 H + 0.5 B_3 (h+h_1)}{B_1 + B_2 + B_3}, \quad \beta_2 = \frac{B_1 H + 0.5 B_3 (h+h_2)}{B_1 + B_2 + B_3}, \quad M = \frac{m^2 \pi^2}{l^2} \\ B_1 &= \frac{E_1 h_1}{1-\nu^2}, \quad B_2 = \frac{E_2 h_2}{1-\nu^2}, \quad B_3 = \frac{Eh}{1-\nu^2} \\ D_1 &= \frac{E_1 h_1^3}{12(1-\nu^2)}, \quad D_2 = \frac{E_2 h_2^3}{12(1-\nu^2)}, \quad D_3 = \frac{Eh^3}{12(1-\nu^2)} \end{aligned} \right\} \quad (3.2)$$

由 $m=1, 2, 3, \dots$ 进行试算, 可求得 $(N_{cr})_{\min}$.

当夹心沿厚度方向的剪切模量很大, 可采用克希荷夫假定, $C_5 \rightarrow \infty$, 由 (3.1) 式可得

$$N_{cr} = (C_3 + C_4) M + \frac{C_1(1-\nu^2)}{R^2 M} \quad (3.3)$$

由 $\frac{\partial N_{cr}}{\partial M} = 0$, 可得

$$m = \frac{l}{\pi} \left[\frac{C_1(1-\nu^2)}{R^2(C_3 + C_4)} \right]^{\frac{1}{2}}, \quad (N_{cr})_{\min} = \frac{2}{R} [C_1(1-\nu^2)(C_3 + C_4)]^{\frac{1}{2}} \quad (3.4)$$

当夹心沿厚度方向的剪切模量较小时, 则 (3.1) 式中, $\frac{M}{C_5} \frac{h}{2} (B_1 \beta_1 + B_2 \beta_2) \gg 1$, 可近似地求得

$$\left. \begin{aligned} m &= \frac{l}{\pi} \left[\frac{C_1}{C_4 R^2} \left(1 - \nu^2 - \frac{G^2 R^2 H^2}{B_1 B_2 h^2} \right) \right]^{\frac{1}{2}} \\ (N_{cr})_{\min} &= \frac{2}{R} \left[C_1 C_4 \left(1 - \nu^2 - \frac{G^2 R^2 H^2}{B_1 B_2 h^2} \right) \right]^{\frac{1}{2}} + G \frac{H^2}{h} \end{aligned} \right\} \quad (3.5)$$

由 (3.4) 和 (3.5) 式, 可确定 m 在上限、下限间一系列的整数值和 $(N_{cr})_{\min}$ 所在的范围. 依次将 m 代入 (3.1) 式, 可较快地算得整体失稳临界载荷的最小值和对应的 m_0 值. 再以 $m_0 \pm 3$ 和临界应力的 0.9~1.1 倍作为试算范围, 代入 (2.32) 式, 可求得临界应力的严格解.

(2) 对于两端简支对称构造($E_1=E_2$, $h_1=h_2$)的夹层圆柱壳在轴压下轴对称失稳时, 设夹心的 $E_{11} \approx E_{22} \approx G_{12} \approx 0$ 。考虑夹心沿壳厚方向的拉伸、压缩和剪切变形, 由文献[3]化简后可得

$$\left. \begin{aligned} \frac{\partial^2 u_\alpha}{\partial z^2} + \frac{\nu}{R} \frac{\partial w_\alpha}{\partial z} &= 0 \\ B \left(\frac{\nu}{R} \frac{\partial u_\alpha}{\partial z} + \frac{w_\alpha}{R^2} \right) + D \frac{\partial^4 w_\alpha}{\partial z^4} - \frac{B}{2} (h+h_1) \left(\frac{\partial^3 u_\beta}{\partial z^3} + \frac{\nu}{R} \frac{\partial^2 w_\beta}{\partial z^2} \right) - T_{1\alpha}^0 \frac{\partial^2 w_\alpha}{\partial z^2} &= 0 \\ B \left(\frac{\nu}{R} \frac{\partial u_\beta}{\partial z} + \frac{w_\beta}{R^2} \right) + D \frac{\partial^4 w_\beta}{\partial z^4} + 2 \frac{E_{33}}{h} w_\beta - T_{1\alpha}^0 \frac{\partial^2 w_\beta}{\partial z^2} &= 0 \\ \frac{Bh}{2G_{13}} \left(\frac{\partial^2 u_\beta}{\partial z^2} + \frac{\nu}{R} \frac{\partial w_\beta}{\partial z} \right) - \frac{Bh^3}{24E_{33}} \left(\frac{\partial^4 u_\beta}{\partial z^4} + \frac{\nu}{R} \frac{\partial^3 w_\beta}{\partial z^3} \right) = u_\beta + \frac{h+h_1}{2} \frac{\partial w_\alpha}{\partial z} \end{aligned} \right\} \quad (3.6)$$

在(3.6)式中, E_{33} 和 G_{13} 为夹心沿厚度方向的弹性模量和剪切模量。此外

$$\left. \begin{aligned} u_\alpha &= \frac{1}{2} (u_{z2} + u_{z1}), \quad u_\beta = \frac{1}{2} (u_{z2} - u_{z1}) \\ w_\alpha &= \frac{1}{2} (u_{r2} + u_{r1}), \quad w_\beta = \frac{1}{2} (u_{r2} - u_{r1}) \\ T_{1\alpha}^0 &= -\frac{N_x^0}{2}, \quad B = \frac{E h_1}{1-\nu^2}, \quad D = \frac{E h_1^3}{12(1-\nu^2)}, \quad R = \frac{R_1 + R_2}{2} \end{aligned} \right\} \quad (3.7)$$

由(2.10)和(2.11)式得

$$\left. \begin{aligned} u_\alpha &= C_1^0 \cos \frac{m\pi z}{l}, \quad u_\beta = C_2^0 \cos \frac{m\pi z}{l} \\ w_\alpha &= C_3^0 \sin \frac{m\pi z}{l}, \quad w_\beta = C_4^0 \sin \frac{m\pi z}{l} \end{aligned} \right\} \quad (3.8)$$

其中

$$\left. \begin{aligned} C_1^0 &= \frac{1}{2} (C_{22} + C_{12}), \quad C_2^0 = \frac{1}{2} (C_{22} - C_{12}) \\ C_3^0 &= \frac{1}{2} (C_{21} + C_{11}), \quad C_4^0 = \frac{1}{2} (C_{21} - C_{11}) \end{aligned} \right\} \quad (3.9)$$

将(3.7), (3.8)代入(3.6)式, 消去 C_i^0 后可得。

$$\left. \begin{aligned} F_1 C_2^0 + \left(F_2 + \frac{N_x^0}{2B} \right) C_3^0 + F_3 C_4^0 &= 0 \\ F_4 C_2^0 + \left(F_5 + \frac{N_x^0}{2B} \right) C_4^0 = 0, \quad F_6 C_3^0 + F_7 C_3^0 + F_8 C_4^0 &= 0 \end{aligned} \right\} \quad (3.10)$$

其中

$$\left. \begin{aligned} F_1 &= \frac{h+h_1}{2} \frac{m\pi}{l}, \quad F_2 = -\left(\frac{1-\nu}{R^2} \frac{l^2}{m^2\pi^2} + \frac{D}{B} \frac{m^2\pi^2}{l^2} \right), \quad F_3 = -\frac{h+h_1}{2} \frac{\nu}{R} \\ F_4 &= \frac{\nu l}{m\pi R}, \quad F_5 = -\left(\frac{l^2}{m^2\pi^2 R^2} + \frac{D}{B} \frac{m^2\pi^2}{l^2} + \frac{2E_{33}}{Bh} \frac{l^2}{m^2\pi^2} \right) \\ F_6 &= 1 + \frac{2G_{13}}{Bh} \frac{l^2}{m^2\pi^2} + \frac{G_{13}h^2}{12E_{33}} \frac{m^2\pi^2}{l^2}, \quad F_7 = (h+h_1) \frac{G_{13}}{Bh} \frac{l}{m\pi} \\ F_8 &= -\left(1 + \frac{G_{13}h^2}{12E_{33}} \frac{m^2\pi^2}{l^2} \right) \frac{\nu}{R} \frac{l}{m\pi} \end{aligned} \right\} \quad (3.11)$$

在(3.10)式中, C_2^0 , C_3^0 和 C_4^0 具有非零解的条件为

$$\begin{vmatrix} F_1 & F_2 + \frac{N_x^0}{2B} & F_3 \\ F_4 & 0 & F_5 + \frac{N_x^0}{2B} \\ F_6 & F_7 & F_8 \end{vmatrix} = 0 \quad (3.12)$$

化简后得

$$F_6 \left(\frac{N_x^0}{2B} \right)^2 + F_9 \left(\frac{N_x^0}{2B} \right) + F_{10} = 0 \quad (3.13)$$

其中

$$F_9 = F_2 F_6 + F_5 F_6 - F_1 F_7 - F_4 F_8 \quad (3.14a)$$

$$F_{10} = F_2 F_5 F_6 + F_3 F_4 F_7 - F_1 F_5 F_7 - F_2 F_4 F_8 \quad (3.14b)$$

则

$$\frac{N_x^0}{2B} = \frac{1}{2F_6} (-F_9 \pm \sqrt{F_9^2 - 4F_6 F_{10}}) \quad (3.15)$$

由(3.11)和(3.14a), 可以发现 F_9 为负值, 因此为了求得 N_x^0 的最小值, (3.15)式中的根号前只能取减号. 取 $m=1, 2, 3, \dots$ 进行试算, 可求得临界载荷的最小值. 为了较快地求得计算结果, 讨论下列情况.

当 $E_{33} \rightarrow \infty$, 则 $F_6 \rightarrow \infty$, 由(3.13)、(3.14)式可得

$$N_x^0 = 2B \left(\frac{F_1 F_7}{F_6} - F_2 \right) = \frac{1-\nu^2}{R^2} \frac{2Bl^2}{m^2 \pi^2} + 2D \frac{m^2 \pi^2}{l^2} + \frac{2B \left(\frac{h+h_1}{2} \right)^2 \frac{m^2 \pi^2}{l^2}}{1 + \frac{Bh}{2G_{13}} \frac{m^2 \pi^2}{l^2}} \quad (3.16)$$

与(3.1)式当 $B_1=B_2=B$, $D_1=D_2=D$, $B_3=D_3=0$ 时所得结果实质上完全相同. 当 $G_{13} \rightarrow \infty$,

仍记 $M = \frac{m^2 \pi^2}{l^2}$, 得

$$N_x^0 = 2 \left[D + B \left(\frac{h+h_1}{2} \right)^2 \right] M + \frac{1-\nu^2}{R^2} \frac{2B}{M} \quad (3.17)$$

由 $\frac{\partial N_x^0}{\partial M} = 0$ 可得

$$m = \frac{l}{\pi} \left\{ \frac{(1-\nu^2)B}{R^2 \left[D + B \left(\frac{h+h_1}{2} \right)^2 \right]} \right\}^{\frac{1}{2}} \quad (3.18)$$

对于 G_{13} 较小的情况, 由(3.5)式的第一式可得

$$m = \frac{l}{\pi} \left[\frac{B(1-\nu^2)}{DR^2} - \frac{G_{13}^2 (h+h_1)^2}{DBh^2} \right]^{\frac{1}{2}} \quad (3.19)$$

由(3.18)和(3.19)式可以确定 m 的上、下限的整数值. 依次将 m 代入(3.16)式, 可较快地算得 $E_{33} \rightarrow \infty$ 时夹层圆柱壳轴对称整体失稳时临界载荷的最小值和相应的 m_0 值. 再以 $m_0 \pm 3$ 和临界应力的0.9~1.1倍作为试算范围, 代入(3.15)式, 可求得临界应力的严格解.

当夹心很弱, $E_{33} \approx G_{13} \approx 0$ 时, 夹层圆柱壳将退化为二个独立的圆柱壳, 失稳时

$$m = \frac{l}{\pi} \left[\frac{B(1-\nu^2)}{DR^2} \right]^{\frac{1}{4}}, \quad N_{cr} = 4 \left[\frac{DB(1-\nu^2)}{R^2} \right]^{\frac{1}{2}} \quad (3.20)$$

以(3.20)式算得的半波数和临界应力作为参考数据, 代入(2.32)式进行试算, 可求得夹层圆柱壳在局部失稳时临界应力的严格解。

四、算例和讨论

算例: 对于对称构造($E_1=E_2$, $h_1=h_2$)的夹层圆柱壳在轴压下轴对称失稳时, 已知数据由表1给出, 其中 $R=(r_1+r_2)/2$ 。由夹层壳理论的(3.1)式, 可算得 N_{cr} 和失稳时的半波数 m , 由另一种夹层壳理论的(3.15)式, 也可算得 N_{cr} 和 m ; 以(3.1)式所得的 N_{cr} 和 m 作为参考值, 由(2.32)式进行多次试算, 可求得临界应力和失稳时的半波数 m , 为了和夹层壳理论算得的结果作比较, 将临界应力也折算成 N_{cr} 。计算结果见表2。

表 1

算例	$E_1=E_2(\text{kg/cm}^2)$	$E_3(\text{kg/cm}^2)$	ν	$r_1(\text{cm})$	$r_2(\text{cm})$	$R(\text{cm})$	$l(\text{cm})$	$h(\text{cm})$	$h_1=h_2(\text{cm})$
(1)	7×10^5	130	0.3	25	26	25.5	80	1.0	0.1
(2)	7×10^5	520	0.3	25	26	25.5	80	1.0	0.1
(3)	7×10^5	26	0.3	27	30	28.5	100	3.0	0.2

表 2

算 例	严 格 解		按(3.1)式的近似解			按(3.15)式的近似解		
	$N_{cr}(\text{kg/cm})$	m	$N_{cr}(\text{kg/cm})$	m	误差(%)	$N_{cr}(\text{kg/cm})$	m	误差(%)
(1)	391.0	29	393.0	29	0.51	390.3	29	-0.18
(2)	564.3	29	574.5	29	1.81	563.7	29	-0.11
(3)	1174.1	24	1223.9	24	4.24	1218.8	24	3.81

讨论: (1) 按(2.32)式求临界应力的严格解, 比按夹层壳理论的(3.1)和(3.15)式求临界载荷的近似解的计算工作量要大得多。算例表明, 能按(2.32)式求得临界应力的严格解。夹层壳的夹心严格按数学弹性力学方法处理(不引入任何假定)求解稳定问题的的工作, 由于难度较大, 目前尚未见到。因此本文的工作具有理论意义。(2) 按夹层壳理论的(3.1)式求得临界载荷的近似解, 因为引入了 $E_z=\infty$ 的假定, 应该比严格解大些; 按(3.15)式求得的近似解, 因为引入了 $E_z=E_y=G_{xy}=0$ 的假定, 应该比严格解小些。算例基本上反映了这种论断。在算例(3)中, 由于夹层壳壳厚和半径 R 之比达到 $1/8.38$, 属于厚壳, 而(3.15)式是按薄壳理论导得的, 按(3.15)式求得的近似解大于严格解, 可能与这个因素有关。本文的少数算例表明, N_{cr} 的近似解和严格解相比, 误差都不大, 半波数 m 也很接近, 说明按夹层壳理论算得的结果是相当精确的。因此本文所作的这种比较具有一定的实际意义。

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A Rigorous Solution for the Axisymmetrical Buckling of Simply Supported Cylindrical Sandwich Shells under Axial Load

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Abstract

In this paper, based on Ref.[1], the axisymmetrical buckling of simply supported cylindrical sandwich shells under the action of uniform axial load is solved by a rigorous method. The classical theory of shells is used for the two face sheets and the core is considered as a three-dimensional elastic body. A series of transcendental equations are obtained, from which the critical loads can be calculated by numerical methods. Numerical examples are given to compare with the solutions of sandwich shell theories.