# 正交各向异性矩形薄板的非线性弯曲*

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**摘 要**

本文应用[1]中提出的摄动方法研究了在各种支承条件下的正交各向异性矩形薄板的非线性弯曲问题．导出了挠度$w$和应力函数$\varphi$的一致有效的$N$阶形式渐近解．

## 一、引 言

在我国，钱伟长对奇异摄动理论的发展是有着开创性的贡献的．他在1948年解决圆板大挠度问题时，即开创了合成展开法[2]．之后，叶开沅和胡海昌又分别研究了在边缘载荷作用下的环板和在复合载荷作用下圆板的大挠度问题[3,4]．江福汝研究了含有小参数的高阶椭圆型方程的混合边值问题，改进了 Comstock 的工作，并研究了环板和圆形薄板在各种支承条件下的非对称、非线性弯曲问题[5]．

本文应用von Kármán 的经典非线性理论，采用江福汝提出的新的构造边界层的方法研究了正交各向异性矩形薄板在各种支承条件下的非线性弯曲问题，构造了挠度$w$和应力函数$\varphi$的$N$阶形式渐近解 $W_N$ 和 $\Phi_N$．

对于在各种支承条件下的各向同性矩形薄板的非线性弯曲分析，只需在文中令$E_1=E_2$，$\mu_1=\mu_2$即得．

## 二、基 本 方 程

假设我们研究的正交各向异性矩形薄板宽为$a$、长为$b$、厚度等于$t$，承受横向均布载荷$q$．$x$和$y$轴沿材料的主方向，$z$轴则垂直于板的中面，如图1所示．

平衡微分方程为

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0 \tag{2.1}$$

$$\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=0 \tag{2.2}$$

* 江福汝推荐．

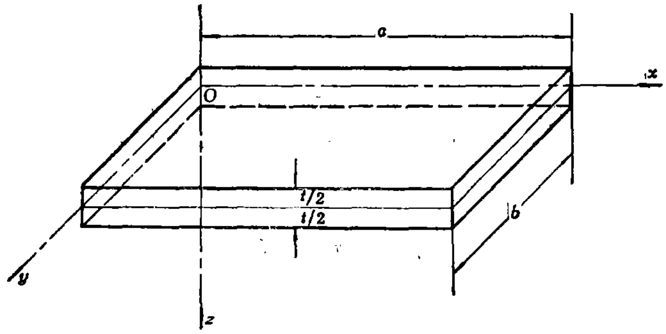


图 1

弹性曲面微分方程为

$$\left(D_1\frac{\partial^4 w}{\partial x^4}+D_2\frac{\partial^4 w}{\partial y^4}+2D_3\frac{\partial^4 w}{\partial x^2\partial y^2}\right)-\left(N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}\right)=q \tag{2.3}$$

式中$w$为挠度；$N_x, N_y$和$N_{xy}$为横向载荷$q$引起的中面内力。

$$N_x=t\sigma_x,\quad N_y=t\sigma_y,\quad N_{xy}=t\tau_{xy}$$

而弯曲刚度

$$D_1=\frac{E_1t^3}{12(1-\mu_1\mu_2)},\quad D_2=\frac{E_2t^3}{12(1-\mu_1\mu_2)},\quad D_K=\frac{Gt^3}{12}$$

$$D_3=\mu_1D_2+2D_K=\mu_2D_1+2D_K$$

中面应变[6]

$$\varepsilon_x=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2,\quad \varepsilon_y=\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2,\quad \gamma_{xy}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{2.4}$$

相容方程为

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}-\frac{\partial^2\gamma_{xy}}{\partial x\partial y}=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \tag{2.5}$$

应力-应变关系为[7]

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}=\begin{pmatrix}S_{11} & S_{12} & 0\\ S_{12} & S_{22} & 0\\ 0 & 0 & S_{66}\end{pmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}$$

或

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}=\frac{1}{t}\begin{pmatrix}\frac{1}{E_1} & -\frac{\mu_1}{E_1} & 0\\ -\frac{\mu_2}{E_2} & \frac{1}{E_2} & 0\\ 0 & 0 & \frac{1}{G}\end{pmatrix}\begin{Bmatrix}N_x\\ N_y\\ N_{xy}\end{Bmatrix} \tag{2.6}$$

式中 $\mu_1/E_1=\mu_2/E_2$。

引入应力函数$\varphi(x,y)$，使

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = t \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = t \begin{Bmatrix} \dfrac{\partial^2 \varphi}{\partial y^2} \\ \dfrac{\partial^2 \varphi}{\partial x^2} \\ -\dfrac{\partial^2 \varphi}{\partial x \partial y} \end{Bmatrix} \tag{2.7}$$

它自然满足平衡微分方程。方程(2.3)和(2.5)化为

$$D_1\frac{\partial^4 w}{\partial x^4}+D_2\frac{\partial^4 w}{\partial y^4}+2D_3\frac{\partial^4 w}{\partial x^2\partial y^2}=t\left(\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}-2\frac{\partial^2\varphi}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}\right)+q \tag{2.8a}$$

$$\frac{1}{E_2}\frac{\partial^4\varphi}{\partial x^4}+\frac{1}{E_1}\frac{\partial^4\varphi}{\partial y^4}+\left(\frac{1}{G}-\frac{\mu_1}{E_1}-\frac{\mu_2}{E_2}\right)\frac{\partial^4\varphi}{\partial x^2\partial y^2}=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \tag{2.8b}$$

引入无量纲量

$$\tilde{w}=\frac{w}{a},\quad \tilde{x}=\frac{x}{a},\quad \tilde{y}=\frac{y}{a},\quad \tilde{\varphi}=\frac{\varphi}{E_1a^2},\quad \tilde{q}=\frac{aq}{tE_1}$$

则方程(2.8a,b)可表为（略去了字母上的“～”号）

$$\Pi_s(w,\varphi)\equiv\varepsilon^2\nabla_1^4 w-L(w,\varphi)=q \tag{2.9a}$$

$$\Pi(w,\varphi)\equiv\nabla_2^4\varphi+\frac{1}{2}L(w,w)=0 \tag{2.9b}$$

式中
$$\varepsilon^2=\frac{t^2}{12(1-\mu_1\mu_2)a^2}$$

微分算子
$$\nabla_1^4=\frac{\partial^4}{\partial x^4}+\frac{D_2}{D_1}\frac{\partial^4}{\partial y^4}+\frac{2D_3}{D_1}\frac{\partial^4}{\partial x^2\partial y^2}$$

$$\nabla_2^4=\frac{E_1}{E_2}\frac{\partial^4}{\partial x^4}+\frac{\partial^4}{\partial y^4}+\left(\frac{E_1}{G}-2\mu_1\right)\frac{\partial^4}{\partial x^2\partial y^2}$$

而

$$L(w,\varphi)\equiv\frac{\partial^2 w}{\partial x^2}\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2\varphi}{\partial x^2}-2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2\varphi}{\partial x\partial y} \tag{2.10}$$

假设挠度$w$和应力函数$\varphi$所需满足的边界条件为

$$w|_{x=0}=f_1(y),\qquad \left.\frac{\partial w}{\partial x}\right|_{x=0}=g_1(y) \tag{2.11}$$

$$w|_{x=1}=f_2(y),\qquad \left.\frac{\partial w}{\partial x}\right|_{x=1}=g_2(y) \tag{2.12}$$

$$w|_{y=0}=f_3(x),\qquad -D_2\left(\frac{\partial^2 w}{\partial y^2}+\mu_1\frac{\partial^2 w}{\partial x^2}\right)\Bigg|_{y=0}=g_3(x) \tag{2.13}$$

$$w|_{y=b/a}=f_4(x),\qquad -D_2\left(\frac{\partial^2 w}{\partial y^2}+\mu_1\frac{\partial^2 w}{\partial x^2}\right)\Bigg|_{y=b/a}=g_4(x) \tag{2.14}$$

$$\left.\frac{\partial^2\varphi}{\partial y^2}\right|_{x=0}=h_1(y), \qquad \left.\frac{\partial^2\varphi}{\partial x\partial y}\right|_{x=0}=I_1(y) \tag{2.15}$$

$$\left.\frac{\partial^2\varphi}{\partial y^2}\right|_{x=l}=h_2(y), \qquad \left.\frac{\partial^2\varphi}{\partial x\partial y}\right|_{x=l}=I_2(y) \tag{2.16}$$

$$\left.\frac{\partial^2\varphi}{\partial x^2}\right|_{y=0}=h_3(x), \qquad \left.\frac{\partial^2\varphi}{\partial x\partial y}\right|_{y=0}=I_3(x) \tag{2.17}$$

$$\left.\frac{\partial^2\varphi}{\partial x^2}\right|_{y=b/a}=h_4(x), \qquad \left.\frac{\partial^2\varphi}{\partial x\partial y}\right|_{y=b/a}=I_4(x) \tag{2.18}$$

不难看出，对于其它形式的边界条件可同样制作。

## 三、微分算子展开式

在边界$x=0$的邻域引进两变量$\xi$和$\eta$，将关于$x$的偏导数$\frac{\partial}{\partial x}$变换成关于$\xi$和$\eta$的偏导数

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial \xi}\ \frac{\partial \xi}{\partial x}+\frac{\partial}{\partial \eta}\ \frac{\partial \eta}{\partial x}$$

式中选取

$$\xi=\frac{u(x,y)}{\varepsilon}, \quad \eta=x \tag{3.1}$$

这里$u(x,y)$是待定函数，由于

$$\frac{\partial}{\partial x^i}=\varepsilon^{-i}(\delta_{i,0}+\varepsilon\delta_{i,1}+\varepsilon^2\delta_{i,2}+\cdots+\varepsilon^i\delta_{i,i}) \qquad (i=1,2,3,4) \tag{3.2}$$

式中

$$\delta_{1,0}=u_x\frac{\partial}{\partial\xi}, \quad \delta_{1,1}=\frac{\partial}{\partial\eta}$$

$$\delta_{2,0}=u_x^2\frac{\partial^2}{\partial\xi^2}, \quad \delta_{2,1}=2u_x\frac{\partial^2}{\partial\xi\partial\eta}+u_{xx}\frac{\partial}{\partial\xi}, \quad \delta_{2,2}=\frac{\partial^2}{\partial\eta^2}$$

$$\delta_{3,0}=u_x^3\frac{\partial^3}{\partial\xi^3}, \quad \delta_{3,1}=3u_x^2\frac{\partial^3}{\partial\xi^2\partial\eta}+3u_x u_{xx}\frac{\partial^2}{\partial\xi^2}$$

$$\delta_{3,2}=3u_x\frac{\partial^3}{\partial\xi\partial\eta^2}+3u_{xx}\frac{\partial^2}{\partial\xi\partial\eta}+u_{xxx}\frac{\partial}{\partial\xi}, \quad \delta_{3,3}=\frac{\partial^3}{\partial\eta^3}$$

$$\delta_{4,0}=u_x^4\frac{\partial^4}{\partial\xi^4}, \quad \delta_{4,1}=4u_x^3\frac{\partial^4}{\partial\xi^3\partial\eta}+6u_x^2 u_{xx}\frac{\partial^3}{\partial\xi^3}$$

$$\delta_{4,2}=6u_x^2\frac{\partial^4}{\partial\xi^2\partial\eta^2}+12u_xu_{xx}\frac{\partial^3}{\partial\xi^2\partial\eta}+4u_xu_{xxx}\frac{\partial^2}{\partial\xi^2}+3u_{xx}^2\frac{\partial^2}{\partial\xi^2}$$

$$\delta_{4,3}=4u_x\frac{\partial^4}{\partial\xi\partial\eta^3}+6u_{xx}\frac{\partial^3}{\partial\xi\partial\eta^2}+4u_{xxx}\frac{\partial^2}{\partial\xi\partial\eta}+u_{xxxx}\frac{\partial}{\partial\xi}, \quad \delta_{4,4}=\frac{\partial^4}{\partial\eta^4}$$

我们得到算子

$$\nabla^4 = c\frac{\partial^4}{\partial x^4} + d\frac{\partial^4}{\partial y^4} + e\frac{\partial^4}{\partial x^2 \partial y^2}$$

的展开式

$$\nabla^4 \equiv \varepsilon^{-4}(D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \varepsilon^3 D_3 + \varepsilon^4 D_4) \tag{3.3}$$

式中

$$D_0 = c\delta_{4,0},\ D_1 = c\delta_{4,1},\ D_2 = c\delta_{4,2} + e\delta_{2,0}\frac{\partial^2}{\partial y^2}$$

$$D_3 = c\delta_{4,3} + e\delta_{2,1}\frac{\partial^2}{\partial y^2},\ D_4 = c\delta_{4,4} + d\frac{\partial^4}{\partial y^4} + e\delta_{2,2}\frac{\partial^2}{\partial y^2}$$

显然，若$\nabla^4 = \nabla_1^4$，则

$$c = 1,\ d = \frac{D_2}{D_1},\quad e = \frac{2D_3}{D_1}$$

若$\nabla^4 = \nabla_2^4$，则

$$c = \frac{E_1}{E_2},\ d = 1,\quad e = \frac{E_1}{G} - 2\mu_1$$

同样，在$x=1$的邻域引进两变量

$$\tilde{\xi} = \frac{\tilde{u}(x,y)}{\varepsilon},\quad \tilde{\eta} = x \tag{3.4}$$

将对$x$的偏导数变换成关于$\tilde{\xi}$和$\tilde{\eta}$的偏导数，

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{\xi}}\frac{\partial \tilde{\xi}}{\partial x} + \frac{\partial}{\partial \tilde{\eta}}\frac{\partial \tilde{\eta}}{\partial x}$$

可得算子展开式

$$\nabla^4 \equiv \varepsilon^{-4}\sum_{i=0}^{4}\varepsilon^i \tilde{D}_i \tag{3.5}$$

式中

$$\tilde{D}_0 = c\tilde{\delta}_{4,0},\ \tilde{\delta}_{4,0} = \tilde{u}_x^4\frac{\partial^4}{\partial \tilde{\xi}^4}$$

在边界$y=0$的邻域引进两变量$\alpha$和$\beta$，将关于$y$的偏导数$\frac{\partial}{\partial y}$变换成关于$\alpha$和$\beta$的偏导数

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial}{\partial \beta}\frac{\partial \beta}{\partial y}$$

式中

$$\alpha = \frac{p(x,y)}{\varepsilon},\quad \beta = y \tag{3.6}$$

这里$p(x,y)$是待定函数，由于

$$\frac{\partial}{\partial y^i} = \varepsilon^{-i}(\gamma_{i,0} + \varepsilon\gamma_{i,1} + \cdots + \varepsilon^i\gamma_{i,i})\qquad (i=1,2,3,4) \tag{3.7}$$

式中

$$\gamma_{1,0} = p_y\frac{\partial}{\partial \alpha},\quad \gamma_{1,1} = \frac{\partial}{\partial \beta}$$

$$\gamma_{2,0}=p_y^2\frac{\partial^2}{\partial\alpha^2},\quad \gamma_{2,1}=2p_y\frac{\partial^2}{\partial\alpha\partial\beta}+p_{yy}\frac{\partial}{\partial\alpha},\quad \gamma_{2,2}=\frac{\partial^2}{\partial\beta^2}$$

$$\gamma_{3,0}=p_y^3\frac{\partial^3}{\partial\alpha^3},\quad \gamma_{3,1}=3p_y^2\frac{\partial^3}{\partial\alpha^2\partial\beta}+3p_yp_{yy}\frac{\partial^2}{\partial\alpha^2}$$

$$\gamma_{3,2}=3p_y\frac{\partial^3}{\partial\alpha\partial\beta^2}+3p_{yy}\frac{\partial^2}{\partial\alpha\partial\beta}+p_{yyy}\frac{\partial}{\partial\alpha},\qquad \gamma_{3,3}=\frac{\partial^3}{\partial\beta^3}$$

$$\gamma_{4,0}=p_y^4\frac{\partial^4}{\partial\alpha^4},\quad \gamma_{4,1}=4p_y^3\frac{\partial^4}{\partial\alpha^3\partial\beta}+6p_y^2p_{yy}\frac{\partial^3}{\partial\alpha^3}$$

$$\gamma_{4,2}=6p_y^2\frac{\partial^4}{\partial\alpha^2\partial\beta^2}+12p_yp_{yy}\frac{\partial^3}{\partial\alpha^2\partial\beta}+4p_yp_{yyy}\frac{\partial^2}{\partial\alpha^2}+3p_{yy}^2\frac{\partial^2}{\partial\alpha^2}$$

$$\gamma_{4,3}=4p_y\frac{\partial^4}{\partial\alpha\partial\beta^3}+6p_{yy}\frac{\partial^3}{\partial\alpha\partial\beta^2}+4p_{yyy}\frac{\partial^2}{\partial\alpha\partial\beta}+p_{yyyy}\frac{\partial}{\partial\alpha},\quad \gamma_{4,4}=\frac{\partial^4}{\partial\beta^4}$$

得到算子$\nabla'^4$的展开式

$$\nabla'^4\equiv\varepsilon^{-4}(D_0'+\varepsilon D_1'+\varepsilon^2D_2'+\varepsilon^3D_3'+\varepsilon^4D_4')\tag{3.8}$$

式中

$$D_0'=d\gamma_{4,0},\quad D_1'=d\gamma_{4,1},\quad D_2'=d\gamma_{4,2}+e\gamma_{2,0}\frac{\partial^2}{\partial x^2}$$

$$D_3'=d\gamma_{4,3}+e\gamma_{2,1}\frac{\partial^2}{\partial x^2},\quad D_4'=c\frac{\partial^4}{\partial x^4}+d\gamma_{4,4}+e\gamma_{2,2}\frac{\partial^2}{\partial x^2}$$

同样，在$y=b/a$的邻域引进两变量$\tilde\alpha$和$\tilde\beta$

$$\tilde\alpha=\frac{\tilde p(x,y)}{\varepsilon},\quad \tilde\beta=y\tag{3.9}$$

将对$y$的偏导数变换成

$$\frac{\partial}{\partial y}=\frac{\partial}{\partial\tilde\alpha}\frac{\partial\tilde\alpha}{\partial y}+\frac{\partial}{\partial\tilde\beta}\frac{\partial\tilde\beta}{\partial y}$$

得到算子展开式

$$\tilde\nabla'^4\equiv\varepsilon^{-4}\sum_{i=0}^{4}\varepsilon^i\tilde D_i'\tag{3.10}$$

式中

$$\tilde D_0'=d\tilde\gamma_{4,0},\quad \tilde\gamma_{4,0}=\tilde p_y^4\frac{\partial^4}{\partial\tilde\alpha^4}$$

最后，从方程(2.10)我们得到关于$L(w(x,y),v(\xi,\eta,y))$，$L(w(x,y),v(x,\alpha,\beta))$，$L(v(\xi,\eta,y),h(\xi,\eta,y))$，$L(v(x,\alpha,\beta),h(x,\alpha,\beta))$和$L(v(\xi,\eta,y),h(x,\alpha,\beta))$的展开式如下：

$$L(w(x,y),\ v^{(1)}(\xi,\eta,y))=\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^iR_i^{(1)}(w,v^{(1)})\tag{3.11}$$

式中

$$R_0^{(1)}=w_{yy}\delta_{2,0}v^{(1)}$$

$$R_1^{(1)}=w_{yy}\delta_{2,1}v^{(1)}-2w_{xy}\delta_{1,0}v_y^{(1)}$$

$$R_2^{(1)}=w_{xx}v_{yy}^{(1)}+w_{yy}\delta_{2,2}v^{(1)}-2w_{xy}\delta_{1,1}v_y^{(1)}$$

$$L(w(x,y),v^{(3)}(x,\alpha,\beta))=\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^i R_i^{(3)}(w,v^{(3)}) \tag{3.12}$$

式中

$$R_0^{(3)}=w_{xx}\gamma_{2,0}v^{(3)}$$

$$R_1^{(3)}=w_{xx}\gamma_{2,1}v^{(3)}-2w_{xy}\gamma_{1,0}v_x^{(3)}$$

$$R_2^{(3)}=w_{yy}v_{xx}^{(3)}+w_{xx}\gamma_{2,2}v^{(3)}-2w_{xy}\gamma_{1,1}v_x^{(3)}$$

$$L(v^{(1)}(\xi,\eta,y),\ h^{(1)}(\xi,\eta,y))=\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^i M_i^{11}(v^{(1)},h^{(1)}) \tag{3.13}$$

式中

$$M_0^{11}=h_{yy}\delta_{2,0}v^{(1)}+v_{yy}\delta_{2,0}h^{(1)}-2\delta_{1,0}v_y^{(1)}\delta_{1,0}h_y^{(1)}$$

$$M_1^{11}=h_{yy}\delta_{2,1}v^{(1)}+v_{yy}\delta_{2,1}h^{(1)}-2\delta_{1,1}v_y^{(1)}\delta_{1,0}h_y^{(1)}-2\delta_{1,0}v_y^{(1)}\delta_{1,1}h_y^{(1)}$$

$$M_2^{11}=h_{yy}\delta_{2,2}v^{(1)}+v_{yy}\delta_{2,2}h^{(1)}-2\delta_{1,1}v_y^{(1)}\delta_{1,1}h_y^{(1)}$$

$$L(v^{(3)}(x,\alpha,\beta),\ h^{(3)}(x,\alpha,\beta))=\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^i N_i^{33}(v^{(3)},h^{(3)}) \tag{3.14}$$

式中

$$N_0^{33}=h_{xx}\gamma_{2,0}v^{(3)}+v_{xx}\gamma_{2,0}h^{(3)}-2\gamma_{1,0}v_x^{(3)}\gamma_{1,0}h_x^{(3)}$$

$$N_1^{33}=h_{xx}\gamma_{2,1}v^{(3)}+v_{xx}\gamma_{2,1}h^{(3)}-2\gamma_{1,1}v_x^{(3)}\gamma_{1,0}h_x^{(3)}-2\gamma_{1,0}v_x^{(3)}\gamma_{1,1}h_x^{(3)}$$

$$N_2^{33}=h_{xx}\gamma_{2,2}v^{(3)}+v_{xx}\gamma_{2,2}h^{(3)}-2\gamma_{1,1}v_x^{(3)}\gamma_{1,1}h_x^{(3)}$$

$$L(v^{(1)}(\xi,\eta,y),h^{(3)}(x,\alpha,\beta))=\varepsilon^{-4}\sum_{i=0}^{4}\varepsilon^i K_i^{13}(v^{(1)},h^{(3)}) \tag{3.15}$$

式中

$$K_0^{13}=\delta_{2,0}v^{(1)}\gamma_{2,0}h^{(3)}$$

$$K_1^{13}=\delta_{2,1}v^{(1)}\gamma_{2,0}h^{(3)}+\delta_{2,0}v^{(1)}\gamma_{2,1}h^{(3)}$$

$$K_2^{13}=\delta_{2,2}v^{(1)}\gamma_{2,0}h^{(3)}+\delta_{2,1}v^{(1)}\gamma_{2,1}h^{(3)}+\delta_{2,0}v^{(1)}\gamma_{2,2}h^{(3)}-2\delta_{1,0}v_y^{(1)}\gamma_{1,0}h_x^{(3)}$$

$$K_3^{13}=\delta_{2,2}v^{(1)}\gamma_{2,1}h^{(3)}+\delta_{2,1}v^{(1)}\gamma_{2,2}h^{(3)}-2(\delta_{1,1}v_y^{(1)}\gamma_{1,0}h_x^{(3)}+\delta_{1,0}v_y^{(1)}\gamma_{1,1}h_x^{(3)})$$

$$K_4^{13}=\delta_{2,2}v^{(1)}\gamma_{2,2}h^{(3)}+v_{yy}^{(1)}h_{xx}^{(3)}-2\delta_{1,1}v_y^{(1)}\gamma_{1,1}h_x^{(3)}$$

以及

$$
\left.\begin{aligned}
&L(w(x,y),v^{(2)}(\xi,\tilde{\eta},y))=\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^{i}R_{i}^{(2)}(w,v^{(2)})\\
&L(w(x,y),v^{(4)}(x,\tilde{\alpha},\tilde{\beta}))=\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^{i}R_{i}^{(4)}(w,v^{(4)})\\
&L(v^{(2)}(\xi,\tilde{\eta},y),h^{(1)}(\xi,\eta,y))=\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^{i}M_{i}^{21}(v^{(2)},h^{(1)})\\
&L(v^{(3)}(x,\alpha,\beta),h^{(4)}(x,\tilde{\alpha},\tilde{\beta}))=\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^{i}N_{i}^{34}(v^{(3)},h^{(4)})
\end{aligned}\right\}\tag{3.16}
$$

式中

$$R_0^{(2)}=w_{yy}\tilde{\delta}_{2,0}v^{(2)},\qquad R_0^{(4)}=w_{xx}\tilde{\gamma}_{2,0}v^{(4)}$$

$$M_1^{21}=h_{yy}\tilde{\delta}_{2,0}v^{(2)}+v_{yy}\delta_{2,0}h^{(1)}-2\tilde{\delta}_{1,0}v_y^{(2)}\delta_{1,0}h_y^{(1)}$$

$$N_0^{34}=h_{xx}\gamma_{2,0}v^{(3)}+v_{xx}\gamma_{2,0}h^{(4)}-2\gamma_{1,0}v_x^{(3)}\gamma_{1,0}h_x^{(4)}$$

…………

## 四、递推方程和边界条件

假设挠度$w$和应力函数$\varphi$的$N$阶近似式为

$$
\begin{aligned}
w_N(x,y,\varepsilon)=&\sum_{n=0}^{N}\varepsilon^n w_n(x,y)+\sum_{n=0}^{N}\varepsilon^{n+\alpha_1}v_n^{(1)}(\xi,\eta,y)+\sum_{n=0}^{N}\varepsilon^{n+\alpha_2}v_n^{(2)}(\xi,\tilde{\eta},y)\\
&+\sum_{n=0}^{N}\varepsilon^{n+\alpha_3}v_n^{(3)}(x,\alpha,\beta)+\sum_{n=0}^{N}\varepsilon^{n+\alpha_4}v_n^{(4)}(x,\tilde{\alpha},\tilde{\beta})
\end{aligned}\tag{4.1}
$$

$$
\begin{aligned}
\Phi_N(x,y,\varepsilon)=&\sum_{n=0}^{N}\varepsilon^n \varphi_n(x,y)+\sum_{n=0}^{N}\varepsilon^{n+\beta_1}h_n^{(1)}(\xi,\eta,y)+\sum_{n=0}^{N}\varepsilon^{n+\beta_2}h_n^{(2)}(\xi,\tilde{\eta},y)\\
&+\sum_{n=0}^{N}\varepsilon^{n+\beta_3}h_n^{(3)}(x,\alpha,\beta)+\sum_{n=0}^{N}\varepsilon^{n+\beta_4}h_n^{(4)}(x,\tilde{\alpha},\tilde{\beta})
\end{aligned}\tag{4.2}
$$

式中 $v_n^{(1)}$，$h_n^{(1)}$和$v_n^{(2)}$，$h_n^{(2)}$分别是$x=0$和$x=1$的邻域的边界层型函数；$v_n^{(3)}$，$h_n^{(3)}$和$v_n^{(4)}$，$h_n^{(4)}$分别是$y=0$和$y=b/a$的邻域的边界层型函数．$\alpha_1,\cdots,\alpha_4$；$\beta_1\cdots,\beta_4$则为待定常数．

将方程(4.1)和(4.2)代入方程(2.9a,b)，并引入展开式(3.11)～(3.16)，得到

$$
\begin{aligned}
\Pi_\varepsilon(W_N,\Phi_N)\equiv&\left\{\varepsilon^2\nabla_1^4\left(\sum_{n=0}^{N}\varepsilon^n w_n\right)-L\left(\sum_{n=0}^{N}\varepsilon^n w_n,\sum_{n=0}^{N}\varepsilon^n\varphi_n\right)\right\}\\
&+\left\{\varepsilon^{-2}\left[\sum_{i=0}^{4}\varepsilon^i D_i\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_1}v_n^{(1)}\right)-\sum_{i=0}^{2}\varepsilon^i M_i^{11}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_1}v_n^{(1)},\ \sum_{n=0}^{N}\varepsilon^{n+\beta_1}h_n^{(1)}\right)\right]\right.
\end{aligned}
$$

$$-\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^{i}\left[R_i^{(1)}\left(\sum_{n=0}^{N}\varepsilon^{n}w_n,\sum_{n=0}^{N}\varepsilon^{n+\beta_1}h_n^{(1)}\right)+R_i^{(1)}\left(\sum_{n=0}^{N}\varepsilon^{n}\varphi_n,\sum_{n=0}^{N}\varepsilon^{n+\alpha_1}v_n^{(1)}\right)\right]\Bigg\}$$

$$+\Bigg\{\varepsilon^{-2}\left[\sum_{i=0}^{4}\varepsilon_i D_i'\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_3}v_n^{(3)}\right)-\sum_{i=0}^{2}\varepsilon^{i}N_i^{33}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_3}v_n^{(3)},\ \sum_{n=0}^{N}\varepsilon^{n+\beta_3}h_n^{(3)}\right)\right]$$

$$-\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^{i}\left[R_i^{(3)}\left(\sum_{n=0}^{N}\varepsilon^{n}w_n,\sum_{n=0}^{N}\varepsilon^{n+\beta_3}h_n^{(3)}\right)+R_i^{(3)}\left(\sum_{n=0}^{N}\varepsilon_n\varphi_n,\sum_{n=0}^{N}\varepsilon^{n+\alpha_3}v_n^{(3)}\right)\right]\Bigg\}$$

$$+\Bigg\{\varepsilon^{-2}\left[\sum_{i=0}^{4}\varepsilon^{i}\widetilde{D}_i\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_2}v_n^{(2)}\right)-\sum_{i=0}^{2}\varepsilon^{i}M_i^{22}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_2}v_n^{(2)},\ \sum_{n=0}^{N}\varepsilon^{n+\beta_2}h_n^{(2)}\right)\right]$$

$$-\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^{i}\left[R_i^{(2)}\left(\sum_{n=0}^{N}\varepsilon^{n}w_n,\sum_{n=0}^{N}\varepsilon^{n+\beta_2}h_n^{(2)}\right)+R_i^{(2)}\left(\sum_{n=0}^{N}\varepsilon^{n}\varphi_n.\sum_{n=0}^{N}\varepsilon^{n+\alpha_2}v_n^{(2)}\right)\right]\Bigg\}$$

$$+\Bigg\{\varepsilon^{-2}\left[\sum_{i=0}^{4}\varepsilon_i\widetilde{D}_i'\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_4}v_n^{(4)}\right)-\sum_{i=0}^{2}\varepsilon^{i}N_i^{44}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_4}v_n^{(4)},\ \sum_{n=0}^{N}\varepsilon^{n+\beta_4}h_n^{(4)}\right)\right]$$

$$-\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^{i}\left[R_i^{(4)}\left(\sum_{n=0}^{N}\varepsilon^{n}w_n,\ \sum_{n=0}^{N}\varepsilon^{n+\beta_4}h_n^{(4)}\right)+R_i^{(4)}\left(\sum_{n=0}^{N}\varepsilon^{n}\varphi_n,\sum_{n=0}^{N}\varepsilon^{n+\alpha_4}v_n^{(4)}\right)\right]\Bigg\}$$

$$-\Bigg\{\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^{i}\left[M_i^{12}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_1}v_n^{(1)},\ \sum_{n=0}^{N}\varepsilon^{n+\beta_2}h_n^{(2)}\right)+M_i^{21}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_2}v_n^{(2)},\sum_{n=0}^{N}\varepsilon^{n+\beta_1}h_n^{(1)}\right)\right.$$

$$\left.+N_i^{34}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_3}v_n^{(3)},\sum_{n=0}^{N}\varepsilon^{n+\beta_4}h_n^{(4)}\right)+N_i^{43}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_4}v_n^{(4)},\sum_{n=0}^{N}\varepsilon^{n+\beta_3}h_n^{(3)}\right)\right]$$

$$+\varepsilon^{-4}\sum_{i=0}^{4}\varepsilon^{i}\left[K_i^{31}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_3}v_n^{(3)},\sum_{n=0}^{N}\varepsilon^{n+\beta_1}h_n^{(1)}\right)+K_i^{32}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_3}v_n^{(3)},\sum_{n=0}^{N}\varepsilon^{n+\beta_2}h_n^{(2)}\right)\right.$$

$$+K_i^{41}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_4}v_n^{(4)},\sum_{n=0}^{N}\varepsilon^{n+\beta_1}h_n^{(1)}\right)+K_i^{42}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_4}v_n^{(4)},\sum_{n=0}^{N}\varepsilon^{n+\beta_2}h_n^{(2)}\right)$$

$$+K_i^{13}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_1}v_n^{(1)},\sum_{n=0}^{N}\varepsilon^{n+\beta_3}h_n^{(3)}\right)+K_i^{14}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_1}v_n^{(1)},\sum_{n=0}^{N}\varepsilon^{n+\beta_4}h_n^{(4)}\right)$$

$$\left.+K_i^{23}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_2}v_n^{(2)},\sum_{n=0}^{N}\varepsilon^{n+\beta_3}h_n^{(3)}\right)+K_i^{24}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_2}v_n^{(2)},\sum_{n=0}^{N}\varepsilon^{n+\beta_4}h_n^{(4)}\right)\right]\Bigg\}\qquad(4.3)$$

$$\Pi\ (W_N,\Phi_N)\equiv\left\{\nabla_2^4\left(\sum_{n=0}^{N}\varepsilon^{n}\varphi_n\right)+\frac{L}{2}\left(\sum_{n=0}^{N}\varepsilon^{n}w_n,\ \sum_{n=0}^{N}\varepsilon^{n}w_n\right)\right\}$$

$$+\left\{\varepsilon^{-4}\sum_{i=0}^{4}\varepsilon^{i}D'_{i}\left(\sum_{n=0}^{N}\varepsilon^{n+\beta_1}h_n^{(1)}\right)+\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^{i}R_i^{(1)}\left(\sum_{n=0}^{N}\varepsilon^{n}w_n,\sum_{n=0}^{N}\varepsilon^{n+\alpha_1}v_n^{(1)}\right)\right.$$

$$\left.+\frac{1}{2}\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^{i}M_i^{11}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_1}v_n^{(1)},\sum_{n=0}^{N}\varepsilon^{n+\alpha_1}v_n^{(1)}\right)\right\}$$

$$+\left\{\varepsilon^{-4}\sum_{i=1}^{4}\varepsilon^{i}D'_{i}\left(\sum_{n=0}^{N}\varepsilon^{n+\beta_3}h_n^{(3)}\right)+\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^{i}R_i^{(3)}\left(\sum_{n=0}^{N}\varepsilon^{n}w_n,\sum_{n=0}^{N}\varepsilon^{n+\alpha_3}v_n^{(3)}\right)\right.$$

$$\left.+\frac{1}{2}\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^{i}N_i^{33}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_3}v_n^{(3)},\sum_{n=0}^{N}\varepsilon^{n+\alpha_3}v_n^{(3)}\right)\right\}$$

$$+\left\{\varepsilon^{-4}\sum_{i=0}^{4}\varepsilon^{i}\widetilde{D}_{i}\left(\sum_{n=0}^{N}\varepsilon^{n+\beta_2}h_n^{(2)}\right)+\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^{i}R_i^{(2)}\left(\sum_{n=0}^{N}\varepsilon^{n}w_n,\sum_{n=0}^{N}\varepsilon^{n+\alpha_2}v_n^{(2)}\right)\right.$$

$$\left.+\frac{1}{2}\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^{i}M_i^{22}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_2}v_n^{(2)},\sum_{n=0}^{N}\varepsilon^{n+\alpha_2}v_n^{(2)}\right)\right\}$$

$$+\left\{\varepsilon^{-4}\sum_{i=0}^{4}\varepsilon_{i}\widetilde{D}'_{i}\left(\sum_{n=0}^{N}\varepsilon^{n+\beta_4}h_n^{(4)}\right)+\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^{i}R_i^{(4)}\left(\sum_{n=0}^{N}\varepsilon^{n}w_n,\sum_{n=0}^{N}\varepsilon^{n+\alpha_4}v_n^{(4)}\right)\right.$$

$$\left.+\frac{1}{2}\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^{i}N_i^{44}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_4}v_n^{(4)},\sum_{n=0}^{N}\varepsilon^{n+\alpha_4}v_n^{(4)}\right)\right\}$$

$$+\left\{\varepsilon^{-2}\sum_{i=0}^{2}\varepsilon^{i}\left[M_i^{12}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_1}v_n^{(1)},\sum_{n=0}^{N}\varepsilon^{n+\alpha_2}v_n^{(2)}\right)+N_i^{34}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_3}v_n^{(3)},\sum_{n=0}^{N}\varepsilon^{n+\alpha_4}v_n^{(4)}\right)\right]\right.$$

$$+\varepsilon^{-4}\sum_{i=0}^{4}\varepsilon^{i}\left[K_i^{31}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_3}v_n^{(3)},\sum_{n=0}^{N}\varepsilon^{n+\alpha_1}v_n^{(1)}\right)+K_i^{32}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_3}v_n^{(3)},\sum_{n=0}^{N}\varepsilon^{n+\alpha_2}v_n^{(2)}\right)\right.$$

$$\left.\left.+K_i^{41}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_4}v_n^{(4)},\sum_{n=0}^{N}\varepsilon^{n+\alpha_1}v_n^{(1)}\right)+K_i^{42}\left(\sum_{n=0}^{N}\varepsilon^{n+\alpha_4}v_n^{(4)},\sum_{n=0}^{N}\varepsilon^{n+\alpha_2}v_n^{(2)}\right)\right]\right\}\tag{4.4}$$

比较等式两边$\varepsilon$的各次幂的系数，我们看到在(4.3)式的第一大括号中，$\varepsilon^0$前的系数应等于$q$，$\varepsilon$的其它幂次的系数应等于零；在(4.4)式的第一大括号中，$\varepsilon$的各次幂的系数皆等于零．于是，便得到$w_n$和$\varphi_n$的递推方程如下：

$$L(w_0,\varphi_0)=-q,\quad \nabla_2^2\varphi_0+\frac{L}{2}(w_0,w_0)=0\tag{4.5}$$

$$\begin{cases} L(w_0,\varphi_n)+L(w_n,\varphi_0)=\nabla_1^4 w_{n-2}-\sum_{i=1}^{n-1}L(w_i,\varphi_{n-i}) & (4.6a) \\ \nabla_2^4\varphi_n+L(w_0,w_n)=-\frac{1}{2}\sum_{i=1}^{n-1}L(w_i,w_{n-i}) \qquad (n=1,2,\cdots,N) & (4.6b) \end{cases}$$

注意，在这里和以后各式中，我们都将负下标的量取为零。

再将(4.1)式代入边界条件(2.11)～(2.14)，得

$$\sum_{n=0}^{N}\varepsilon^n w_n\Big|_{x=0}+\varepsilon^{\alpha_1}\sum_{n=0}^{N}\varepsilon^n v_n^{(1)}\Big|_{\eta=0}=f_1(y) \tag{4.7}$$

$$\sum_{n=0}^{N}\varepsilon^n w_{n,x}\Big|_{x=0}+\varepsilon^{\alpha_1}\cdot\varepsilon^{-1}(\delta_{1,0}+\varepsilon\delta_{1,1})\sum_{n=0}^{N}\varepsilon^n v_n^{(1)}\Big|_{\eta=0}=g_1(y) \tag{4.8}$$

$$\sum_{n=0}^{N}\varepsilon^n w_n\Big|_{x=1}+\varepsilon^{\alpha_2}\sum_{n=0}^{N}\varepsilon^n v_n^{(2)}\Big|_{\tilde{\eta}=1}=f_2(y) \tag{4.9}$$

$$\sum_{n=0}^{N}\varepsilon^n w_{n,x}\Big|_{x=1}+\varepsilon^{\alpha_2}\cdot\varepsilon^{-1}(\tilde{\delta}_{1,0}+\varepsilon\tilde{\delta}_{1,1})\sum_{n=0}^{N}\varepsilon^n v_n^{(2)}\Big|_{\tilde{\eta}=1}=g_2(y) \tag{4.10}$$

$$\sum_{n=0}^{N}\varepsilon^n w_n\Big|_{y=0}+\varepsilon^{\alpha_3}\sum_{n=0}^{N}\varepsilon^n v_n^{(3)}\Big|_{\beta=0}=f_3(x) \tag{4.11}$$

$$-D_2\left\{\left[\sum_{n=0}^{N}\varepsilon^n w_{n,yy}+\mu_1\sum_{n=0}^{N}\varepsilon^n w_{n,xx}\right]_{y=0}+\varepsilon^{\alpha_3}\cdot\varepsilon^{-2}(\gamma_{2,0}+\varepsilon\gamma_{2,1}\right.$$
$$\left.+\varepsilon^2\gamma_{2,2})\sum_{n=0}^{N}\varepsilon^n v_n^{(3)}\Big|_{\beta=0}\right\}=g_3(x) \tag{4.12}$$

$$\sum_{n=0}^{N}\varepsilon^n w_n\Big|_{y=\frac{b}{a}}+\varepsilon^{\alpha_4}\sum_{n=0}^{N}\varepsilon^n v_n^{(4)}\Big|_{\tilde{\beta}=\frac{b}{a}}=f_4(x) \tag{4.13}$$

$$-D_2\left\{\left[\sum_{n=0}^{N}\varepsilon^n w_{n,yy}+\mu_1\sum_{n=0}^{N}\varepsilon^n w_{n,xx}\right]_{y=\frac{b}{a}}+\varepsilon^{\alpha_4}\cdot\varepsilon^{-2}(\tilde{\gamma}_{2,0}+\varepsilon\tilde{\gamma}_{2,1}\right.$$
$$\left.+\varepsilon^2\tilde{\gamma}_{2,2})\sum_{n=0}^{N}\varepsilon^n v_n^{(4)}\Big|_{\tilde{\beta}=\frac{b}{a}}\right\}=g_4(x) \tag{4.14}$$

从(4.8)和(4.10)式我们看出应取$\alpha_1=\alpha_2=1$，从(4.12)和(4.14)式我们看出应取$\alpha_3=\alpha_4=2$。逐次地比较等式两端$\varepsilon$的各次幂的系数，得到关于$w_n$，$v_n^{(1)}$，$v_n^{(2)}$，$v_n^{(3)}$和$v_n^{(4)}$的边界条件：

$$w_0|_{x=0}=f_1(y), \qquad w_n|_{x=0}+v^{(1)}_{n-1}|_{\eta=0}=0 \tag{4.15}$$

$$w_{0,x}|_{x=0}+\delta_{1,0}v^{(1)}_0|_{\eta=0}=g_1(y), \quad w_{n,x}|_{x=0}+(\delta_{1,0}v^{(1)}_n+\delta_{1,1}v^{(1)}_{n-1})|_{\eta=0}=0 \tag{4.16}$$

$$w_0|_{x=1}=f_2(y), \qquad w_n|_{x=1}+v^{(2)}_{n-1}|_{\bar\eta=1}=0 \tag{4.17}$$

$$w_{0,x}|_{x=1}+\tilde\delta_{1,0}v^{(2)}_0|_{\bar\eta=1}=g_2(y), \quad w_{n,x}|_{x=1}+(\tilde\delta_{1,0}v^{(2)}_n+\tilde\delta_{1,1}v^{(2)}_{n-1})|_{\bar\eta=1}=0 \tag{4.18}$$

$$w_0|_{y=0}=f_3(x), \qquad w_n|_{y=0}+v^{(3)}_{n-2}|_{\beta=0}=0 \tag{4.19}$$

$$\left.\begin{aligned}&-D_2\left\{\left[w_{0,yy}+\mu_1 w_{0,xx}\right]_{y=0}+\gamma_{2,0}\,v^{(3)}_0\Big|_{\beta=0}\right\}=g_3(x)\\&-D_2\left\{\left[w_{n,yy}+\mu_1 w_{n,xx}\right]_{y=0}+(\gamma_{2,0}\,v^{(3)}_n+\gamma_{2,1}\,v^{(3)}_{n-1}+\gamma_{2,2}\,v^{(3)}_{n-2})\Big|_{\beta=0}\right\}=0\end{aligned}\right\} \tag{4.20}$$

$$w_0|_{y=\frac{b}{a}}=f_4(x), \qquad w_n|_{y=\frac{b}{a}}+v^{(4)}_{n-2}\Big|_{\bar\beta=\frac{b}{a}}=0 \tag{4.21}$$

$$\left.\begin{aligned}&-D_2\left\{\left[w_{0,yy}+\mu_1\,w_{0,xx}\right]_{y=\frac{b}{a}}+\tilde\gamma_{2,0}v^{(4)}_0\Big|_{\bar\beta=\frac{b}{a}}\right\}=g_4(x)\\&-D_2\left\{\left[w_{n,yy}+\mu_1\,w_{n,xx}\right]_{y=\frac{b}{a}}+\left(\tilde\gamma_{2,0}v^{(4)}_n+\tilde\gamma_{2,1}v^{(4)}_{n-1}+\tilde\gamma_{2,2}v^{(4)}_{n-2}\right)\Big|_{\bar\beta=\frac{b}{a}}\right\}=0\end{aligned}\right\}$$

$$(n=1,2,\cdots,N) \tag{4.22}$$

然后，我们在(4.3)和(4.4)式的第二至第五大括号中逐个比较$\varepsilon$最低次幂的系数，看到应取$\beta_1=\beta_2=3$，$\beta_3=\beta_4=4$．在(4.3)式的第二至第五大括号中逐个地令$\varepsilon$的各次幂的系数为零，得到边界层函数$v^{(1)}_n, v^{(2)}_n, v^{(3)}_n$和$v^{(4)}_n$的递推方程

$$D_0v^{(1)}_0-R^{(1)}_0(\varphi_0,\ v^{(1)}_0)=0 \tag{4.23}$$

$$\begin{aligned}D_0v^{(1)}_n-R^{(1)}_0(\varphi_0,v^{(1)}_n)=&\sum_{\substack{j+k=n\\(j\neq0)}}R^{(1)}_0(\varphi_j,v^{(1)}_k)+\sum_{i=1}^{2}\sum_{j+k=n-i}R^{(1)}_i(\varphi_j,v^{(1)}_k)\\&+\sum_{i=0}^{2}\sum_{j+k=n-2-i}R^{(1)}_i(w_j,h^{(1)}_k)+\sum_{i=0}^{2}\sum_{j+k=n-3-i}M^{(1)}_i(v^{(1)}_j,h^{(1)}_k)-\sum_{i=1}^{4}D_iv^{(1)}_{n-i}\end{aligned} \tag{4.24}$$

$$D'_0v^{(3)}_0-R^{(3)}_0(\varphi_0,v^{(3)}_0)=0 \tag{4.25}$$

$$\begin{aligned}D'_0v^{(3)}_n-R^{(3)}_0(\varphi_0,v^{(3)}_n)=&\sum_{\substack{j+k=n\\(j\neq0)}}R^{(3)}_0(\varphi_j,v^{(3)}_k)+\sum_{i=1}^{2}\sum_{j+k=n-i}R^{(3)}_i(\varphi_j,v^{(3)}_k)\\&+\sum_{i=0}^{2}\sum_{j+k=n-2-i}R^{(3)}_i(w_j,h^{(3)}_k)+\sum_{i=0}^{2}\sum_{j+k=n-4-i}N^{(3)}_i(v^{(3)}_j,h^{(3)}_k)-\sum_{i=1}^{4}D'_iv^{(3)}_{n-i}\end{aligned} \tag{4.26}$$

$$\tilde{D}_0 v_0^{(2)} - R_0^{(2)}(\varphi_0, v_0^{(2)}) = 0 \tag{4.27}$$

$$\tilde{D}_0 v_n^{(2)} - R_0^{(2)}(\varphi_0, v_n^{(2)}) = \sum_{\substack{j+k=n \\ (j\neq 0)}} R_0^{(2)}(\varphi_j, v_k^{(2)}) + \sum_{i=1}^{2} \sum_{j+k=n-i} R_i^{(2)}(\varphi_j, v_k^{(2)})$$

$$+ \sum_{i=0}^{2} \sum_{j+k=n-2-i} R_i^{(2)}(w_j, h_k^{(2)}) + \sum_{i=0}^{2} \sum_{j+k=n-3-i} M_i^{22}(v_j^{(2)}, h_k^{(2)}) - \sum_{i=1}^{4} \tilde{D}_i v_{n-i}^{(2)} \tag{4.28}$$

$$\tilde{D}_0' v_0^{(4)} - R_0^{(4)}(\varphi_0, v_0^{(4)}) = 0 \tag{4.29}$$

$$\tilde{D}_0' v_n^{(4)} - R_0^{(4)}(\varphi_0, v_n^{(4)}) = \sum_{\substack{j+k=n \\ (j\neq 0)}} R_0^{(4)}(\varphi_j, v_k^{(4)}) + \sum_{i=1}^{2} \sum_{j+k=n-i} R_i^{(4)}(\varphi_j, v_k^{(4)})$$

$$+ \sum_{i=0}^{2} \sum_{j+k=n-2-i} R_i^{(4)}(w_j, h_k^{(4)}) + \sum_{i=0}^{2} \sum_{j+k=n-4-i} N_i^{44}(v_j^{(4)}, h_k^{(4)}) - \sum_{i=1}^{4} \tilde{D}_i' v_{n-i}^{(4)}$$

$$(n=1,2,\cdots,N) \tag{4.30}$$

再在(4.4)式的第二至第五大括号中逐个令 $\varepsilon$ 的各次幂的系数为零，得 到$h_n^{(1)}, h_n^{(2)}, h_n^{(3)}$ 和 $h_n^{(4)}$ 的递推方程

$$D_0 h_0^{(1)} = - R_0^{(1)}(w_0, v_0^{(1)}) \tag{4.31}$$

$$D_0 h_n^{(1)} = -\sum_{i=0}^{2} \sum_{j+k=n-i} R_i^{(1)}(w_j, v_k^{(1)})$$

$$-\frac{1}{2}\sum_{i=0}^{2} \sum_{j+k=n-1-i} M_i^{11}(v_j^{(1)}, v_k^{(1)}) - \sum_{i=1}^{4} D_i h_{n-i}^{(1)} \tag{4.32}$$

$$D_0' h_0^{(3)} = - R_0^{(3)}(w_0, v_0^{(3)}) \tag{4.33}$$

$$D_0' h_n^{(3)} = -\sum_{i=0}^{2} \sum_{j+k=n-i} R_i^{(3)}(w_j, v_k^{(3)})$$

$$-\frac{1}{2}\sum_{i=0}^{2} \sum_{j+k=n-2-i} N_i^{33}(v_j^{(3)}, v_k^{(3)}) - \sum_{i=1}^{4} D_i' h_{n-i}^{(3)} \tag{4.34}$$

$$\tilde{D}_0 h_0^{(2)} = - R_0^{(2)}(w_0, v_0^{(2)}) \tag{4.35}$$

$$\tilde{D}_0 h_n^{(2)} = -\sum_{i=0}^{2} \sum_{j+k=n-i} R_i^{(2)}(w_j, v_k^{(2)})$$

$$-\frac{1}{2}\sum_{i=0}^{2} \sum_{j+k=n-1-i} M_i^{22}(v_j^{(2)}, v_k^{(2)}) - \sum_{i=1}^{4} \tilde{D}_i h_{n-i}^{(2)} \tag{4.36}$$

$$\tilde{D}_0^{\prime}h_0^{(4)}=-R_0^{(4)}(w_0,v_0^{(4)}) \tag{4.37}$$

$$\tilde{D}_0^{\prime}h_n^{(4)}=-\sum_{i=0}^{2}\sum_{j+k=n-i}R_i^{(4)}(w_j,v_k^{(4)})-\frac{1}{2}\sum_{i=0}^{2}\sum_{j+k=n-2-i}N_i^{44}(v_j^{(4)},v_k^{(4)})-\sum_{i=1}^{4}\tilde{D}_i^{\prime}h_{n-i}^{(4)}$$

$$(n=1,2,\cdots,N) \tag{4.38}$$

最后，将(4.2)式代入方程(2.15)～(2.18)，得到关于$\varphi_n$，$h_n^{(1)}$，$h_n^{(2)}$，$h_n^{(3)}$和$h_n^{(4)}$的边界条件

$$\varphi_{0,yy}\Big|_{x=0}=h_1(y),\qquad \varphi_{n,yy}\Big|_{x=0}+h_{n-3,yy}^{(1)}\Big|_{\eta=0}=0 \tag{4.39}$$

$$\varphi_{0,xy}\Big|_{x=0}=I_1(y),\qquad \varphi_{n,xy}\Big|_{x=0}+[\delta_{1,0}h_{n-2,y}^{(1)}+\delta_{1,1}h_{n-3,y}^{(1)}]_{\eta=0}=0 \tag{4.40}$$

$$\varphi_{0,yy}\Big|_{x=1}=h_2(y),\qquad \varphi_{n,yy}\Big|_{x=1}+h_{n-3,yy}^{(2)}\Big|_{\bar{\eta}=1}=0 \tag{4.41}$$

$$\varphi_{0,xy}\Big|_{x=1}=I_2(y),\qquad \varphi_{n,xy}\Big|_{x=1}+[\tilde{\delta}_{1,0}h_{n-2,y}^{(2)}+\tilde{\delta}_{1,1}h_{n-3,y}^{(2)}]_{\bar{\eta}=1}=0 \tag{4,42}$$

$$\varphi_{0,xx}\Big|_{y=0}=h_3(x),\qquad \varphi_{n,xx}\Big|_{y=0}+h_{n-4,xx}^{(3)}\Big|_{\beta=0}=0 \tag{4.43}$$

$$\varphi_{0,xy}\Big|_{y=0}=I_3(x),\qquad \varphi_{n,xy}\Big|_{y=0}+[\gamma_{1,0}h_{n-3,x}^{(3)}+\gamma_{1,1}h_{n-4,x}^{(3)}]_{\beta=0}=0 \tag{4.44}$$

$$\varphi_{0,xx}\Big|_{y=\frac{b}{a}}=h_4(x),\quad \varphi_{n,xx}\Big|_{y=\frac{b}{a}}+h_{n-4,xx}^{(4)}\Big|_{\bar{\beta}=\frac{b}{a}}=0 \tag{4.45}$$

$$\varphi_{0,xy}\Big|_{y=\frac{b}{a}}=I_4(x),\quad \varphi_{n,xy}\Big|_{y=\frac{b}{a}}+\left[\tilde{\gamma}_{1,0}h_{n-3,x}^{(4)}+\tilde{\gamma}_{1,1}h_{n-4,x}^{(4)}\right]_{\bar{\beta}=\frac{b}{a}}=0$$

$$(n=1,2,\cdots,N) \tag{4.46}$$

## 五、N阶形式渐近解的导出

方程(4.5)和边界条件(4.15)～(4.21)，(4.39)～(4.45)给出

$$L(w_0,\varphi_0)=-q,\quad \nabla_2^4\varphi_0+L(w_0,w_0)/2=0 \tag{5.1}$$

$$\left.\begin{array}{ll} w_0|_{x=0}=f_1(y), & w_0|_{x=1}=f_2(y) \\ w_0|_{y=0}=f_3(x), & w_0|_{y=\frac{b}{a}}=f_4(x) \\ \varphi_{0,yy}|_{x=0}=h_1(y), & \varphi_{0,yy}|_{x=1}=h_2(y) \\ \varphi_{0,xx}|_{y=0}=h_3(x), & \varphi_{0,xx}|_{y=\frac{b}{a}}=h_4(x) \end{array}\right\} \tag{5.2}$$

式中$w_0$和$\varphi_0$为薄膜理论的解．将求得的$w_0$和$\varphi_0$代入方程(4.23)和(4.27)（这里$c=1$），得

$$u_x^4\ \frac{\partial^4 v_0^{(1)}}{\partial\xi^4}-\varphi_{0,yy}\cdot u_x^2\ \frac{\partial^2 v_0^{(1)}}{\partial\xi^2}=0 \tag{5.3a}$$

$$\tilde{u}_x^4\ \frac{\partial^4 v_0^{(2)}}{\partial\bar{\xi}^4}-\varphi_{0,yy}\cdot \tilde{u}_x^2\ \frac{\partial^2 v_0^{(2)}}{\partial\bar{\xi}^2}=0 \tag{5.3b}$$

选取待定函数

$$u(x,y)=\int_0^x\sqrt{\varphi_{0,yy}(x,y)}\,dx\,,\quad \tilde{u}(x,y)=\int_x^1\sqrt{\varphi_{0,yy}(x,y)}\,dx \tag{5.4}$$

并假设$h_1(y)>0, h_2(y)>0$. 则方程(5.3a,b)化为

$$\frac{\partial^4 v_0^{(1)}}{\partial\xi^4}-\frac{\partial^2 v_0^{(1)}}{\partial\xi^2}=0,\quad \frac{\partial^4 v_0^{(2)}}{\partial\tilde{\xi}^4}-\frac{\partial^2 v_0^{(2)}}{\partial\tilde{\xi}^2}=0 \tag{5.5}$$

其解为

$$v_0^{(1)}(\xi,\eta,y)=C_0^{(1)}(\eta,y)e^{-\xi}=C_0^{(1)}(x,y)\exp\left[-\frac{1}{\varepsilon}\int_0^x\sqrt{\varphi_{0,yy}(x,y)}\,dx\right] \tag{5.6a}$$

$$v_0^{(2)}(\tilde{\xi},\tilde{\eta},y)=C_0^{(2)}(\tilde{\eta},y)e^{-\tilde{\xi}}=C_0^{(2)}(x,y)\exp\left[-\frac{1}{\varepsilon}\int_x^1\sqrt{\varphi_{0,yy}(x,y)}\,dx\right] \tag{5.6b}$$

式中 $C_0^{(1)}(\eta,y)$ 和 $C_0^{(2)}(\tilde{\eta},y)$ 是任意函数，将分别由下面导出的关于 $C_0^{(1)}(\eta,y)$和 $C_0^{(2)}(\tilde{\eta},y)$ 的一阶线性偏微分方程和边值条件确定.

从方程(4.16)，我们得到 $C_0^{(1)}(\eta,y)$的边界条件是

$$C_0^{(1)}(\eta,y)\Big|_{\eta=0}=-\frac{g_1(y)-w_{0,x}(0,y)}{\sqrt{h_1(y)}} \tag{5.7}$$

同样，从方程(4.18)得到 $C_0^{(2)}(\tilde{\eta},y)$ 的边界条件是

$$C_0^{(2)}(\tilde{\eta},y)\Big|_{\tilde{\eta}=1}=-\frac{g_2(y)-w_{0,x}(1,y)}{\sqrt{h_2(y)}} \tag{5.8}$$

将$\varphi_0$和$w_0$代入方程(4.25)和(4.29)，有

$$d\cdot p_y^4\frac{\partial^4 v_0^{(3)}}{\partial\alpha^4}-\varphi_{0,xx}\cdot p_y^2\frac{\partial^2 v_0^{(3)}}{\partial\alpha^2}=0 \tag{5.9a}$$

$$d\cdot\tilde{p}_y^4\frac{\partial^4 v_0^{(4)}}{\partial\tilde{\alpha}^4}-\varphi_{0,xx}\cdot\tilde{p}_y^2\frac{\partial^2 v_0^{(4)}}{\partial\tilde{\alpha}^2}=0 \tag{5.9b}$$

式中$d=D_2/D_1=E_2/E_1$. 取待定函数

$$p(x,y)=\int_0^y\sqrt{\varphi_{0,xx}(x,y)}\,dy,\quad \tilde{p}(x,y)=\int_y^{b/a}\sqrt{\varphi_{0,xx}(x,y)}\,dy \tag{5.10}$$

并假设$h_3(x)>0, h_4(x)>0$. 方程(5.9a.b)成为

$$d\frac{\partial^4 v_0^{(3)}}{\partial\alpha^4}-\frac{\partial^2 v_0^{(3)}}{\partial\alpha^2}=0,\quad d\frac{\partial^4 v_0^{(4)}}{\partial\tilde{\alpha}^4}-\frac{\partial^2 v_0^{(4)}}{\partial\tilde{\alpha}^2}=0 \tag{5.11}$$

$$\therefore\quad v_0^{(3)}(x,\alpha,\beta)=C_0^{(3)}(x,\beta)\exp\left[-\frac{\alpha}{\sqrt{d}}\right]=C_0^{(3)}(x,y)\exp\left[-\frac{1}{\varepsilon\sqrt{d}}\int_0^y\sqrt{\varphi_{0,xx}(x,y)}\,dy\right] \tag{5.12a}$$

$$\begin{aligned}v_0^{(4)}(x,\tilde{\alpha},\tilde{\beta})&=C_0^{(4)}(x,\tilde{\beta})\exp\left[-\frac{\tilde{\alpha}}{\sqrt{d}}\right]\\&=C_0^{(4)}(x,y)\exp\left[-\frac{1}{\varepsilon\sqrt{d}}\int_y^{b/a}\sqrt{\varphi_{0,xx}(x,y)}\,dy\right]\end{aligned} \tag{5.12b}$$

从方程(4.20)，可求得 $C_0^{(3)}(x,\beta)$的边界条件是

$$C_0^{(3)}(x,\ \beta)\Big|_{\beta=0}=-\frac{\dfrac{1}{D_2}g_3(x)+[w_{0,yy}(x,0)+\mu_1 w_{0,xx}(x,0)]}{h_3(x)} \tag{5.13}$$

同样，从方程(4.22)得到 $C_0^4(x,\tilde{\beta})$ 的边界条件是

$$C_0^{(4)}(x,\ \tilde{\beta})\Big|_{\tilde{\beta}=\frac{b}{a}}=-\frac{\dfrac{1}{D_2}g_4(x)+[w_{0,yy}\,(x,\dfrac{b}{a})+\mu_1\,w_{0,xx}\,(x,\dfrac{b}{a})]}{h_4(x)} \tag{5.14}$$

再将所求得 的 $v_0^{(1)},v_0^{(2)},v_0^{(3)}$和$v_0^{(4)}$分别代入(4.31)，(4.35)，(4.33)和(4.37)式，得

$$h_0^{(1)}(\xi,\ \eta,\ y)=-\frac{1}{H(\eta,y)}w_{0,yy}C_0^{(1)}(\eta,y)e^{-\xi} \tag{5.15}$$

$$h_0^{(2)}(\tilde{\xi},\tilde{\eta},y)=-\frac{1}{H(\tilde{\eta},y)}w_{0,yy}C_0^{(2)}(\tilde{\eta},y)e^{-\tilde{\xi}} \tag{5.16}$$

$$h_0^{(3)}(x,\ \alpha,\beta)=-\frac{1}{d\cdot H'(x,\alpha)}w_{0,xx}C_0^{(3)}(x,\alpha)e^{-\alpha} \tag{5.17}$$

$$h_0^{(4)}(x,\tilde{\alpha},\tilde{\beta})=-\frac{1}{d\cdot H'(x,\tilde{\alpha})}w_{0,xx}C_0^{(4)}(x,\tilde{\alpha})e^{-\tilde{\alpha}} \tag{5.18}$$

式中$H(x,y)=\varphi_{0,yy}, H'(x,y)=\varphi_{0,xx}$。将求得的$w_0,\varphi_0,v_0^{(1)},\cdots,\ v_0^{(4)},h_0^{(1)},\ \cdots,h_0^{(4)}$代入方程(4.6a,b) 和边界条件 (4.15)～(4.21)，(4.39)～(4.45)并取$n=1$，得到关于$w_1$和$\varphi_1$的线性边值问题：

$$L(w_0,\varphi_1)+L(w_1,\varphi_0)=0,\ \ \nabla_2^4\varphi_1+L(w_0,w_1)=0 \tag{5.19}$$

$$\left.\begin{aligned}
&w_1|_{x=0}=-v_0^{(1)}|_{\eta=0}=-C_0^{(1)}(0,y)\\
&w_1|_{x=1}=-v_0^{(2)}|_{\tilde{\eta}=1}=-C_0^{(2)}(1,y)\\
&w_1|_{y=0}=0,\qquad w_1|_{y=\frac{b}{a}}=0\\
&\varphi_{1,yy}|_{x=0}=0,\qquad \varphi_{1,yy}|_{x=1}=0\\
&\varphi_{1,xx}|_{y=0}=0,\qquad \varphi_{1,xx}|_{y=\frac{b}{a}}=0
\end{aligned}\right\} \tag{5.20}$$

于是可求得$w_1$和$\varphi_1$。求得$w_1$和$\varphi_1$后再代入(4.24)式，并令等式右端为零，有

$$2H\frac{\partial C_0^{(1)}}{\partial\eta}+2\varphi_{0,xy}\frac{\partial C_0^{(1)}}{\partial y}+\left[\frac{5}{2}H_x+\varphi_{1,yy}\sqrt{H}\right]C_0^{(1)}=0 \tag{5.21}$$

从(4.28)式，得到$C_0^{(2)}(\tilde{\eta},y)$的一阶线性方程

$$2H\frac{\partial C_0^{(2)}}{\partial\tilde{\eta}}+2\varphi_{0,xy}\frac{\partial C_0^{(2)}}{\partial y}+\left[\frac{5}{2}H_x+\varphi_{1,yy}\sqrt{H}\right]C_0^{(2)}=0 \tag{5.22}$$

同样，由(4.26)和(4.30)式，可分别得到$C_0^{(3)}(x,\beta)$和$C_0^{(4)}(x,\tilde{\beta})$的一阶线性方程

$$2H'\frac{\partial C_0^{(3)}}{\partial\beta}+2(2d-1)\varphi_{0,xy}\frac{\partial C_0^{(3)}}{\partial x}+\left(\frac{6d-1}{2}H'_x+\varphi_{1,xx}\sqrt{H'}\right)C_0^{(3)}=0 \tag{5.23}$$

$$2H'\frac{\partial C_0^{(4)}}{\partial\tilde{\beta}}+2(2d-1)\varphi_{0,xy}\frac{\partial C_0^{(4)}}{\partial x}+\left(\frac{6d-1}{2}H'_x+\varphi_{1,xx}\sqrt{H'}\right)C_0^{(4)}=0 \tag{5.24}$$

只要$H(x,y)>0$，$H'(x,y)>0$，便可以根据Cauchy条件(5.7)，(5.8)，(5.13)和(5.14)唯一地解得$C_0^{(1)}$，$C_0^{(2)}$，$C_0^{(3)}$和$C_0^{(4)}$，由此，便完全确定了边界层型函数$v_0^{(1)},\cdots,v_0^{(4)},h_0^{(1)},\cdots,h_0^{(4)}$。而方程(4.24)，(4.28)，(4.26)和(4.30)化为齐次方程（取$n=1$）

$$\frac{\partial^4 v_1^{(1)}}{\partial\xi^4}-\frac{\partial^2 v_1^{(1)}}{\partial\xi^2}=0,\qquad \frac{\partial^4 v_1^{(2)}}{\partial\tilde{\xi}^4}-\frac{\partial^2 v_1^{(2)}}{\partial\tilde{\xi}^2}=0 \tag{5.25a}$$

$$d\cdot\frac{\partial^4 v_1^{(3)}}{\partial\alpha^4}-\frac{\partial^2 v_1^{(3)}}{\partial\alpha^2}=0,\qquad d\cdot\frac{\partial^4 v_1^{(4)}}{\partial\tilde{\alpha}^4}-\frac{\partial^2 v_1^{(4)}}{\partial\tilde{\alpha}^2}=0 \tag{5.25b}$$

求得

$$v_1^{(1)}(\xi,\eta,y)=C_1^{(1)}(\eta,y)e^{-\xi},\quad v_1^{(2)}(\tilde{\xi},\tilde{\eta},y)=C_1^{(2)}(\tilde{\eta},y)e^{-\tilde{\xi}} \tag{5.26a}$$

$$v_1^{(3)}(x,\alpha,\beta)=C_1^{(3)}(x,\beta)\exp\left[-\frac{\alpha}{\sqrt{d}}\right],\qquad v_1^{(4)}(x,\tilde{\alpha},\tilde{\beta})=C_1^{(4)}(x,\tilde{\beta})\exp\left[-\frac{\tilde{\alpha}}{\sqrt{d}}\right] \tag{5.26b}$$

从方程(4.16)，(4.18)，(4.20)和(4.22)(取$n=1$)，又得到$C_1^{(1)},\cdots,C_1^{(4)}$的边界条件

$$C_1^{(1)}(\eta,y)\Big|_{\eta=0}=h_1^{-\frac{1}{2}}\left[w_{1,x}(0,y)+\frac{\partial C_0^{(1)}}{\partial\eta}(0,y)\right] \tag{5.27}$$

$$C_1^{(2)}(\tilde{\eta},y)\Big|_{\tilde{\eta}=1}=h_2^{-\frac{1}{2}}\left[w_{1,x}(1,y)+\frac{\partial C_0^{(2)}}{\partial\tilde{\eta}}(1,y)\right] \tag{5.28}$$

$$C_1^{(3)}(x,\beta)\Big|_{\beta=0}=-h_3^{-1}[w_{1,yy}(x,0)+\mu_1 w_{1,xx}(x,0)]+\left[2h_3^{-\frac{1}{2}}\frac{\partial C_0^{(3)}}{\partial\beta}+\frac{1}{2}h_3^{-\frac{3}{2}}h_{3,y}C_0^{(3)}\right]_{\beta=0} \tag{5.29}$$

$$C_1^{(4)}(x,\tilde{\beta})\Big|_{\tilde{\beta}=\frac{b}{a}}=-h_4^{-1}\left[w_{1,yy}\left(x,\frac{b}{a}\right)+\mu_1 w_{1,xx}\left(x,\frac{b}{a}\right)\right]+\left[2h_4^{-\frac{1}{2}}\frac{\partial C_0^{(4)}}{\partial\tilde{\beta}}+\frac{1}{2}h_4^{-\frac{3}{2}}h_{4,y}C_0^{(4)}\right]_{\tilde{\beta}=\frac{b}{a}} \tag{5.30}$$

如此继续下去，可逐次求得$w_n,\varphi_n,v_n^{(1)},\cdots,v_n^{(4)},h_n^{(1)},\cdots,h_n^{(4)}$ $(n=1,2,\cdots,N)$，将求得的$w_n$，$v_n^{(1)},\cdots,v_n^{(4)}$；$\varphi_n,h_n^{(1)},\cdots,h_n^{(4)}$分别代入(4.1)和(4.2)式，便得到正交各向异性矩形薄板非线性弯曲的$N$阶形式渐近解$W_N$，$\Phi_N$。求得应力函数$\Phi_N$后，利用(2.7)式就可求出各个中面应力分量，求得挠度$W_N$后，便可求得弯曲应力和剪应力，从而可求得板上任一点的总应力。

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# Nonlinear Bendings for the Orthotropic Rectangular Thin Plates under Various Supports

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### Abstract

In this paper, the nonlinear bendings for the orthotropic rectangular thin plates under various supports are studied.

The uniformity valid asymptotic solutions of the displacement $w$ and stress function $\varphi$ are derived by means of the perturbation method offered in [1].