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连续介质中杆件系统的振动问题*

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摘　要

弹性连续介质中杆件系统的振动问题，是工程中经常遇到的，这是弹性动力学和结构动力学的混合求解问题．按照一般的方法进行求解，似乎有很多困难，且很复杂．

本文采用Lagrange乘子法[1][2]，给出了这种类型平面问题的广义泛函．并通过实例说明本文方法的应用．

一、序　　言

在地下工程中，当地表面受到冲击波荷载作用时，介质和结构将产生振动．对这种问题的分析，一般有两种方法：一是将介质和杆件结构作为复合弹性体进行分析；另一种方法，是隔离体的分析方法，即将杆件结构隔离出来，将外荷载作用在结构表面上，对于结构周围介质的作用，有的处理为单向弹簧，有的处理为波动阻抗，也有的处理为两者的结合．不论采用那种分析方法，问题的分析都是十分复杂的，这也是防护工程研究的重要课题．

从地下工程这一问题出发，我们把地层介质视为弹性体，分析时采用弹性动力学的方法；将介质中的结构视为杆件结构系统，分析时采用结构动力学的分析方法；而将弹性体和杆件结构系统一起看做是一个混合体系．本文给出了这种混合体系的广义变分原理，即连续介质中杆件结构系统的广义变分原理，可作为求解这类问题的理论依据．

作为文中广义变分原理应用的一个例子，我们对半无限弹性介质中具有加强环的圆柱形孔穴进行了动力分析，给出了求解问题的振动微分方程组．

二、弹性平面问题和平面曲杆结构的基本方程

在弹性体平面应变问题中，我们用u，v表示沿x，y方向上的位移；X，Y表示体力；ρ表示质量密度；E表示弹性模量；μ表示泊松比，则以位移分量表示的平衡方程为

$$\frac{E}{1+\mu}\left[\frac{1-\mu}{1-2\mu}\frac{\partial^2 u}{\partial x^2}+\frac{1}{2(1-2\mu)}\cdot\frac{\partial^2 v}{\partial x\partial y}+\frac{1}{2}\frac{\partial^2 u}{\partial y^2}\right]+X-\rho\frac{\partial^2 u}{\partial t^2}=0$$

$$\frac{E}{1+\mu}\left[\frac{1-\mu}{1-2\mu}\frac{\partial^2 v}{\partial y^2}+\frac{1}{2(1-2\mu)}\frac{\partial^2 u}{\partial x\partial y}+\frac{1}{2}\frac{\partial^2 v}{\partial x^2}\right]+Y-\rho\frac{\partial^2 v}{\partial t^2}=0$$

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在位移边界问题中，若弹性体外边界的位移分量是已知的，则在边界上应有

$$u-\bar{u}=0,\quad v-\bar{v}=0$$

其中$\bar{u}$和$\bar{v}$在弹性体外边界上是坐标的已知函数。

在应力边界问题中，若弹性体外边界上所受的面力是已知的，也就是说，弹性体外边界上的面力分量$\bar{X}$和$\bar{Y}$是坐标的已知函数，则弹性体外边界上的应力边界条件为

$$l\sigma_x+m\tau_{yx}=\bar{X},\quad m\sigma_y+l\tau_{xy}=\bar{Y}$$

式中：σ_x，σ_y，τ_{xy}表示弹性体内的应力分量；l，m表示方向余弦。

在平面曲杆结构中，我们用w，τ分别表示径向位移和切向位移；M，N，Q分别表示弯矩、轴力和剪力；$\bar{m}$表示单位长度的质量；R，r分别表示变形前和变形后的曲率半径；并用$P_w(t)$和$P_\tau(t)$分别表示作用在曲杆结构上的径向外荷载和切向外荷载，则曲杆结构的平衡方程为：

$$\frac{\partial N}{\partial s}-\frac{Q}{R}-\bar{m}\frac{\partial^2\tau}{\partial t^2}+P_\tau(t)=0$$

$$\frac{\partial^2 M}{\partial s^2}+\frac{N}{R}-\bar{m}\frac{\partial^2 w}{\partial t^2}+P_w(t)=0$$

根据几何关系，可得结构轴向应变ε、结构截面转角φ和截面在变形后曲率的改变量κ分别为

$$\varepsilon=\frac{\partial\tau}{\partial s}-\frac{w}{R},\quad \varphi=\frac{\partial w}{\partial s}+\frac{\tau}{R},\quad \kappa=\frac{1}{r}-\frac{1}{R}=\frac{\partial\varphi}{\partial s}$$

结构截面内力与变位之间存在下述关系

$$M=-EJ\kappa=-EJ\left(\frac{\partial^2 w}{\partial s^2}+\frac{1}{R}\frac{\partial\tau}{\partial s}\right)$$

$$N=EF\varepsilon=EF\left(\frac{\partial\tau}{\partial s}-\frac{w}{R}\right)$$

$$Q=\frac{\partial M}{\partial s}=-EJ\left(\frac{\partial^3 w}{\partial s^3}+\frac{1}{R}\frac{\partial^2\tau}{\partial s^2}\right)$$

式中：EJ为截面抗弯刚度，F为截面面积。

曲杆结构两端的位移边界条件为

$$w(0,t)-\bar{w}(0,t)=0;\ \tau(0,t)-\bar{\tau}(0,t)=0;\ \varphi(0,t)-\bar{\varphi}(0,t)=0$$
$$w(l,t)-\bar{w}(l,t)=0;\ \tau(l,t)-\bar{\tau}(l,t)=0;\ \varphi(l,t)-\bar{\varphi}(l,t)=0$$

三、连续介质中杆件结构系统的广义变分原理

如图1所示，连续介质中的杆件系统，在平面应变问题的情况下，其广义变分原理为：

在弹性体D域内服从虎克定律和应变与位移之间的关系；在杆件L上满足杆件结构的内力与位移关系；且在$t=t_1$和$t=t_2$时，弹性体内和结构上位移为已知的条件下，在所有的许可位移u，v，w，τ中使下述泛函取驻值，必导出问题的真实解。

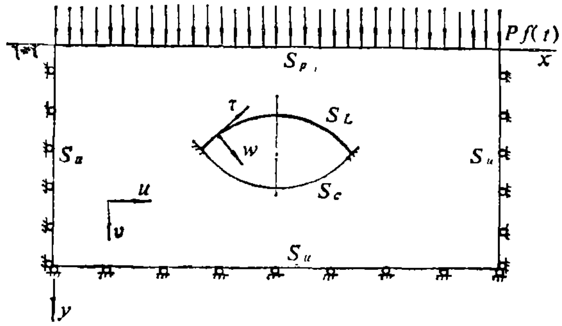

图 1　计算体系

$$\Pi=\int_{t_1}^{t_2}\left\{\iint_D \frac{1}{2}\rho\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2\right]dxdy+\int_{S_L}\frac{1}{2}\bar{m}\left[\left(\frac{\partial w}{\partial t}\right)^2+\left(\frac{\partial \tau}{\partial t}\right)^2\right]ds\right.$$

$$-\frac{E}{2(1+\mu)}\iint_D\left[\frac{\mu}{1-2\mu}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right.$$

$$\left.+\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right]dxdy-\int_{S_L}\frac{EJ}{2}\left(\frac{\partial^2 w}{\partial s^2}+\frac{1}{R}\frac{\partial \tau}{\partial s}\right)^2ds-\int_{S_L}\frac{EF}{2}\left(\frac{\partial \tau}{\partial s}-\frac{w}{R}\right)^2ds$$

$$+\iint_D(Xu+Yv)dxdy+\int_{S_p}(\bar{X}u+\bar{Y}v)ds+\int_{S_u}\bar{X}_u(u-\bar{u})ds+\int_{S_u}\bar{Y}_u(v-\bar{v})ds$$

$$+\int_{S_L}P_w(w'-w)ds-Q(0,\ t)[w(0,\ t)-\bar{w}(0,\ t)]-N(0,\ t)[\tau(0,\ t)-\bar{\tau}(0,\ t)]$$

$$+M(0,\ t)[\varphi(0,\ t)-\bar{\varphi}(0,\ t)]+Q(l,\ t)[w(l,\ t)-\bar{w}(l,\ t)]+N(l,\ t)[\tau(l,\ t)$$

$$\left.-\bar{\tau}(l,\ t)]-M(l,\ t)[\varphi(l,\ t)-\bar{\varphi}(l,\ t)]\right\}dt \qquad (3.1)$$

其中：$\bar{X}_u$，$\bar{Y}_u$，P_w，$Q(0,\ t)$，$N(0,\ t)$，$M(0,\ t)$，$Q(l,\ t)$，$N(l,\ t)$，$M(l,\ t)$ 在进行变分时，都是独立变量.

证明

弹性体在振动过程中的动能为

$$T_1=\frac{1}{2}\iint_D\rho\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2\right]dx\,dy$$

弹性体的体力势能和面力势能为

$$W=\iint_D(Xu+Yv)dx\,dy+\int_{S_p}(\bar{X}u+\bar{Y}v)ds$$

弹性体的应变能为

$$U_1=\frac{E}{2(1+\mu)}\iint_D\left[\frac{\mu}{1-2\mu}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right.$$

$$\left.+\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right]dx\,dy$$

曲杆结构的动能为

$$T_2=\int_{S_L}\frac{1}{2}\bar{m}\left[\left(\frac{\partial w}{\partial t}\right)^2+\left(\frac{\partial \tau}{\partial t}\right)^2\right]ds$$

曲杆结构的位能为

$$U_2=\int_{S_L}\frac{EJ}{2}\left(\frac{\partial^2 w}{\partial s^2}+\frac{1}{R}\frac{\partial \tau}{\partial s}\right)^2ds+\int_{S_L}\frac{EF}{2}\left(\frac{\partial \tau}{\partial s}-\frac{w}{R}\right)^2ds$$

弹性体内边界上的径向位移可写成下面形式

$$w'=\varphi_1(\theta)u+\varphi_2(\theta)v$$

式中：$\varphi_1(\theta)$和$\varphi_2(\theta)$为确定的三角函数。

我们假设弹性体与曲杆结构的接触面上为"零宽度光滑接触"，则要求介质和结构径向位移协调，即

$$w'-w=0$$

我们采用Lagrange乘子解除所取体系的全部约束，于是整个体系的泛函为

$$\begin{aligned}\Pi^*=\int_{t_1}^{t_2}\{&T_1+T_2+W-U_1-U_2+\int_{S_u}\mu_1(u-\bar{u})ds+\int_{S_u}\mu_2(v-\bar{v})ds+\int_{S_L}\mu_3(w'-w)ds\\&+\lambda_1[w(0,\ t)-\bar{w}(0,\ t)]+\lambda_2[\tau(0,\ t)-\bar{\tau}(0,\ t)]+\lambda_3[\varphi(0,\ t)-\bar{\varphi}(0,\ t)]\\&+\lambda_4[w(l,\ t)-\bar{w}(l,\ t)]+\lambda_5[\tau(l,\ t)-\bar{\tau}(l,\ t)]+\lambda_6[\varphi(l,\ t)-\bar{\varphi}(l,\ t)]\}dt\end{aligned}\tag{3.2}$$

式中：μ_1，μ_2，μ_3，$\lambda_i(i=1,\ 2,\ \cdots,\ 6)$为Lagrange乘子。

当泛函Π^*达到驻值时，有$\delta\Pi^*=0$，于是

$$\begin{aligned}\delta\Pi^*=&\int_{t_1}^{t_2}\iint_D\rho\left(\frac{\partial u}{\partial t}\frac{\partial\delta u}{\partial t}+\frac{\partial v}{\partial t}\frac{\partial\delta v}{\partial t}\right)dx\,dy\,dt-\int_{t_1}^{t_2}\frac{E}{2(1+\mu)}\iint_D\left\{\left[\frac{2\mu}{1-2\mu}\left(\frac{\partial u}{\partial x}\right.\right.\right.\\&\left.\left.+\frac{\partial v}{\partial y}\right)+2\frac{\partial u}{\partial x}\right]\frac{\partial\delta u}{\partial x}+\left[\frac{2\mu}{1-2\mu}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\frac{\partial v}{\partial y}\right]\frac{\partial\delta v}{\partial y}+\left(\frac{\partial v}{\partial x}\right.\\&\left.\left.+\frac{\partial u}{\partial y}\right)\frac{\partial\delta v}{\partial x}+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\frac{\partial\delta u}{\partial y}\right\}dxdydt+\int_{t_1}^{t_2}\iint_D(X\delta u+Y\delta v)dxdydt\\&+\int_{t_1}^{t_2}\int_{S_p}(\bar{X}\delta u+\bar{Y}\delta v)dsdt+\int_{t_1}^{t_2}\int_{S_u}(\mu_1\delta u+\mu_2\delta v)\,dsdt+\int_{t_1}^{t_2}\int_{S_u}(u-\bar{u})\delta\mu_1dsdt\\&+\int_{t_1}^{t_2}\int_{S_u}(v-\bar{v})\,\delta\mu_2dsdt+\int_{t_1}^{t_2}\int_{S_L}\left[\frac{\partial^2M}{\partial s^2}+\frac{N}{R}-\bar{m}\frac{\partial^2w}{\partial t^2}\right]\delta wdsdt\\&+\int_{t_1}^{t_2}\int_{S_L}\left[\frac{\partial N}{\partial s}-\frac{Q}{R}-\bar{m}\frac{\partial^2\tau}{\partial t^2}\right]\delta\tau dsdt+\int_{t_1}^{t_2}\int_{S_L}\mu_3[\varphi_1(\theta)\delta u+\varphi_2(\theta)\delta v-\delta w]dsdt\\&+\int_{t_1}^{t_2}\int_{S_L}(w'-w)\delta\mu_3dsdt+\int_{t_1}^{t_2}\{[\lambda_1+Q(0,\ t)]\delta w(0,\ t)+[\lambda_2+N(0,t)]\delta\tau(0,\ t)\\&+[\lambda_3-M(0,\ t)]\delta\varphi(0,\ t)+[\lambda_4-Q(l,\ t)]\delta w(l,\ t)+[\lambda_5-N(l,\ t)]\delta\tau(l,t)\\&+[\lambda_6+M(l,\ t)]\delta\varphi(l,\ t)+[w(0,\ t)-\bar{w}(0,\ t)]\delta\lambda_1+[\tau(0,\ t)-\bar{\tau}(0,\ t)]\delta\lambda_2\\&+[\varphi(0,\ t)-\bar{\varphi}(0,\ t)]\delta\lambda_3+[w(l,\ t)-\bar{w}(l,\ t)]\delta\lambda_4+[\tau(l,\ t)-\bar{\tau}(l,\ t)]\delta\lambda_5\\&+[\varphi(l,\ t)-\bar{\varphi}(l,\ t)]\delta\lambda_6\}dt\end{aligned}\tag{3.3}$$

因为

$$\begin{aligned}&\frac{E}{2(1+\mu)}\iint_D\left\{\left[\frac{2\mu}{1-2\mu}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\frac{\partial u}{\partial x}\right]\frac{\partial\delta u}{\partial x}+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\frac{\partial\delta u}{\partial y}\right\}dxdy\\&=\frac{E}{2(1+\mu)}\int_{S_p+S_u+S+S_L}\left\{l\left[\frac{2\mu}{1-2\mu}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\frac{\partial u}{\partial x}\right]\right.\\&\qquad\left.+m\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right\}\delta uds\end{aligned}$$

$$-\frac{E}{2(1+\mu)}\iint_D\left[\frac{2\mu}{1-2\mu}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}\right)+2\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}\right]\delta u dxdy$$

$$=\int_{S_p+S_u+S_c+S_l}(l\sigma_x+m\tau_{yx})\delta u ds$$

$$-\frac{E}{2(1+\mu)}\iint_D\left[\frac{2\mu}{1-2\mu}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}\right)+2\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}\right]\delta u dxdy$$

和

$$\frac{E}{2(1+\mu)}\iint_D\left\{\left[\frac{2\mu}{1-2\mu}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\frac{\partial v}{\partial y}\right]\frac{\partial\delta v}{\partial y}+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\frac{\partial\delta v}{\partial x}\right\}dxdy$$

$$=\int_{S_p+S_u+S_c+S_l}(m\sigma_y+l\tau_{xy})\,\delta v ds$$

$$-\frac{E}{2(1+\mu)}\iint_D\left[\frac{2\mu}{1-2\mu}\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}\right)+2\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}\right]\delta v dxdy$$

于是，可将(3.3)式写成下面的形式

$$\delta\Pi^*=\iint_D\rho\frac{\partial u}{\partial t}\delta u dxdy\Big|_{t_1}^{t_2}+\int_{t_1}^{t_2}\left\{\frac{E}{2(1+\mu)}\iint_D\left[\frac{2\mu}{1-2\mu}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}\right)+2\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}\right.\right.$$

$$\left.\left.+\frac{\partial^2 u}{\partial y^2}\right]dxdy+\iint_D\left[X-\rho\frac{\partial^2 u}{\partial t^2}\right]dxdy\right\}\delta u dt+\iint_D\rho\frac{\partial v}{\partial t}\delta v dxdy\Big|_{t_1}^{t_2}$$

$$+\int_{t_1}^{t_2}\left\{\frac{E}{2(1+\mu)}\iint_D\left[\frac{2\mu}{1-2\mu}\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}\right)+2\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}\right]dxdy\right.$$

$$\left.+\iint_D\left[Y-\rho\frac{\partial^2 v}{\partial t^2}\right]dxdy\right\}\delta v dt-\int_{t_1}^{t_2}\left\{\int_{S_p}[(l\sigma_x+m\tau_{yx})-\bar{X}]ds\cdot\delta u\right\}dt$$

$$-\int_{t_1}^{t_2}\left\{\int_{S_p}[(m\sigma_y+l\tau_{xy})-\bar{Y}]ds\cdot\delta v\right\}dt-\int_{t_1}^{t_2}\left\{\int_{S_u}[(l\sigma_x+m\tau_{yx})-\mu_1]ds\cdot\delta u\right\}dt$$

$$-\int_{t_1}^{t_2}\left\{\int_{S_u}[(m\sigma_y+l\tau_{xy})-\mu_2]ds\cdot\delta v\right\}dt+\int_{t_1}^{t_2}\left\{\int_{S_u}(u-\bar{u})\delta\mu_1 ds\right\}dt$$

$$+\int_{t_1}^{t_2}\left\{\int_{S_u}(v-\bar{v})\delta\mu_2 ds\right\}dt-\int_{t_1}^{t_2}\left\{\int_{S_c}[l\sigma_x+m\tau_{yx}]\delta u ds\right\}dt-\int_{t_1}^{t_2}\left\{\int_{S_c}[m\sigma_y\right.$$

$$\left.+l\tau_{xy}]\delta v ds\right\}dt+\int_{t_1}^{t_2}\left\{\int_{S_L}\left[\left(\frac{\partial^2 M}{\partial s^2}+\frac{N}{R}-\bar{m}\frac{\partial^2 w}{\partial t^2}\right)-\mu_3\right]\delta w ds\right\}dt$$

$$+\int_{t_1}^{t_2}\left\{\int_{S_L}\left[\frac{\partial N}{\partial s}-\frac{Q}{R}-\bar{m}\frac{\partial^2\tau}{\partial t^2}\right]\delta\tau ds\right\}dt-\int_{t_1}^{t_2}\left\{\int_{S_L}[(l\sigma_x+m\tau_{yx})\right.$$

$$\left.-\mu_3\varphi_1(\theta)]ds\cdot\delta u\right\}dt-\int_{t_1}^{t_2}\left\{\int_{S_L}[(m\sigma_y+l\tau_{xy})-\mu_3\varphi_2(\theta)]ds\cdot\delta v\right\}dt$$

$$+\int_{t_1}^{t_2}\left\{\int_{S_L}(w'-w)\,\delta\mu_3 ds\right\}dt+\int_{t_1}^{t_2}\{[\lambda_1+Q(0,\ t)]\delta w(0,\ t)\}dt$$

$$+\int_{t_1}^{t_2}\{[\lambda_2+N(0,t)]\delta\tau(0,t)\}dt+\int_{t_1}^{t_2}\{[\lambda_3-M(0,t)]\delta\varphi(0,t)\}dt$$
$$+\int_{t_1}^{t_2}\{[\lambda_4-Q(l,t)]\delta w(l,t)\}dt+\int_{t_1}^{t_2}\{[\lambda_5-N(l,t)]\delta\tau(l,t)\}dt$$
$$+\int_{t_1}^{t_2}\{[\lambda_6+M(l,t)]\delta\varphi(l,t)\}dt+\int_{t_1}^{t_2}\{[w(0,t)-\bar{w}(0,t)]\delta\lambda_1\}dt$$
$$+\int_{t_1}^{t_2}\{[\tau(0,t)-\bar{\tau}(0,t)]\delta\lambda_2\}dt+\int_{t_1}^{t_2}\{[\varphi(0,t)-\bar{\varphi}(0,t)]\delta\lambda_3\}dt$$
$$+\int_{t_1}^{t_2}\{[w(l,t)-\bar{w}(l,t)]\delta\lambda_4\}dt+\int_{t_1}^{t_2}\{[\tau(l,t)-\bar{\tau}(l,t)]\delta\lambda_5\}dt$$
$$+\int_{t_1}^{t_2}\{[\varphi(l,t)-\bar{\varphi}(l,t)]\delta\lambda_6\}dt \tag{3.4}$$

由$\delta\Pi^*=0$可得

弹性体平衡方程：

$$\left.\begin{aligned}&\frac{E}{1+\mu}\left[\frac{1-\mu}{1-2\mu}\frac{\partial^2 u}{\partial x^2}+\frac{1}{2(1-2\mu)}\frac{\partial^2 v}{\partial x\partial y}+\frac{1}{2}\frac{\partial^2 u}{\partial y^2}\right]+X-\rho\frac{\partial^2 u}{\partial t^2}=0\\&\frac{E}{1+\mu}\left[\frac{1-\mu}{1-2\mu}\frac{\partial^2 v}{\partial y^2}+\frac{1}{2(1-2\mu)}\frac{\partial^2 u}{\partial x\partial y}+\frac{1}{2}\frac{\partial^2 v}{\partial x^2}\right]+Y-\rho\frac{\partial^2 v}{\partial t^2}=0\end{aligned}\right\}\text{在}D\text{域内}$$

曲杆结构的平衡方程：

$$\left.\begin{aligned}&\frac{\partial^2 M}{\partial s^2}+\frac{N}{R}-\bar{m}\frac{\partial^2 w}{\partial t^2}+P_w=0\\&\frac{\partial N}{\partial s}-\frac{Q}{R}-\bar{m}\frac{\partial^2 \tau}{\partial t^2}=0\end{aligned}\right\}\text{在}S_L\text{上}$$

Lagrange乘子的物理意义：

$$\left.\begin{aligned}&\mu_1=l\sigma_x+m\tau_{yx}=\bar{X}_u\\&\mu_2=m\sigma_y+l\tau_{xy}=\bar{Y}_u\end{aligned}\right\}\text{在}S_u\text{上}$$

$$\mu_3=\frac{\partial^2 M}{\partial s^2}+\frac{N}{R}-\bar{m}\frac{\partial^2 w}{\partial t^2}=-P_w\quad\text{在}S_L\text{上}$$

$$\left.\begin{aligned}&\lambda_1=-Q(0,t);\ \lambda_2=-N(0,t);\ \lambda_3=M(0,t)\\&\lambda_4=Q(l,t);\ \lambda_5=N(l,t);\ \lambda_6=-M(l,t)\end{aligned}\right\}\text{在}S_L\text{两端}$$

力边界条件：

$$\left.\begin{aligned}&l\sigma_x+m\tau_{yx}=\bar{X}\\&m\sigma_y+l\tau_{xy}=\bar{Y}\end{aligned}\right\}\text{在}S_p\text{上}$$

$$\left.\begin{aligned}&l\sigma_x+m\tau_{yx}=-P_w\varphi_1(\theta)\\&m\sigma_y+l\tau_{xy}=-P_w\varphi_2(\theta)\end{aligned}\right\}\text{在}S_L\text{上}$$

$$\left.\begin{aligned}&l\sigma_x+m\tau_{yx}=0\\&m\sigma_y+l\tau_{xy}=0\end{aligned}\right\}\text{在}S_o\text{上}$$

位移边界条件：

$$\left.\begin{aligned}&u-\bar{u}=0\\&v-\bar{v}=0\end{aligned}\right\}\text{在}S_u\text{上}$$

$$w'-w=0\quad\text{在}S_L\text{上}$$

$$\left.\begin{array}{l} w(0,\ t)-\bar{w}(0,\ t)=0;\ \tau(0,\ t)-\bar{\tau}(0,\ t)=0 \\ \varphi(0,\ t)-\bar{\varphi}(0,\ t)=0;\ w(l,\ t)-\bar{w}(l,\ t)=0 \\ \tau(l,\ t)-\bar{\tau}(l,\ t)=0;\ \varphi(l,\ t)-\bar{\varphi}(l,\ t)=0 \end{array}\right\} \text{在} S_L \text{两端}$$

由此可见变分后满足一切应该满足的条件.

四、实　　例

设有一个大面积的平面冲击波作用在一个半无限大的弹性均匀介质的上表面，在距离上表面一定深度处有一个无限长的圆柱形结构，其轴线平行于介质的上表面，该问题是一个平面应变问题. 我们取图 2 作为计算简图，许多实际经验表明，在动荷载作用下，位移边界对孔穴的影响范围一般取为 5 倍以上孔穴的计算跨度.

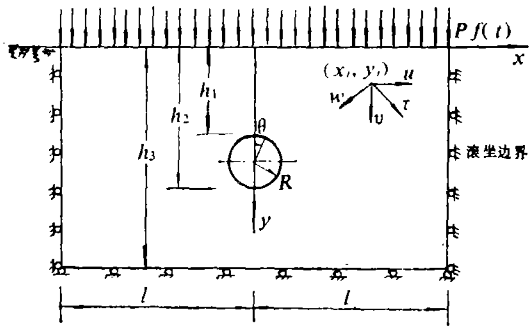

图 2　计算简图

(1)　能量的计算和运动微分方程的建立

根据所取力学模型边界点上的位移是已知的，我们可首先在整个平面弹性介质模型中选取$i\times m$个点，这$i\times m$个点的位移是随时间变化的，并将其取为广义坐标. 然后，应用插值公式，通过上述$i\times m$个点的瞬时位移做两个多项式，表示平面弹性介质模型中各点的垂直位移和水平位移，即分别表示成下列形式

$$u(x,\ y,\ t)=\sum_i\sum_m f_i(x)f_m(y)u_{im}(t) \tag{4.1}$$

$$v(x,\ y,\ t)=\sum_i\sum_m f_i(x)f_m(y)v_{im}(t) \tag{4.2}$$

$$(i=1,\ 2,\ \cdots;\ m=1,\ 2,\ \cdots)$$

式中：$f_i(x)$和$f_m(y)$均是位置坐标函数，它的形式是确定的，见(2)；$u_{im}(t)$和$v_{im}(t)$是平面上$(x_i,\ y_m)$点的待求广义坐标. 为书写方便，以下用u_{im}和v_{im}分别代替$u_{im}(t)$和$v_{im}(t)$. 由上述可知，(4.1)、(4.2)两式必然满足所取的介质模型的位移边界条件.

圆柱形结构采用结构力学的方法. 在以弯曲变形为主的杆件系统中，可以忽略结构的轴向变形，将圆形结构的径向位移和切向位移可分别表示成

$$w(\theta,\ t)=\sum_\lambda a_\lambda(t)\cos\lambda\theta \tag{4.3}$$

$$\tau(\theta,\ t)=\sum_\lambda \frac{1}{\lambda}a_\lambda(t)\sin\lambda\theta \tag{4.4}$$

$$(\lambda=1,\ 2,\ \cdots)$$

式中：$a_\lambda(t)$是与时间有关的广义坐标.

在圆柱形结构周边处的介质的径向位移，可以表示成

$$w'(\theta,\ t)=\sum_i\sum_m f_i(R\sin\theta)f_m(h_1+R-R\cos\theta)v_{im}\cos\theta$$

$$-\sum_{i}\sum_{n} f_i(R\sin\theta) f_m(h_1+R-R\cos\theta) u_{im}\sin\theta \tag{4.5}$$

体系的动能包括弹性介质的动能和结构的动能，即

$$T=\frac{1}{2}\rho\iint_D(\dot{u}^2+\dot{v}^2)dxdy+\int_0^{2\pi}\frac{1}{2}\bar{m}(\dot{w}^2+\dot{\tau}^2)Rd\theta$$

$$=\frac{1}{2}\sum_i\sum_m\sum_j\sum_n A_{\substack{i\,m\\j\,n}}(\dot{u}_{im}\dot{u}_{jn}+\dot{v}_{im}\dot{v}_{jn})+\frac{1}{2}\sum_\lambda A_\lambda\dot{a}_\lambda^2$$

式中：

$$A_{\substack{i\,m\\j\,n}}=2\rho\left[\int_0^l f_i(x)f_j(x)dx\int_0^{h_3}f_m(y)f_n(y)dy\right.$$
$$\left.-\int_{h_1}^{h_2}f_m(y)f_n(y)dy\int_0^{\sqrt{R^2-(h_1+R-y)^2}}f_i(x)f_j(x)dx\right];$$

$A_\lambda=\pi\bar{m}R\left(1+\frac{1}{\lambda^2}\right)$；$\rho$——介质的密度；$\bar{m}$——结构单位长度的质量。

体系的总应变能包括介质的应变能和结构的应变能（只计结构的弯矩位能）为

$$U=\frac{1}{2}\iint_D\left\{(a+b)\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]+\frac{b}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]+2a\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right)\right.$$
$$\left.+b\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right)\right\}dxdy+\int_0^{2\pi}\frac{M^2}{2EJ}Rd\theta$$

$$=\frac{1}{2}\sum_i\sum_m\sum_j\sum_n(B_{\substack{i\,m\\j\,n}}u_{im}u_{jn}+C_{\substack{i\,m\\j\,n}}v_{im}v_{jn}+2D_{\substack{i\,m\\j\,n}}u_{im}v_{jn})+\sum_\lambda\sum_k\frac{1}{2}E_{\lambda k}a_\lambda a_k$$

式中：

$$a=\frac{\mu E}{(1+\mu)(1-2\mu)};\qquad b=\frac{E}{1+\mu};$$

$$B_{\substack{i\,m\\j\,n}}=2\int_0^l dx\int_0^{h_3}\left[(a+b)f_i'(x)f_j'(x)f_m(y)f_n(y)+\frac{b}{2}f_i(x)f_j(x)f_m'(y)f_n'(y)\right]dy$$
$$-2\int_{h_1}^{h_2}dy\int_0^{\sqrt{R^2-(h_1+R-y)^2}}\left[(a+b)f_i'(x)f_j'(x)f_m(y)f_n(y)\right.$$
$$\left.+\frac{b}{2}f_i(x)f_j(x)f_m'(y)f_n'(y)\right]dx$$

$$C_{\substack{i\,m\\j\,n}}=2\int_0^l dx\int_0^{h_3}\left[(a+b)f_i(x)f_j(x)f_m'(y)f_n'(y)+\frac{b}{2}f_i'(x)f_j'(x)f_m(y)f_n(y)\right]dy$$
$$-2\int_{h_1}^{h_2}dy\int_0^{\sqrt{R^2-(h_1+R-y)^2}}\left[(a+b)f_i(x)f_j(x)f_m'(y)f_n'(y)\right.$$
$$\left.+\frac{b}{2}f_i'(x)f_j'(x)f_m(y)f_n(y)\right]dx$$

$$D_{\substack{i\,m\\j\,n}}=\int_0^l dx\int_0^{h_3}2af_i'(x)f_m(y)f_j(x)f_n'(y)dy-\int_{h_1}^{h_2}dy\int_0^{\sqrt{R^2-(h_1+R-y)^2}}2a$$

$$\cdot f_i'(x)\, f_m(y) f_j(x) f_n'(y)\, dx + \int_0^l dx \int_0^{h_3} b f_i'(x) f_n(y) f_i(x) f_m'(y)\, dy$$

$$-\int_{h_1}^{h_2} dy \int_0^{\sqrt{R^2-(h_1+R-y)^2}} b f_i'(x) f_n(y) f_i(x) f_m'(y)\, dx$$

$$E_{\lambda k}=\frac{\pi EJ}{R^3}(1-\lambda^2)^2$$

外界动荷载做的功为

$$W=2\int_0^l Pf(t)v(x,\ y_0,\ t)dx=\sum_i\sum_m Q_{im}f(t)v_{im}$$

式中：

$$Q_{im}=2Pf_m(y_0)\int_0^l f_i(x)dx$$

体系的总位能是

$$V=U-W$$

Lagrange函数（也称作用量）为

$$L=T-V$$

根据混合广义变分原理[1]，即：在弹性体D域内服从虎克定律和应变与位移之间的关系；在杆件L上满足杆件结构内力与位移关系；且在$t=t_1$和$t=t_2$时，弹性体内和结构上位移为已知的条件下，在所有的许可位移u，v，w，τ中，使下述泛函取驻值，必导出问题的真实解。

$$\begin{aligned}\Pi=\int_{t_1}^{t_2}\{ & L+\int_{S_u}\bar{X}_u(u-\bar{u})ds+\int_{S_u}\bar{Y}_u(v-\bar{v})ds+\int_{S_L}P_w(w'-w)ds\\ & -Q(0,\ t)[w(0,\ t)-\bar{w}(0,\ t)]-N(0,\ t)[\tau(0,\ t)-\bar{\tau}(0,\ t)]\\ & +M(0,\ t)[\varphi(0,\ t)-\bar{\varphi}(0,\ t)]+Q(l,\ t)[w(l,\ t)-\bar{w}(l,\ t)]\\ & +N(l,\ t)[\tau(l,\ t)-\bar{\tau}(l,\ t)]-M(l,\ t)[\varphi(l,\ t)-\bar{\varphi}(l,\ t)]\}dt\end{aligned} \tag{4.6}$$

由于描述弹性体内各点位移的位移函数(4.1)、(4.2)是通过包括边界点在内插值而得到的，所以它满足位移边界条件$u=\bar{u}$，$v=\bar{v}$；对于圆环来说，我们所选择的位移函数也满足位移边界条件。于是泛函中只剩下Lagrange函数和$\int_{S_L}P_w(w'-w)ds$两项。Lagrange函数前面已经求得，下面对$\int_{S_L}P_w(w'-w)ds$可计算如下：

我们假设弹性体与加强环为“零宽度光滑接触”，于是在接触面上就要求介质和圆环的径向位移协调，即

$$w'-w=0 \tag{4.7}$$

在介质和圆环接触面上引进一个力函数

$$P_w=P(\theta,\ t)=\sum_\xi R_\xi(t)\psi_\xi(\theta) \qquad (\xi=1,\ 2,\ \cdots) \tag{4.8}$$

其中：

$R_\xi(t)$为待定系数；$\psi_\xi(\theta)$为自选的坐标函数。

于是

$$\int_{S_L}P_w\ (w'-w)\ ds=\int_0^{2\pi}P(\theta,\ t)(w'-w)Rd\theta$$

$$=\sum_i\sum_m\sum_\xi F_{im\xi}v_{im}R_\xi+\sum_i\sum_m\sum_\xi G_{im\xi}u_{im}R_\xi+\sum_\lambda\sum_\xi H_{\lambda\xi}a_\lambda R_\xi \tag{4.9}$$

式中：

$$F_{im\xi}=\int_0^{2\pi}f_i(R\sin\theta)f_m(h_1+R-R\cos\theta)\cos\theta\psi_\xi(\theta)Rd\theta$$

$$G_{im\xi}=-\int_0^{2\pi}f_i(R\sin\theta)f_m(h_1+R-R\cos\theta)\sin\theta\psi_\xi(\theta)Rd\theta$$

$$H_{\lambda\xi}=-\int_0^{2\pi}\cos\lambda\theta\psi_\xi(\theta)Rd\theta$$

将Lagrange函数L和(4.9)式代入(4.6)式中，经整理得：

$$\Pi=\int_{t_1}^{t_2}\Big\{\frac{1}{2}\sum_i\sum_m\sum_j\sum_n A_{\substack{im\\jn}}(\dot u_{im}\dot u_{jn}+\dot v_{im}\dot v_{jn})+\frac{1}{2}\sum_\lambda A_\lambda\dot a_\lambda^2$$

$$-\frac{1}{2}\sum_i\sum_m\sum_j\sum_n(B_{\substack{im\\jn}}u_{im}u_{jn}+C_{\substack{im\\jn}}v_{im}v_{jn}+2D_{\substack{im\\jn}}u_{im}v_{jn})$$

$$-\sum_\lambda\sum_k\frac{1}{2}E_{\lambda k}a_\lambda a_k-\sum_i\sum_m\sum_\xi F_{im\xi}v_{im}R_\xi-\sum_i\sum_m\sum_\xi G_{im\xi}u_{im}R_\xi$$

$$-\sum_\lambda\sum_\xi H_{\lambda\xi}a_\lambda R_\xi+\sum_i\sum_m Q_{im}f(t)v_{im}\Big\}dt \tag{4.10}$$

由$\delta\Pi=0$，可得下列微分方程组：

$$\sum_i\sum_m(A_{\substack{im\\jn}}\ddot v_{im}+C_{\substack{im\\jn}}v_{im}+D_{\substack{im\\jn}}u_{im})+\sum_\xi F_{jn\xi}R_\xi=Q_{jn}f(t) \tag{4.11}$$

$$\sum_i\sum_m(A_{\substack{im\\jn}}\ddot u_{im}+B_{\substack{im\\jn}}u_{im}+D_{\substack{im\\jn}}v_{im})+\sum_\xi G_{jn\xi}R_\xi=0 \tag{4.12}$$

$$A_\lambda\ddot a_\lambda+\sum_k E_{\lambda k}a_k+\sum_\xi H_{\lambda\xi}R_\xi=0 \tag{4.13}$$

$$\sum_i\sum_m F_{im\xi}v_{im}+\sum_i\sum_m G_{im\xi}u_{im}+\sum_\lambda H_{\lambda\xi}a_\lambda=0 \tag{4.14}$$

（2） 关于位置坐标函数$f_i(x)$，$f_m(y)$的确定

$f_i(x)$和$f_m(y)$是利用双向插值公式得到的。它将随着所选取的点数$i\times m$的增多，多项式$f_i(x)$和$f_m(y)$中的幂次也增高。若选取的点数如图3所示，可得下式的表达式。

$f_i(x)$：

$$f_1(x)=\frac{1}{24a^4}(x^4-10ax^3+35a^2x^2-50a^3x+24a^4)$$

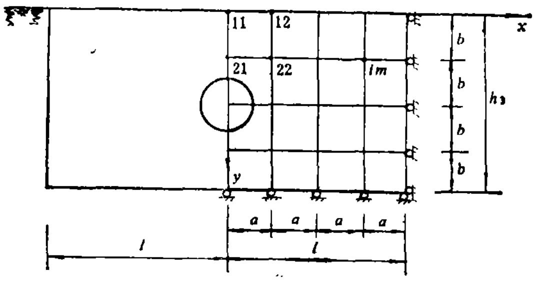

图3 选点示意图

$$f_2(x)=-\frac{1}{6a^4}(x^4-9ax^3+26a^2x^2-24a^3x)$$

$$f_3(x)=\frac{1}{4a^4}(x^4-8ax^3+19a^2x^2-12a^3x)$$

$$f_4(x)=-\frac{1}{6a^4}(x^4-7ax^3+14a^2x^2-8a^3x)$$

$$f_5(x)=\frac{1}{24a^4}(x^4-6ax^3+11a^2x^2-6a^3x)$$

$f_m(y)$：

$$f_1(y)=\frac{1}{24b^4}(y^4-10by^3+35b^2y^2-50b^3y+24b^4)$$

$$f_2(y)=-\frac{1}{6b^4}(y^4-9by^3+26b^2y^2-24b^3y)$$

$$f_3(y)=\frac{1}{4b^4}(y^4-8by^3+19b^2y^2-12b^3y)$$

$$f_4(y)=-\frac{1}{6b^4}(y^4-7by^3+14b^2y^2-8b^3y)$$

$$f_5(y)=\frac{1}{24b^4}(y^4-6by^3+11b^2y^2-6b^3y)$$

参 考 文 献

[1] 钱伟长，《变分法及有限元》，科学出版社，(1980).
[2] 胡海昌，《弹性力学的变分原理及其应用》，科学出版社，(1981).

The Vibration Problem of Rod System in the Continuous Elastic Medium

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Abstract

The vibration of rod system in the elastic continuous medium is a problem which is often met and also a composite solution problem of elastic dynamics and structural dynamics It seems rather difficult and complex to find solutions by general method. Using Lagrangian method of multipliers, we give here the generalized functionals concerning this kind of plane problem and show how to apply the method presented here through examples.