应用数学和力学，第5卷第4期（1984年7月）
Applied Mathematics and Mechanics

应用数学和力学编委会编
四川科学技术出版社出版

以边框加固的承重预制墙

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（1983年7月26日收到）

摘　要

这承重预制墙，由一薄板及沿其周界有一加固用的框架所组成．当荷载作用于框架时，板的边界条件，既不是已知的力也不是已知的位移．在满足板的应力函数的双调和方程的基础上，以总的应变能为最小，来保证这两弹性体的位移协调．于是得到一组无穷联立方程．所得的结果表明这解法是有效的．

在北京的房屋建筑中，曾采用以边框加固的薄板所组成的预制墙（图1）．荷载P由上层传来，而两直杆起了柱的作用．由两边的地板传来均布荷载q．这两种荷载的组合组成设计荷载．当这种预制构件开始被使用时，曾作实体试验以显示裂缝的开展．但由于应力分布为未知，钢筋的需要量就成为问题．这就促使当时对这问题的研究．

引用以下的记号：

h	板的厚度	I	框架杆的横截面对中性轴的惯性矩
d	框架杆的横截面的宽度	F	框架杆的横截面积
$2h_0$	框架杆的横截面的高度	G,μ	各为板的剪切弹性系数与泊松系数
E,E_1	各为板与框架的弹性系数		

一、板的分析

我们先解图2所示这问题．对于通常的薄板问题，在板的边界上将给出力或位移．现由于板的边界与框架相联结，板的边界力与位移均为未知．对于板须满足方程

$$\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}=0$$

并且

$$\sigma_x=\frac{\partial^2\phi}{\partial y^2},\quad \sigma_y=\frac{\partial^2\phi}{\partial x^2},\quad \tau_{xy}=-\frac{\partial^2\phi}{\partial x\partial y}$$

板的边界条件可叙述如下：

（1）板的边界力由框架所引起，而板则以大小相等方向相反的反作用力，作用在框架上．

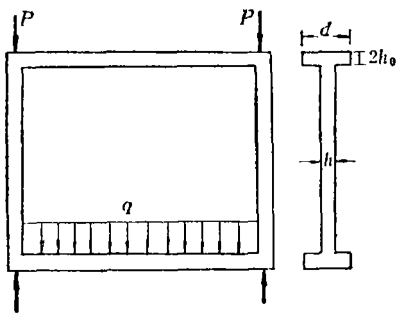

图 1

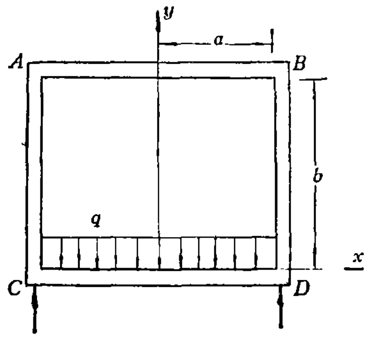

图 2

（2） 板与框架的接触点的位移必须协调。

（3） 板角点的剪应变 $\gamma_{xy}=\frac{\tau_{xy}}{G}$，应等于框架相应的节点的角变形$\theta$，即$\gamma_{xy}=\theta$。对于框架的刚节点，则$\gamma_{xy}=0$。

以图 2 所示这坐标系统，取应力函数为

$$\phi=\sum_{m=1}\frac{\cos\alpha_m x}{\mathrm{ch}\alpha_m b}\left\{C_{m1}\,\mathrm{ch}\alpha_m y+C_{m2}y\,\mathrm{sh}\alpha_m y+C_{m3}\,\mathrm{sh}\alpha_m y+C_{m4}y\,\mathrm{ch}\alpha_m y\right\}$$

$$+\sum_{n=1}\frac{\cos\beta_n y}{\mathrm{ch}\beta_n a}\left\{D_{n1}\,\mathrm{ch}\beta_n x+D_{n2}x\,\mathrm{sh}\beta_n x\right\}+\frac{1}{2}(Ax^2+By^2) \tag{1.1}$$

式中的 $\alpha_m=\frac{m\pi}{a}$，$\beta_n=\frac{n\pi}{b}$

应力分量为：

$$\sigma_x=\sum_{m=1}\frac{\alpha_m\cos\alpha_m x}{\mathrm{ch}\alpha_m b}\left\{C_{m1}\alpha_m\,\mathrm{ch}\alpha_m y+C_{m2}(2\mathrm{ch}\alpha_m y+\alpha_m y\,\mathrm{sh}\alpha_m y)+C_{m3}\alpha_m\,\mathrm{sh}\alpha_m y\right.$$

$$\left.+C_{m4}(2\mathrm{sh}\,\alpha_m y+\alpha_m y\,\mathrm{ch}\,\alpha_m y)\right\}-\sum_{n=1}\frac{\beta_n^2\cos\beta_n y}{\mathrm{ch}\,\beta_n a}\left\{D_{n1}\,\mathrm{ch}\,\beta_n x+D_{n2}x\,\mathrm{sh}\,\beta_n x\right\}+B \tag{1.2}$$

$$\sigma_y=-\sum_{m=1}\frac{\alpha_m^2\cos\alpha_m x}{\mathrm{ch}\alpha_m b}\left\{C_{m1}\,\mathrm{ch}\alpha_m y+C_{m2}y\,\mathrm{sh}\alpha_m y+C_{m3}\,\mathrm{sh}\alpha_m y+C_{m4}y\,\mathrm{ch}\alpha_m y\right\}$$

$$+\sum_{n=1}\frac{\beta_n\cos\beta_n y}{\mathrm{ch}\beta_n a}\left\{D_{n1}\beta_n\,\mathrm{ch}\beta_n x+D_{n2}(2\mathrm{ch}\beta_n x+\beta_n x\,\mathrm{sh}\beta_n x)\right\}+A \tag{1.3}$$

$$\tau_{xy}=\sum_{m=1}\frac{\alpha_m\sin\alpha_m x}{\mathrm{ch}\alpha_m b}\left\{C_{m1}\alpha_m\,\mathrm{sh}\alpha_m y+C_{m2}(\mathrm{sh}\alpha_m y+\alpha_m y\,\mathrm{ch}\alpha_m y)+C_{m3}\alpha_m\,\mathrm{ch}\alpha_m y\right.$$

$$+C_{m4}(\mathrm{ch}\alpha_m y+\alpha_m y\,\mathrm{sh}\alpha_m y)\Big\}+\sum_{n=1}\frac{\beta_n\sin\beta_n y}{\mathrm{ch}\alpha_m b}\Big\{D_{n1}\beta_n\,\mathrm{sh}\beta_n x$$

$$+D_{n2}(\mathrm{sh}\beta_n x+\beta_n x\,\mathrm{ch}\beta_n x)\Big\} \tag{1.4}$$

在板的角点$\tau_{xy}=0$。作用在板边界的力，可以很容易得到，例如$(h\sigma_y)_{y=b}$，$(h\tau_{xy})_{y=b}$等。其中的h为板的厚度。

二、框架的分析

在图3中表示出框架杆AB,BD,CD。只须使作用在板边界上的力反向，即$(-h\sigma_y)_{y=b}$，$(-h\tau_{xy})_{y=b},\cdots$，即得作用在框架内壁的力（图3）。这框架为二次静不定，并以M_1与M_2为静不定量。以M_{AB}，M_{DB}，M_{CD}，S_{AB}，S_{CD}，S_{BD}为杆的弯矩与轴力。

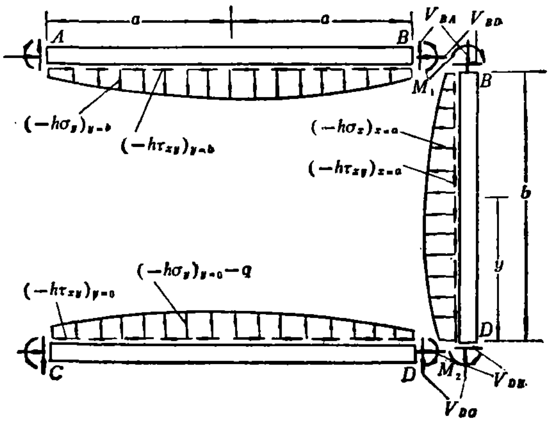

图 3

对于顶部杆AB（图3）

$$V_{AB}=\int_0^a(-h\sigma_y)_{y=b}dx=hAa+h\sum_{n=1}\cos n\pi\Big\{D_{n1}\beta_n\,\mathrm{th}\,\beta_n a+D_{n2}(\mathrm{th}\,\beta_n a+\beta_n a)\Big\} \tag{2.1}$$

$$M_{AB}=M_1+V_{AB}(a-x)-\int_x^a(-h\sigma_y)_{\substack{y=b\\x=x_1}}(x_1-x)\,dx_1-h_0\int_x^a(-h\tau_{xy})_{y=b}dx$$

$$=M_1+\frac{1}{2}Ah(a^2-x^2)+h\sum_{m=1}(\cos m\pi-\cos\alpha_m x)\{C_{m1}(1+h_0\alpha_m\,\mathrm{th}\,\alpha_m b)$$

$$+C_{m2}[(b+h_0)\mathrm{th}\,\alpha_m b+h_0\alpha_m b]+C_{m3}(\mathrm{th}\,\alpha_m b+h_0\alpha_m)+C_{m4}[b+h_0(1+\alpha_m b\,\mathrm{th}\,\alpha_m b)]\}$$

$$+h\sum_{n=1}\frac{\cos n\pi}{\mathrm{ch}\beta_n a}[D_{n1}(\mathrm{ch}\,\beta_n a-\mathrm{ch}\,\beta_n x)+D_{n2}(a\,\mathrm{sh}\,\beta_n a-x\,\mathrm{sh}\,\beta_n x)] \tag{2.2}$$

对于底部杆CD（图3）

$$V_{DC}=\int_0^a[-q+(-h\sigma_y)_{y=0}]dx=-qa+h\sum_{n=1}\{D_{n1}\beta_n\,\text{th}\,\beta_n a+D_{n2}(\text{th}\,\beta_n a+\beta_n a)\} \quad (2.3)$$

$$M_{OD}=M_2+\frac{1}{2}(a^2-x^2)(hA-q)+h\sum_{m=1}(\cos m\pi-\cos\alpha_m x)\frac{1}{\text{ch}\alpha_m b}(C_{m1}-h_0\alpha_m C_{m3}$$

$$-h_0C_{m4})+h\sum_{n=1}\frac{1}{\text{ch}\beta_n a}[(\text{ch}\,\beta_n a-\text{ch}\beta_n x)D_{n1}+(a\,\text{sh}\,\beta_n a-x\,\text{sh}\,\beta_n x)D_{n2}] \quad (2.4)$$

对于竖杆BD

$$V_{BD}=\frac{1}{b}(M_2-M_1)+\frac{h}{b}\sum_{m=1}\cos m\pi\Big\{C_{m1}\Big(\alpha_m b\,\text{th}\,\alpha_m b+\frac{1}{\text{ch}\alpha_m b}-1\Big)+C_{m2}\alpha_m b^2$$

$$+C_{m3}(\alpha_m b-\text{th}\alpha_m b)+C_{m4}\alpha_m b^2\,\text{th}\alpha_m b\Big\}-\frac{h}{b}\sum_{n=1}(\cos n\pi-1)\Big\{D_{n1}(1+h_0\beta_n\,\text{th}\,\beta_n a)$$

$$+D_{n2}[a\,\text{th}\,\beta_n a+h_0(\text{th}\,\beta_n a+\beta_n a)]\Big\}+\frac{1}{2}hBb \quad (2.5)$$

$$V_{DB}=\frac{1}{b}(M_1-M_2)+\frac{h}{b}\sum_{m=1}\cos m\pi\Big\{C_{m1}\Big(1-\frac{1}{\text{ch}\,\alpha_m b}\Big)+C_{m2}b\text{th}\,\alpha_m b$$

$$+C_{m3}\Big(\text{th}\,\alpha_m b-\frac{\alpha_m b}{\text{ch}\alpha_m b}\Big)+C_{m4}b\Big(1-\frac{1}{\text{ch}\,\alpha_m b}\Big)\Big\}+\frac{h}{b}\sum_{n=1}(\cos n\pi-1)$$

$$\cdot\Big\{D_{n1}(1+h_0\beta_n\,\text{th}\,\beta_n a)+D_{n2}[a\,\text{th}\,\beta_n a+h_0(\text{th}\,\beta_n a+\beta_n a)]\Big\}+\frac{1}{2}hBb \quad (2.6)$$

$$M_{DB}=M_2+\frac{y}{b}(M_1-M_2)+\frac{1}{2}hBy(b-y)+h\sum_{m=1}\frac{\cos m\pi}{\text{ch}\alpha_m b}\Big\{C_{m1}\Big[\frac{y}{b}(\text{ch}\,\alpha_m b-1)$$

$$-\text{ch}\alpha_m b+1\Big]+C_{m2}y(\text{sh}\alpha_m b-\text{sh}\alpha_m y)+C_{m3}\Big(\frac{y}{b}\text{sh}\,\alpha_m b-\text{sh}\,\alpha_m y\Big)$$

$$+C_{m4}y(\text{ch}\alpha_m b-\text{ch}\alpha_m y)\Big\}+h\sum_{n=1}\Big[\frac{y}{b}(\cos n\pi-1)-\cos\beta_n y+1\Big]$$

$$\cdot\{D_{n1}(1+h_0\beta_n\,\text{th}\,\beta_n a)+D_{n2}[a\,\text{th}\,\beta_n a+h_0(\text{th}\,\beta_n a+\beta_n a)]\} \quad (2.7)$$

框架杆的轴力各为:

$$S_{AB}=-V_{BD}-\int_x^a(-h\tau_{xy})_{y=b}dx=-V_{BD}-h\sum_{m=1}\cos\alpha_m x\{C_{m1}\alpha_m\text{th}\alpha_m b$$

$$+C_{m2}(\text{th}\alpha_m b+\alpha_m b)+C_{m3}\alpha_m+C_{m4}(1+\alpha_m b\text{th}\alpha_m b)\} \quad (2.8)$$

$$S_{OD}=-V_{DB}+\int_x^a(-h\tau_{xy})_{y=0}dx=-V_{DB}+h\sum_{m=1}\frac{\cos\alpha_m x}{\text{ch}\alpha_m b}(\alpha_m C_{m3}+C_{m4}) \quad (2.9)$$

$$S_{DB}=-V_{BA}-\int_y^b(-h\tau_{xy})_{x=a}dy=-V_{BA}+h\sum_{n=1}(\cos n\pi-\cos\beta_n y)\{D_{n1}\beta_n\text{th}\beta_n a$$

$$+D_{n2}(\text{th}\beta_n a+\beta_n a)\} \quad (2.10)$$

在以上的计算中，我们已符合了板的边界条件(1.1)。

三、以最小应变能原理求解

由于板与框架相联结，两者之间相互作用的力系大小相等而方向相反，两者的接触点的位移应一致。这将导致板与框架两部分的余能的一次变分为零。对于符合虎克定律的材料，真正的应力分量将使通常的应变能为最小。总的应变能包括板与框架的应变能。对于框架，我们将考虑弯曲与轴力的应变能。

$$U=2\frac{h}{2E}\int_0^b\int_0^a\{\sigma_x^2+\sigma_y^2-2\mu\sigma_x\sigma_y+2(1+\mu)\tau_{xy}^2\}dxdy+\frac{1}{E_1I}\left\{\int_0^a(M_{AB}^2+M_{CD}^2)dx+\int_0^b M_{DB}^2dy\right\}+\frac{1}{E_1F}\left\{\int_0^a(S_{AB}^2+S_{CD}^2)dx+\int_0^b S_{DB}^2dy\right\} \tag{3.1}$$

将应力分量，弯矩，轴力代入(3.1)式，得到未知系数C_{m1}，C_{m2}，$\cdots$，D_{n1}，D_{n2}，A，B，M_1及M_2的二次式。要使应变能最小，计算U对以上系数的导数并使它们各等于零。例如由U对C_{m1}的导数，得到：

$$\frac{\partial U}{\partial C_{m1}}=\frac{2h}{E}\int_0^b\int_0^a\left\{\sigma_x\frac{\partial\sigma_x}{\partial C_{m1}}+\sigma_y\frac{\partial\sigma_y}{\partial C_{m1}}-\mu\left(\sigma_x\frac{\partial\sigma_y}{\partial C_{m1}}+\sigma_y\frac{\partial\sigma_x}{\partial C_{m1}}\right)+2(1+\mu)\tau_{xy}\frac{\partial\tau_{xy}}{\partial C_{m1}}\right\}dxdy+\frac{2}{E_1I}\left\{\int_0^a\left(M_{AB}\frac{\partial M_{AB}}{\partial C_{m1}}+M_{CD}\frac{\partial M_{CD}}{\partial C_{m1}}\right)dx+\int_0^b M_{DB}\frac{\partial M_{DB}}{\partial C_{m1}}dy\right\}+\frac{2}{E_1F}\left\{\int_0^a\left(S_{AB}\frac{\partial S_{AB}}{\partial C_{m1}}+S_{CD}\frac{\partial S_{CD}}{\partial C_{m1}}\right)dx+\int_0^b S_{DB}\frac{\partial S_{DB}}{\partial C_{m1}}dy\right\}=0 \tag{3.2}$$

于是，所得的方程的个数与未知系数C_{m1}，C_{m2}，C_{m3}，C_{m4}，D_{n1}，D_{n2}，A，B相同。至于尚有两个未知量M_1与M_2，可用角点条件：框架在节点B与D的角度的改变，等于板在这两角点的剪应变。由于在这两角点的剪应变为零，框架相应的角变形亦应等于零。于是得到：

$$\int_0^a\frac{\partial M_{AB}}{\partial M_1}M_{AB}dx+\int_0^b\frac{\partial M_{BD}}{\partial M_1}M_{BD}dy=0,\quad \int_0^a\frac{\partial M_{CD}}{\partial M_2}M_{CD}dx+\int_0^b\frac{\partial M_{BD}}{\partial M_2}M_{BD}dy=0 \tag{3.3}$$

方程(3.2)与(3.3)表达了板的边界条件(1.2)与(1.3)。这就是体现了板与框架位移的一致。

方程(3.2)与(3.3)组成了一组无穷联立方程，例如

$$\frac{\partial U}{\partial A}=\frac{ab}{E}(A-\mu B)+\frac{1}{E_1I}\left[\frac{a^3}{3}(M_1+M_2)+\frac{4}{15}a^5hA-\frac{2}{15}a^5q\right]+\frac{ha}{E_1I}\sum_{m=1}\left(\frac{a^2}{3}+\frac{1}{\alpha_m^2}\right)\cos m\pi\left\{\left(1+h_0\alpha_m\,\mathrm{th}\,\alpha_m b+\frac{1}{\mathrm{ch}\,\alpha_m b}\right)C_{m1}+[(b+h_0)\mathrm{th}\,\alpha_m b+h_0\alpha_m b]C_{m2}+\left[\mathrm{th}\,\alpha_m b+h_0\alpha_m\left(1-\frac{1}{\mathrm{ch}\,\alpha_m b}\right)\right]C_{m3}+\left[b+h_0\left(1+\alpha_m b\,\mathrm{th}\,\alpha_m b-\frac{1}{\mathrm{ch}\,\alpha_m b}\right)\right]C_{m4}\right\}+\frac{h}{E_1I}\sum_{n=1}(\cos n\pi+1)\left\{\left(\frac{a^3}{3}-\frac{a}{\beta_n^2}+\frac{1}{\beta_n^3}\mathrm{th}\,\beta_n a\right)D_{n1}\right.$$

$$+\left[\frac{3a}{\beta_n^3}+\left(\frac{a^4}{3}-\frac{a^2}{\beta_n^2}-\frac{3}{\beta_n^4}\right)\text{th}\,\beta_n a\right]D_{n2}\Big\}+\frac{a^2bhA}{E_1F}=0 \tag{3.4}$$

$$\frac{\partial U}{\partial C_{m1}}=\frac{1+\mu}{E}a\alpha_m^2\Big\{C_{m1}\alpha_m\text{th}\alpha_m b+\frac{1}{2}C_{m2}[\text{th}\alpha_m b+\alpha_m b(1+\text{th}^2\alpha_m b)]+C_{m3}\alpha_m\text{th}^2\alpha_m b$$

$$+C_{m4}\text{th}\alpha_m b\left(\alpha_m b+\frac{1}{2}\text{th}\alpha_m b\right)\Big\}-\frac{2(1+\mu)}{E}\alpha_m^3\text{th}\alpha_m b$$

$$\cdot\cos m\pi\sum_{n=1}\frac{\beta_n^2}{(\alpha_m^2+\beta_n^2)^2}\text{th}\,\beta_n a\cos n\pi D_{n2}+\frac{1}{E_1I}\Big\{\Big[a(1+h_0\alpha_m\text{th}\alpha_m b)$$

$$+\left(\frac{b}{3}+\frac{1}{\alpha_m^2 b}\right)\left(1-\frac{1}{\text{ch}\alpha_m b}\right)-\frac{1}{\alpha_m}\text{th}\alpha_m b\Big]M_1\cos m\pi+\Big[\frac{b}{6}-\frac{1}{\alpha_m^2 b}$$

$$+\frac{1}{\text{ch}\alpha_m b}\left(a+\frac{b}{3}+\frac{1}{\alpha_m^2 b}\right)\Big]M_2\cos m\pi+a\left(\frac{a^2}{3}+\frac{1}{\alpha_m^2}\right)\Big(1+h_0\alpha_m\text{th}\alpha_m b$$

$$+\frac{1}{\text{ch}\alpha_m b}\Big)hA\cos m\pi+\frac{h}{2}\Big[b\left(\frac{b^2}{12}-\frac{1}{\alpha_m^2}\right)\left(1+\frac{1}{\text{ch}\alpha_m b}\right)+\frac{2}{\alpha_m^3}\text{th}\alpha_m b\Big]B\cos m\pi$$

$$-aq\left(\frac{a^2}{3}+\frac{1}{\alpha_m^2}\right)\frac{1}{\text{ch}\alpha_m b}\cos m\pi\Big\}+\frac{ha}{E_1I}\sum_{n=1}\left(\cos m\pi\cos n\pi+\frac{1}{2}\delta_{mn}\right)$$

$$\cdot\Big\{C_{n1}\Big[(1+h_0\alpha_m\text{th}\alpha_m b)(1+h_0\alpha_n\text{th}\alpha_n b)+\frac{1}{\text{ch}\alpha_m b\,\text{ch}\alpha_n b}\Big]+C_{n2}(1+h_0\alpha_m\text{th}\alpha_m b)$$

$$\cdot[(b+h_0)\text{th}\alpha_n b+h_0\alpha_n b]+C_{n3}\Big[(1+h_0\alpha_m\text{th}\alpha_m b)(\text{th}\alpha_n b+h_0\alpha_n)$$

$$-\frac{h_0\alpha_m}{\text{ch}\alpha_m b\,\text{ch}\alpha_n b}\Big]+C_{n4}\Big[(1+h_0\alpha_m\text{th}\alpha_m b)(b+h_0+k_0\alpha_n b\,\text{th}\,\alpha_n b)-\frac{h_0}{\text{ch}\alpha_m b\,\text{ch}\alpha_n b}\Big]\Big\}$$

$$+\frac{h}{E_1I}\cos m\pi\sum_{n=1}\cos n\pi\Big\{C_{n1}\Big[\left(\frac{b}{2}-\frac{1}{\alpha_m}\text{th}\,\alpha_m b-\frac{1}{\alpha_m^2 b}-\frac{1}{\alpha_m^2 b\,\text{ch}\alpha_m b}\right)\left(1-\frac{1}{\text{ch}\alpha_n b}\right)$$

$$+\left(\frac{b}{2}+\frac{b}{2\text{ch}\alpha_m b}-\frac{1}{\alpha_m}\text{th}\alpha_m b\right)\frac{1}{\text{ch}\alpha_n b}+\frac{1}{\alpha_n^2 b}\left(1-\alpha_n b\,\text{th}\alpha_n b-\frac{1}{\text{ch}\alpha_n b}\right)\left(1-\frac{1}{\text{ch}\alpha_m b}\right)$$

$$-\frac{1}{\alpha_n\text{ch}\alpha_m b}\text{th}\,\alpha_n b+\frac{1}{\alpha_m^2-\alpha_n^2}(\alpha_m\text{th}\alpha_m b-\alpha_n\text{th}\alpha_n b)\Big]+C_{n2}\Big[\Big(\frac{b^2}{2}+\frac{1}{\alpha_m^2}-\frac{b}{\alpha_m}\text{th}\alpha_m b$$

$$-\frac{1}{\alpha_m^2\text{ch}\alpha_m b}\Big)\text{th}\,\alpha_n b-\left(1-\frac{1}{\text{ch}\alpha_m b}\right)\frac{1}{\alpha_n}\left(b-\frac{2}{\alpha_n}\text{th}\alpha_n b+\frac{2}{\alpha_n^2 b}-\frac{2}{\alpha_n^2 b\,\text{ch}\alpha_n b}\right)$$

$$-\frac{1}{\alpha_n^2\text{ch}\,\alpha_m b}(\alpha_n b-\text{th}\alpha_n b)+\frac{1}{\alpha_m(\alpha_m^2-\alpha_n^2)}(\alpha_m\text{th}\alpha_m b-\alpha_n\text{th}\alpha_n b)\Big]$$

$$+C_{n3}\Big[\left(\frac{b}{2}-\frac{1}{\alpha_m}\text{th}\,\alpha_m b-\frac{1}{\alpha_m^2 b\,\text{ch}\alpha_m b}+\frac{1}{\alpha_m^2 b}\right)\text{th}\alpha_n b-\frac{1}{\alpha_n^2}(\alpha_n b$$

$$-\text{th}\,\alpha_n b)\frac{1}{\text{ch}\,\alpha_m b}+\frac{1}{\alpha_m^2-\alpha_n^2}\left(\alpha_m\text{th}\alpha_m b\,\text{th}\,\alpha_n b-\alpha_n+\frac{1}{\text{ch}\alpha_m b\,\text{ch}\alpha_n b}\right)\Big]$$

$$+C_{n4}\Big[\frac{b^2}{2}-\frac{b}{\alpha_m}\text{th}\,\alpha_m b+\frac{1}{\alpha_m^2}\left(1-\frac{1}{\text{ch}\alpha_m b}\right)-\frac{1}{\alpha_n^3 b}(\alpha_n^2 b^2\text{th}\alpha_n b+2\text{th}\alpha_n b$$

$$-2\alpha_n b\Big)\left(1-\frac{1}{\mathrm{ch}\alpha_m b}\right)-\frac{1}{\alpha_n^2}\left(\alpha_n b\,\mathrm{th}\alpha_n b-1+\frac{1}{\mathrm{ch}\alpha_n b}\right)\frac{1}{\mathrm{ch}\alpha_n b}+\frac{1}{\alpha_m^2-\alpha_n^2}(\alpha_m\mathrm{th}\alpha_m b$$

$$-\alpha_n\mathrm{th}\,\alpha_n b)-\frac{1}{2(\alpha_m^2+\alpha_n^2)}\left(1+\mathrm{th}\,\alpha_m b\,\mathrm{th}\,\alpha_n b-\frac{1}{\mathrm{ch}\alpha_m b\,\mathrm{ch}\alpha_n b}\right)$$

$$-\frac{1}{2(\alpha_m-\alpha_n)^2}\left(1-\mathrm{th}\alpha_m b\,\mathrm{th}\alpha_n b-\frac{1}{\mathrm{ch}\alpha_m b\,\mathrm{ch}\alpha_n b}\right)\Big]\Big\}$$

$$+\frac{h}{E_1 I}\sum_{n=1}\left[(1+h_0\alpha_m\mathrm{th}\alpha_m b)\cos n\pi+\frac{1}{\mathrm{ch}\alpha_m b}\right]\left[\left(a-\frac{\alpha_m^2}{\beta_n(\alpha_m^2+\beta_n^2)}\mathrm{th}\beta_n a\right)D_{n1}\cos m\pi\right.$$

$$\left.+\left(a^2+\frac{\alpha_m^2(\alpha_m^2+3\beta_n^2)}{\beta_n^2(\alpha_m^2+\beta_n^2)}\,\mathrm{th}\,\beta_n a-\frac{\alpha_m^2}{\beta_n^2(\alpha_m^2+\beta_n^2)}\beta_n a\right)D_{n2}\cos m\pi\right]$$

$$+\frac{h}{E_1 I}\sum_{n=1}\left[\left(\frac{b}{3}+\frac{1}{\alpha_m^2 b}-\frac{1}{\beta_n^2 b}\right)\left(1-\frac{1}{\mathrm{ch}\alpha_m b}\right)(1-\cos n\pi)+\frac{b}{2}\left(\cos n\pi+\frac{1}{\mathrm{ch}\alpha_m b}\right)\right.$$

$$\left.+\frac{2\alpha_m^2+\beta_n^2}{\alpha_m(\alpha_m^2+\beta_n^2)}\mathrm{th}\alpha_m b\right]\{(1+h_0\beta_n\mathrm{th}\beta_n a)D_{n1}+[(a+h_0)\mathrm{th}\beta_n a+h_0\beta_n a]D_{n2}\}$$

$$+\frac{1}{E_1 F}\left\{\frac{2a}{b^2}\left(1-\frac{1}{\mathrm{ch}\alpha_m b}\right)(M_1-M_2)\cos m\pi+\frac{1}{2}ha\alpha_m\mathrm{th}\alpha_m b[\alpha_m\mathrm{th}\alpha_m b\,C_{m1}+(\alpha_m b\right.$$

$$+\mathrm{th}\alpha_m b)C_{m2}+\alpha_m C_{m3}+(1+\alpha_m b\,\mathrm{th}\alpha_m b)C_{m4}]+\frac{1}{b}\,2ha\left(1-\frac{1}{\mathrm{ch}\alpha_m b}\right)$$

$$\cdot\cos m\pi\sum_{n=1}\cos n\pi\left[\frac{1}{b}\left(1-\frac{1}{\mathrm{ch}\alpha_n b}\right)C_{n1}+C_{n2}\,\mathrm{th}\,\alpha_n b+C_{n3}\frac{1}{b}\mathrm{th}\alpha_n b+C_{n4}\right]$$

$$+\frac{1}{b^2}2ha\left(1-\frac{1}{\mathrm{ch}\alpha_m b}\right)\cos m\pi\sum_{n=1}(\cos n\pi-1)[(1+h_0\beta_n\mathrm{th}\beta_n a)D_{n1}+(h_0\beta_n a$$

$$\left.+a\,\mathrm{th}\beta_n a+h_0\mathrm{th}\beta_n a)D_{n2}]\right\}=0 \tag{3.5}$$

式中的

$$\delta_{mn}=\begin{cases}1 & m=n\\ 0 & m\neq n\end{cases}$$

自(3.3)式的第一式，得到：

$$\left(2a+\frac{b}{3}\right)M_1+\frac{b}{6}M_2+\frac{2}{3}a^3hA+\frac{b^3}{24}hB+h\sum_{m=1}\cos m\pi\left\{\left[2a+\frac{b}{3}+\frac{1}{\alpha_m^2 b}\right.\right.$$

$$\left.+\left(2ah_0\alpha_m-\frac{1}{\alpha_m}\right)\mathrm{th}\,\alpha_m b+\left(\frac{b}{6}-\frac{1}{\alpha_m^2 b}\right)\frac{1}{\mathrm{ch}\alpha_m b}\right]C_{m1}+\left[2abh_0\alpha_m-\frac{b}{\alpha_m}\right.$$

$$\left.-\frac{2}{\alpha_m^3 b}+\left(2ab+2ah_0+\frac{b^2}{3}+\frac{2}{\alpha_m^2}\right)\mathrm{th}\alpha_m b+\frac{2}{\alpha_m^3 b\,\mathrm{ch}\alpha_m b}\right]C_{m2}+\left[2ah_0\alpha_m\right.$$

$$\left.-\frac{1}{\alpha_m}+\left(2a+\frac{b}{3}+\frac{1}{\alpha_m^2 b}\right)\mathrm{th}\alpha_m b\right]C_{m3}+\left[2ab+2ah_0+\frac{b^2}{3}+\frac{2}{\alpha_m^2}+\left(2abh_0\alpha_m\right.\right.$$

$$\left.\left.\left.-\frac{b}{\alpha_m}-\frac{2}{\alpha_m^3 b}\right)\mathrm{th}\alpha_m b\right]C_{m4}\right\}+h\sum_{n=1}\left\{\left[\frac{b}{2}+\left(\frac{b}{3}-\frac{1}{\beta_n^2 b}\right)\left(\cos n\pi-1\right)\right]\right.$$

$$\cdot(1+h_0\beta_n\mathrm{th}\beta_n)+2\left(a-\frac{1}{\beta_n}\mathrm{th}\beta_n a\right)\cos n\pi\Big\}D_{n1}+h\sum_{n=1}\left\{\left[\frac{b}{2}+\left(\frac{b}{3}-\frac{1}{\beta_n^2 b}\right)\right.\right.$$

$$\cdot(\cos n\pi-1)\Big][(a+h_0)\mathrm{th}\beta_n a+h_0\beta_n a]+2\Big[\Big(a^2+\frac{1}{\beta_n^2}\Big)\mathrm{th}\beta_n a-\frac{a}{\beta_n}\Big]\cos n\pi\Big\}D_{n2}=0 \tag{3.6}$$

同样的，可以得到其它的方程，它们总共有十个，其中有些是很冗长的，因而也不在这里列出了。从这组无穷联立方程，可以取足够的方程，使解收敛到满意的结果。

四、一个数字例题

现以真正的墙板作为一个例子。图2所示这墙的尺寸为：$a=270\text{cm}$，$b=300\text{cm}$，$d=24\text{cm}$，$h=h_0=5\text{cm}$，$E=E_1$，$\mu=0.25$，$q\text{kg/cm}$。

以逐次近似解这问题。对于未知量 C_{m1}，C_{m2}，C_{m3}，C_{m4}，D_{n1}，D_{n2} 所取的项数例如 C_{11}，C_{21}，C_{31}，…为4，6，8，10项…。相应的方程的个数为28，40，52，64，…。相应于所取的项数，对板的几个点作应力计算。逐渐增多未知量的项数，直至对各点所得的应力值一致地收敛。计算结果表明：每个系数取10项就够好了。将 C_{m1}，C_{m2}，…代入框架杆的弯矩与轴力的算式，用这方程得到应力：

$$\sigma_{\substack{\max\\ \min}}=\frac{S}{F}\pm\frac{M}{W}$$

以表1，2，3列出应力σ_x，σ_y，τ_{xy}，并在图4，5，6给出它们的分布。

表1　　$\sigma_y(q/h)$在几个截面内的值

截　面	$x=0$	$a/4$	$a/2$	$3a/4$	$7a/8$	$15a/16$	a
$y=0$	1.146	0.974	0.726	1.536	−0.463	−1.123	−0.125
$y=b/4$	0.900	0.920	0.794	0.199	−0.521	−0.961	−1.725
$y=b/2$	0.650	0.615	0.437	−0.063	−0.472	−0.699	−0.826
$y=3b/4$	0.301	0.275	0.168	−0.052	−0.195	−0.276	−0.317
$y=b$	0.027	0.006	−0.004	0.026	−0.011	0.360	0.343

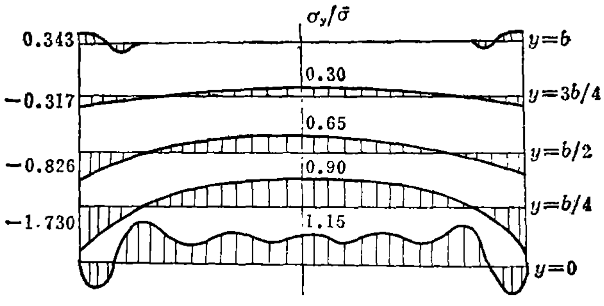

图4　σ_y 的曲线 $\bar{\sigma}=q/h$

表2　　$\sigma_x(q/h)$在几个截面内的值

截　面	$y=0$	$y=b/4$	$y=b/2$	$y=3b/4$	$y=b$
$x=0$	0.713	0.382	−0.177	−0.490	−0.880
$x=a/4$	1.732	0.283	−0.210	−0.464	−0.778
$x=a/2$	0.993	0.046	−0.270	−0.374	−0.519
$x=3a/4$	0.027	−0.236	−0.234	−0.210	−0.197
$x=a$	−3.183	−0.076	−0.003	−0.050	0.465

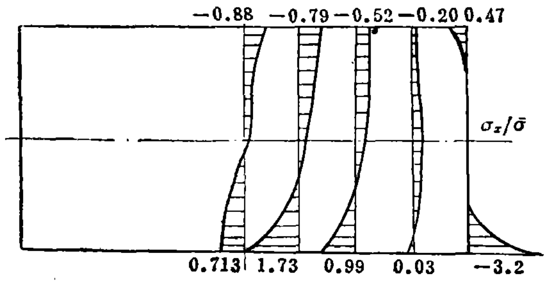

图5 σ_x 的曲线

在截面内的σ_x与杆内的轴力与弯矩，将平衡截面内的弯矩。σ_x随弯矩的减小而小起来。但在板的边界上，出现较大的σ_x以平衡杆内的轴力。

表 3 在几个截面内的$\tau_{xy}(q/h)$

截面	$y=0$	$y=b/4$	$y=b/2$	$y=3b/4$	$y=b$
$x=0$	0.000	0.000	0.000	0.000	0.000
$x=a/4$	0.280	0.175	0.302	0.302	0.167
$x=a/2$	0.379	0.402	0.607	0.552	0.323
$x=3a/4$	0.189	0.886	0.822	0.641	0.332
$x=a$	0.000	0.971	0.551	0.401	0.000

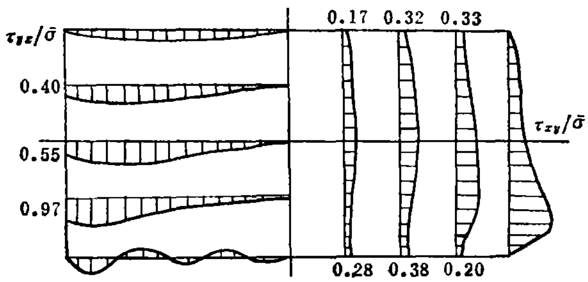

图6 τ_{xy}曲线

表 4 给出框架的轴力与弯矩。

表 4

	$x=0$	$x=a/4$	$x=a/2$	$x=3a/4$	$x=a$
S_{AB}	$-0.260qa$	−0.237	−0.176	−0.090	−0.023
S_{CD}	$0.322qa$	0.277	0.287	0.330	0.212
M_{AB}	$0.0856q$(kg-m)	0.0523	0.0212	0.0398	−0.8015
M_{CD}	0.947(kg-m)	−0.4747	−1.0182	1.1389	5.2021
	$y=0$	$y=b/4$	$y=b/2$	$y=3b/4$	$y=b$
S_{DB}	$-0.811qa$	−0.453	−0.234	−0.097	−0.035
M_{DB}	$5.325q$(kg-m)	0.1007	0.0572	−0.0443	−0.9729

现对所得的解进行校核：平行于x及y轴的截面内的力，包括板内的及杆内的正应力所引起的力，是否保持部分墙的平衡？所作的计算如下：

$$\frac{-\sum_{i=1}^{32}\frac{h}{2}(\sigma_{x_i}+\sigma_{x_{i+1}})\Delta y_i}{S_{AB}+S_{CD}},\qquad \Delta y_i=\frac{b}{32}\qquad（在x方向）\tag{4.1}$$

$$\frac{-\sum_{i=1}^{32}\frac{h}{2}(\sigma_{y_i}+\sigma_{y_{i+1}})\Delta x_i}{S_{DB}}, \quad \Delta x_i=\frac{a}{32} \quad (\text{在}y\text{方向}) \tag{4.2}$$

如果这比值各近于1，就符合部分墙的平衡条件．将结果列于表5．

表 5

x	0	$a/4$	$a/2$	$3a/4$	a
比值(4.1)	1.0178	0.9697	1.0003	0.9964	1.0136
y	0	$b/4$	$b/2$	$3b/4$	b
比值(4.2)	1.0012	0.9988	1.0002	1.0002	0.9919

可以看出，对所有的截面均满足平衡．

本文表明：最小应变能原理可以成功地解用框架加固的预制墙．使应变能最小，保证了板与框架变形的协调．其实，这解法可以推广到两个弹性体的接触问题．可以列举多个有实用价值的例子，例如镶边拉杆的力的传递，木桁架的榫接，及双金属条整温器的接触热应力等．剪力墙当亦为一例．

参 考 文 献

[1] Timoshenko, S., *Theory of Elasticity*.

[2] von Kármán, Th. , *Festschrift August Föppl* (1923), 114.

[3] Hoff, N. J., Stresses in a reinforced monocoque cylinder under concentrated symmetric transuerse loads, *J. Appl. Mech.*, 11, 4 (1944).

[4] 张福范、燕公溥, The analysis of a load supporting prefabricated wall, Ninth U. S. National Congress of Applied Mechanics, Preliminary Program, June,(1982), 21—25.

[5] 张福范, 双金属条整温器接触热应力, 应用数学和力学, 4,3 (1983),347—360.

[6] 燕公溥, 承重预制墙的应力分析, 硕士论文, 清华大学, (1981,10).

Load Supporting Prefabricated Wall Reinforced by a Frame

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Abstract

This load supporting prefabricated wall consists of a thin plate reinforced along its boundaries by a frame. With the loads acting on the frame, the plate has its boundary conditions defined neither by forces nor by displacements. On the basis of satisfying the bi-harmonic equation of the stress function for the plate, we make the total strain energy minimum to ensure the compatibility of displacements for the two elastic bodies. Thus we get a set of infinite simultaneous equations. The results indicate the method of solution is effective.