

常数速度梯度时均匀湍流的 二元速度关联函数

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摘 要

在湍流脉动速度比较小的条件下, 本文得到了富氏变换过后脉动速度方程的解. 它所代表的涡旋, 在平均速度梯度为小量时, 化为具有常数平均速度梯度的、组成后期均匀各向同性湍流场的涡旋和组成后期各向异性湍流场的涡旋. 利用不同时刻的这种涡旋解, 组成定常的有常数平均速度梯度的湍流场, 这个湍流场可以近似地表达槽流和管流近中心区域的湍流场. 我们求得了这种湍流场的二元速度关联函数, 包括纵向的关联系数 $f\left(\frac{r}{\lambda}\right)$ 和横向的关联系数 $g\left(\frac{r}{\lambda}\right)$. 并且和均匀各向同性湍流实验中的前期和后期的 $f\left(\frac{r}{\lambda}\right)$ 和 $g\left(\frac{r}{\lambda}\right)$ 进行了比较. 并且弄清楚了速度梯度对关联系数 $f\left(\frac{r}{\lambda}\right)$ 所产生的影响. 最后还得到了雷诺应力和涡旋粘性系数的表达式.

一、引 言

在过去我们先求解 Navier-Stokes 方程后平均的湍流理论里, 主要讨论了均匀各向同性(或接近各向同性)湍流的衰变问题^[1-4]. 对有速度梯度的情形, 虽然曾经一度接触过^[5], 但后来因为某种原因, 没有继续下去. 本文是接着从前的工作, 往后继续下去. 本来湍流场的能量是 Reynolds 应力和平均流动速度梯度相互作用, 从平均流动中获得的. 然后从强的大涡旋再破碎成弱的小涡旋. 能量也从大的涡旋逐步转移到小的涡旋里. 但是槽流和管流的中心区域, 湍流场的涡旋主要是由近壁区域向中心区域扩散和运动而得来的, 主要并不是当地产生的. 所以中心区域的湍流的能量相应于单位时间单位体积里, 从外部向中心扩散过来了多少个涡旋的能量. 由于不均匀湍流场讨论起来过于复杂, 我们就先讨论一个平均速度梯度为常数的均匀湍流场, 这样可避免涡旋扩散等麻烦问题, 同时假定在单位时间单位体积里产生某种类型涡旋的个数刚好能弥补湍流场由粘性消耗引起的能量减少, 从而维持整个运动为定常状态. 同时由于在中心区域, 平均速度梯度较小, 所以我们假定平均速度梯度是小量, 而仅仅保留它的一次项. 从这个图案出发, 我们计算了平均速度梯度小的时候的二元速度关联. 对各向同性涡旋和各向异性涡旋都进行了平均计算. 同时讨论了平均速度梯度对纵向速度关联系数 $f\left(\frac{r}{\lambda}\right)$ 的影响. 最后还得到了 Reynolds 剪应力和涡旋粘性系数的表达式.

二、涨落速度方程的线性化和它的 Fourier 变换后的解

不可压缩流体的涨落速度所满足的方程式为文献[5]中的(2.1)式,即

$$\left. \begin{aligned} \frac{\partial W_i}{\partial t} + U_\beta W_{i,\beta} + (W_i W_\beta)_{,\beta} + W_\beta U_{i,\beta} &= -\frac{1}{\rho} w_{,i} + (\overline{W_i W_\beta})_{,\beta} + \nu \nabla^2 W_i \\ W_{\beta,\beta} &= 0 \end{aligned} \right\} \quad (2.1)$$

式中 W_i 为涨落速度, U_i 为平均速度, w 为脉动压力, $-\rho \overline{W_i W_j}$ 为 Reynolds 应力. 假使我们考虑流场的湍流度很低, 我们就可以把方程(2.1)中非线性项和 Reynolds 应力项略去, 而得到

$$\left. \begin{aligned} \frac{\partial W_i}{\partial t} + U_\beta W_{i,\beta} + W_\beta U_{i,\beta} &= -\frac{1}{\rho} w_{,i} + \nu \nabla^2 W_i \\ W_{\beta,\beta} &= 0 \end{aligned} \right\} \quad (2.2)$$

如果我们假定平均速度梯度为常数. 流速方向沿着 x 方向, 速度变化沿着 y 方向. 我们就得到方程(2.3)

$$\left. \begin{aligned} \frac{\partial W_i}{\partial t} + \frac{U}{h} x_2 \frac{\partial W_i}{\partial x_1} + W_2 \frac{U}{h} \delta_{i1} &= -\frac{1}{\rho} \frac{\partial w}{\partial x_i} + \nu \nabla^2 W_i \\ \frac{\partial W_j}{\partial x_j} &= 0 \end{aligned} \right\} \quad (2.3)$$

对(2.3)式中第一式取散度, 就得到

$$-\frac{1}{\rho} \frac{\partial^2}{\partial x_i \partial x_i} w = 2 \frac{U}{h} \frac{\partial W_2}{\partial x_1} \quad (2.4)$$

式中 U 为特征速度, h 为特征宽度, $\frac{U}{h}$ 就是平均速度梯度 (为一个常数).

引入富氏变换

$$\begin{aligned} W_i &= \iiint e^{i\vec{k} \cdot \vec{x}} \phi_i dk_1 dk_2 dk_3 \\ w &= \iiint e^{i\vec{k} \cdot \vec{x}} \sigma dk_1 dk_2 dk_3 \end{aligned}$$

它们的逆变换为

$$\begin{aligned} \phi_i &= \frac{1}{(2\pi)^3} \iiint e^{-i\vec{k} \cdot \vec{x}} W_i dx_1 dx_2 dx_3 \\ \sigma &= \frac{1}{(2\pi)^3} \iiint e^{-i\vec{k} \cdot \vec{x}} w dx_1 dx_2 dx_3 \end{aligned}$$

于是连续方程成为

$$k_j \phi_j = 0 \quad (2.5)$$

把(2.5)式对 k_n 微分

$$\frac{\partial}{\partial k_n} (k_i \phi_i) = \phi_n + k_i \frac{\partial \phi_i}{\partial k_n} = 0 \quad (2.6)$$

(2.4) 式的富氏变换为

$$\frac{1}{\rho} k^2 \sigma = 2i \frac{U}{h} k_1 \phi_2 \quad (2.7)$$

(2.3) 式第一式的富氏变换为

$$\frac{\partial \phi_i}{\partial t} - \frac{U}{h} k_1 \frac{\partial \phi_i}{\partial k_2} + \frac{U}{h} \phi_2 \delta_{i1} = -\frac{1}{\rho} (i k_i \sigma) - \nu k^2 \phi_i \quad (2.8)$$

消去 σ , 就得到 ϕ_i 所满足的方程式

$$\frac{\partial \phi_i}{\partial t} - \frac{U}{h} k_1 \frac{\partial \phi_i}{\partial k_2} + \frac{U}{h} \phi_2 \left(\delta_{i1} - \frac{2k_1 k_i}{k^2} \right) + \nu k^2 \phi_i = 0 \quad (2.9)$$

必须注意方程 (2.9) 的解还必须满足附加条件 (2.5), 即 $k_j \phi_j = 0$. 把 (2.9) 写成标量形式为

$$\left. \begin{aligned} \frac{\partial \phi_1}{\partial t} + \nu k^2 \phi_1 - \frac{U}{h} k_1 \frac{\partial \phi_1}{\partial k_2} + \frac{U}{h} \phi_2 \left(1 - 2 \frac{k_1^2}{k^2} \right) &= 0 \\ \frac{\partial \phi_2}{\partial t} + \nu k^2 \phi_2 - \frac{U}{h} k_1 \frac{\partial \phi_2}{\partial k_2} + \frac{U}{h} \phi_2 \left(-2 \frac{k_1 k_2}{k^2} \right) &= 0 \\ \frac{\partial \phi_3}{\partial t} + \nu k^2 \phi_3 - \frac{U}{h} k_1 \frac{\partial \phi_3}{\partial k_2} + \frac{U}{h} \phi_2 \left(-2 \frac{k_1 k_3}{k^2} \right) &= 0 \end{aligned} \right\} \quad (2.9)'$$

现在求方程 (2.9) 的解. 我们先求定常解, 这时化为常微分方程组. 积分出来

$$\left. \begin{aligned} \phi_2 &= \frac{c_2}{k^2} \exp \left[\frac{\nu h}{U} \left(\frac{k_1^2 + k_3^2}{k_1} k_2 + \frac{k_2^3}{3k_1} \right) \right] \\ \phi_1 &= \left\{ c_1 + c_2 \left[\frac{\operatorname{tg}^{-1} \frac{k_2}{\sqrt{k_1^2 + k_3^2}}}{k_1 \sqrt{k_1^2 + k_3^2}} - k_1 \left(\frac{k_2}{k^2 (k_1^2 + k_3^2)} + \frac{1}{(k_1^2 + k_3^2)^{3/2}} \right. \right. \right. \\ &\quad \left. \left. \cdot \operatorname{tg}^{-1} \frac{k_2}{\sqrt{k_1^2 + k_3^2}} \right) \right] \right\} \exp \left[\frac{\nu h}{U} \left(\frac{k_1^2 + k_3^2}{k_1} k_2 + \frac{k_2^3}{3k_1} \right) \right] \\ \phi_3 &= \left\{ c_3 - c_2 k_3 \left[\frac{k_2}{k^2 (k_1^2 + k_3^2)} + \frac{1}{(k_1^2 + k_3^2)^{3/2}} \operatorname{tg}^{-1} \frac{k_2}{\sqrt{k_1^2 + k_3^2}} \right] \right\} \\ &\quad \cdot \exp \left[\frac{\nu h}{U} \left(\frac{k_1^2 + k_3^2}{k_1} k_2 + \frac{k_2^3}{3k_1} \right) \right] \end{aligned} \right\} \quad (2.10)$$

式中 c_1, c_2, c_3 为 k_1, k_3 的任意函数. 由于要满足连续方程

$$k_i \phi_i = 0$$

所以

$$k_1 c_1 + k_3 c_3 = 0$$

为了书写方便起见, 我们用 A, B_{12}, B_{32} 分别代表下列表达式

$$\begin{aligned} A &\equiv \frac{\nu h}{U} \left(\frac{k_1^2 + k_3^2}{k_1} k_2 + \frac{k_2^3}{3k_1} \right) \\ B_{12} &\equiv \left\{ \frac{\operatorname{tg}^{-1} \frac{k_2}{\sqrt{k_1^2 + k_3^2}}}{k_1 \sqrt{k_1^2 + k_3^2}} - k_1 \left[\frac{k_2}{k^2 (k_1^2 + k_3^2)} + \frac{1}{(k_1^2 + k_3^2)^{3/2}} \operatorname{tg}^{-1} \frac{k_2}{\sqrt{k_1^2 + k_3^2}} \right] \right\} \\ B_{32} &\equiv k_3 \left[\frac{k_2}{k^2 (k_1^2 + k_3^2)} + \frac{1}{(k_1^2 + k_3^2)^{3/2}} \operatorname{tg}^{-1} \frac{k_2}{\sqrt{k_1^2 + k_3^2}} \right] \end{aligned}$$

而有恒等式

$$\bullet \quad k_1 B_{12} + \frac{k_2}{k^2} - k_3 B_{32} \equiv 0$$

于是

$$\left. \begin{aligned} \phi_1 &= e^A [c_1 + c_2 B_{12}] \\ \phi_2 &= \frac{e^A}{k^2} c_2 \\ \phi_3 &= e^A [c_3 - c_2 B_{32}] \end{aligned} \right\} \quad (2.10)'$$

现在来求解非定常的情形。按通常一阶线性偏微分方程的解法，我们可以直接求解方程 (2.9)'。为了篇幅简短起见，我们只写下最后的结果。我们仍和参考文献[4]那样，把整个解分成各向同性部分和各向异性部分。我们引入符号 ()^{*} 代表 () 内的 k_2 用 $k_2 + \frac{U}{h} k_1 t$ 代入。并且令

$$E = \exp[A - (A)^*] = \exp\left[-\nu k^2 t - k_1 k_2 \frac{U}{h} \nu t^2 - \frac{1}{3} \nu \left(\frac{U}{h}\right)^2 k_1^2 t^3\right]$$

于是我们得到各向同性湍流部分涡旋的解式。

$$\left. \begin{aligned} \phi_1^I &= -iBE(k_n)^* \left[\left(\frac{k_m k_1}{k^2} - \delta_{m1} \right)^s + (k_m k_2 - k^2 \delta_{m2})^s (B_{12} - B_{12}^i) \right] l_m l_n \\ \phi_2^I &= -iBE(k_n)^* \frac{1}{k^2} (k_m k_2 - k^2 \delta_{m2})^s l_m l_n \\ \phi_3^I &= -iBE(k_n)^* \left[\left(\frac{k_m k_3}{k^2} - \delta_{m3} \right)^s - (k_m k_2 - k^2 \delta_{m2})^s (B_{32} - B_{32}^i) \right] l_m l_n \end{aligned} \right\} \quad (2.11)$$

式中 B 为一常数， $B = \frac{1}{3} \frac{A_0 a^4}{\beta^{3/2}} \frac{1}{\pi^{3/2}}$ 。 A_0 , a^2 , β 的含义见文献[1]中的 (3.12) 式。

至于各向异性部分的解式为

$$\left. \begin{aligned} \phi_1^N &= -\lambda_1 A_1 E [(k_1^2 - k^2)^s + (B_{12} - B_{12}^i) (k^2 k_1 k_2)^s] \\ \phi_2^N &= -\lambda_1 A_1 E \frac{(k^2 k_1 k_2)^s}{k^2} \\ \phi_3^N &= -\lambda_1 A_1 E [(k_1 k_3)^s + (B_{32}^i - B_{32}) (k^2 k_1 k_2)^s] \end{aligned} \right\} \quad (2.12)$$

式中 λ_1 , A_1 的意义和文献[4]中一样。

三、二元速度关联函数的富氏变换

在引言中我们曾经说过，一般湍流，它的能量由 Reynolds 应力和平均速度梯度相互作用得来，而由平均流动供给。但是对于我们讨论的这种弱湍流场为了要维持定常运动，必须假定单位时间单位体积里产生相当于 t_0 时刻的涡旋的个数。我们假定每单位时间单位体积产生各向同性的涡旋 n 个，各向异性的涡旋 n_1 个。于是仍旧和文献[4]中第3节一样，二元速度关联函数，可以先求两个速度的富氏变换相乘，得到二元速度关联函数的富氏变换，然后再求富氏逆变换来得到。于是我们有

$$\overline{W_i^* W_j^*} = (2\pi)^3 \iiint e^{i\vec{k} \cdot \vec{r}} dk_1 dk_2 dk_3 \int_{t_0}^{\infty} dt \left[\frac{n}{4\pi} \iint \phi_i^I \phi_j^I d\Omega_i + n_1 \phi_i^N \phi_j^N \right] \quad (3.1)$$

式中对 Ω_i 积分为对各个涡旋的轴 l_i 求平均。 $\phi_i^!$ 为纯虚数， $\bar{\phi}_i^!$ 为它的共轭，所以有

$$\phi_i^! \bar{\phi}_i^! = \bar{\phi}_i^! \phi_i^!$$

ϕ_i^N 为实数，所以没有取共轭的必要。

通过平均，我们最后求得(3.1)中有关的表达式

$$\left. \begin{aligned} \frac{1}{4\pi} \oint \phi_i^! \bar{\phi}_i^! d\Omega_i &= \frac{B^2 E^2}{15} \{ (k^2 - k_1^2)^{\circ} + (B_{12} - B_{1,2}^{\circ})^2 (k^{\circ} - k^4 k_2^{\circ})^{\circ} \\ &\quad - 2(k^2 k_1 k_2)^{\circ} (B_{12} - B_{1,2}^{\circ}) \} \\ \frac{1}{4\pi} \oint \phi_i^! \bar{\phi}_i^! d\Omega_i &= \frac{B^2 E^2}{15} \frac{(k^4)^{\circ}}{k^4} (k^2 - k_2^{\circ})^{\circ} \\ \frac{1}{4\pi} \oint \phi_i^! \bar{\phi}_i^! d\Omega_i &= \frac{B^2 E^2}{15} \{ (k^2 - k_3^{\circ})^{\circ} + (B_{32} - B_{3,2}^{\circ})^2 (k^{\circ} - k^4 k_2^{\circ})^{\circ} \\ &\quad + 2(k^2 k_3 k_2)^{\circ} (B_{32} - B_{3,2}^{\circ}) \} \\ \frac{1}{4\pi} \oint \phi_i^! \bar{\phi}_i^! d\Omega_i &= \frac{B^2 E^2}{15} \{ -(k_1 k_3) - (B_{12} - B_{1,2}^{\circ}) (B_{32} - B_{3,2}^{\circ}) (k^{\circ} - k^4 k_2^{\circ})^{\circ} \\ &\quad - (B_{12} - B_{1,2}^{\circ}) (k^2 k_2 k_3)^{\circ} + (B_{32} - B_{3,2}^{\circ}) (k^2 k_1 k_2)^{\circ} \} \\ \frac{1}{4\pi} \oint \phi_i^! \bar{\phi}_i^! d\Omega_i &= \frac{B^2 E^2}{15} \frac{1}{k^2} \{ -(k^2 k_1 k_2)^{\circ} + (B_{12} - B_{1,2}^{\circ}) (k^{\circ} - k^4 k_2^{\circ})^{\circ} \} \\ \frac{1}{4\pi} \oint \phi_i^! \bar{\phi}_i^! d\Omega_i &= \frac{1}{15} \frac{B^2 E^2}{k^2} \{ -(k_3 k_2 k^2)^{\circ} - (B_{32} - B_{3,2}^{\circ}) (k^{\circ} - k^4 k_2^{\circ})^{\circ} \} \end{aligned} \right\} \quad (3.2)$$

各向异性部分

$$\left. \begin{aligned} \phi_1^N \phi_1^N &= \lambda_1^2 A_1^2 E^2 \{ (k_1^2 - k^2)^{\circ} + (B_{12} - B_{1,2}^{\circ}) (k^2 k_1 k_2)^{\circ} \}^2 \\ \phi_2^N \phi_2^N &= \lambda_2^2 A_2^2 E^2 \frac{(k^4 k_1^{\circ} k_2^{\circ})^{\circ}}{k^4} \\ \phi_3^N \phi_3^N &= \lambda_3^2 A_3^2 E^2 [(k_1 k_3)^{\circ} + (B_{3,2}^{\circ} - B_{32}) (k^2 k_1 k_2)^{\circ}]^2 \\ \phi_1^N \phi_3^N &= \lambda_1^2 A_1^2 E^2 [(k_1^2 - k^2)^{\circ} + (B_{12} - B_{1,2}^{\circ}) (k^2 k_1 k_2)^{\circ}] \\ &\quad \cdot [(k_1 k_3)^{\circ} + (B_{3,2}^{\circ} - B_{32}) (k^2 k_1 k_2)^{\circ}] \\ \phi_1^N \phi_2^N &= \lambda_1^2 A_1^2 E^2 [(k_1^2 - k^2)^{\circ} + (B_{12} - B_{1,2}^{\circ}) (k^2 k_1 k_2)^{\circ}] \frac{(k^2 k_1 k_2)^{\circ}}{k^2} \\ \phi_3^N \phi_2^N &= \lambda_3^2 A_3^2 E^2 [(k_1 k_3)^{\circ} + (B_{3,2}^{\circ} - B_{32}) (k^2 k_1 k_2)^{\circ}] \frac{(k^2 k_1 k_2)^{\circ}}{k^2} \end{aligned} \right\} \quad (3.3)$$

把(3.2)和(3.3)代入(3.1)式积分，就能得到二元速度关联函数。但是要把这样复杂的式子积出解析式子当然是不可能的。

四、二元速度关联函数的一级近似

在速度梯度小的时候，我们可以把(3.2)和(3.3)式对速度梯度展开。这时

$$\left. \begin{aligned} B_{1,2}^{\circ} &\approx B_{12} + \left(\frac{1}{k^2 k_1} - \frac{2k_1}{k^4} \right) k_1 \frac{Ut}{h} \\ B_{3,2}^{\circ} &\approx B_{32} + \frac{2k_3 k_1}{k^4} \frac{Ut}{h} \\ (k^2)^{\circ} &\approx k^2 + 2k_2 k_1 \frac{Ut}{h} \end{aligned} \right\} \quad (4.1)$$

$$\left. \begin{aligned} (k^2 k_2)^0 &\approx k^2 k_2 + (2k_1^2 k_1 + k_1 k^2) \frac{Ut}{h} \\ (k^4)^0 &\approx k^4 + 4k^2 k_1 k_2 \frac{Ut}{h} \\ E^2 &\approx e^{-2\nu k^2 t} \left[1 - 2k_1 k_2 \nu t \frac{Ut}{h} \right] \end{aligned} \right\}$$

于是(3.2)就成了

$$\left. \begin{aligned} \frac{1}{4\pi} \iint \phi_1^1 \bar{\phi}_1^1 d\Omega_1 &= \frac{B^2}{15} e^{-2\nu t k^2} \left[k^2 - k_1^2 + 2k_1 k_2 \frac{Ut}{h} \left(2 - 2\frac{k_1^2}{k^2} - \nu t k^2 + \nu t k_1^2 \right) \right] \\ \frac{1}{4\pi} \iint \phi_2^1 \bar{\phi}_2^1 d\Omega_1 &= \frac{B^2}{15} e^{-2\nu t k^2} (k^2 - k_2^2) \left[1 + \left(4\frac{k_1 k_2}{k^2} - 2k_1 k_2 \nu t \right) \frac{Ut}{h} \right] \\ \frac{1}{4\pi} \iint \phi_3^1 \bar{\phi}_3^1 d\Omega_1 &= \frac{B^2}{15} e^{-2\nu t k^2} \left[k^2 - k_3^2 + 2k_1 k_2 \frac{Ut}{h} \left(1 - \frac{2k_3^2}{k^2} \right. \right. \\ &\quad \left. \left. - k^2 \nu t + k_3^2 \nu t \right) \right] \\ \frac{1}{4\pi} \iint \phi_1^1 \bar{\phi}_3^1 d\Omega_1 &= \frac{B^2}{15} e^{-2\nu t k^2} \left\{ -k_1 k_3 + \left(k_2 k_3 - \frac{4k_1^2 k_2 k_3}{k} \right. \right. \\ &\quad \left. \left. + 2k_1^2 k_2 k_3 \nu t \right) \frac{Ut}{k} \right\} \\ \frac{1}{4\pi} \iint \phi_1^1 \bar{\phi}_2^1 d\Omega_1 &= \frac{B^2}{15} e^{-2\nu t k^2} \left\{ -k_1 k_2 - \frac{Ut}{h} \left(k^2 + \frac{4k_1^2 k_2^2}{k^2} - 2k_1^2 k_2^2 \nu t \right) \right\} \\ \frac{1}{4\pi} \iint \phi_2^1 \bar{\phi}_3^1 d\Omega_1 &= \frac{B^2}{15} e^{-2\nu t k^2} \left\{ -k_3 k_2 + \frac{Ut}{h} \left(k_1 k_3 - \frac{4k_2^2 k_1 k_3}{k^2} \right. \right. \\ &\quad \left. \left. + 2k_2^2 k_1 k_3 \nu t \right) \right\} \end{aligned} \right\} \quad (4.2)$$

而(3.3)式也就变成

$$\left. \begin{aligned} \phi_1^N \phi_1^N &= \lambda_1^2 A_1^2 e^{-2\nu t k^2} \left\{ (k_1^2 - k^2)^2 - \left[2k_1 k_2 \nu t (k_1^2 - k^2)^2 + 2k_1 k_2 (k_1^2 - k^2) \right. \right. \\ &\quad \left. \left. \cdot \left(3 - \frac{2k_1^2}{k^2} \right) \right] \frac{Ut}{h} \right\} \\ \phi_2^N \phi_2^N &= \lambda_2^2 A_2^2 e^{-2\nu t k^2} \left\{ k_1^2 k_2^2 + 2 \left[\frac{k_1^2 k_2}{k^2} (k^2 k_1 + 2k_2^2 k_1) - k_2^2 k_1^2 \nu t \right] \frac{Ut}{h} \right\} \\ \phi_3^N \phi_3^N &= \lambda_3^2 A_3^2 e^{-2\nu t k^2} \left[k_1^2 k_3^2 + \left(4\frac{k_1^2 k_2 k_3^2}{k^2} - 2k_1^2 k_2 k_3^2 \nu t \right) \frac{Ut}{h} \right] \\ \phi_1^N \phi_3^N &= \lambda_1^2 A_1^2 e^{-2\nu t k^2} \left[k_1 k_3 (k_1^2 - k^2) - \left(5k_1^2 k_2 k_3 - \frac{4k_1^4 k_2 k_3}{k^2} \right) \frac{Ut}{h} \right. \\ &\quad \left. - 2k_1^2 k_2 k_3 \nu t (k_1^2 - k^2) \frac{Ut}{h} \right] \\ \phi_1^N \phi_2^N &= \lambda_1^2 A_1^2 e^{-2\nu t k^2} \left[k_1 k_2 (k_1^2 - k^2) - \left(5k_1^2 k_2^2 - \frac{4k_1^4 k_2^2}{k^2} - k_1^4 + k_1^2 k^2 \right. \right. \\ &\quad \left. \left. + 2k_1^4 k_2^2 \nu t - 2k_1^2 k_2^2 k^2 \nu t \right) \frac{Ut}{h} \right] \end{aligned} \right\} \quad (4.3)$$

$$\phi_3^N \phi_2^N = \lambda_1^2 A_1^2 e^{-2\nu t k^2} \left[k_1^2 k_2 k_3 + \left(4 \frac{k_1^3 k_2^2 k_3}{k^2} + k_1^3 k_3 - 2k_1^3 k_2^2 k_3 \nu t \right) \frac{U t}{h} \right]$$

把(4.2)和(4.3)代入(3.1)式积分, 就得到下列关联函数表达式

$$\begin{aligned} \overline{W_1^i W_1^i} &= \frac{8}{15} B^2 n t_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} (2\pi)^3 \int_0^1 \left[\frac{\eta^2}{8\nu t_0} - (r^2 - x_1^2) \left(\frac{\eta^2}{8\nu t_0} \right)^2 \right] \exp\left(\frac{-\eta^2 r^2}{8\nu t_0} \right) d\eta \\ &+ 32 \lambda_1^2 A_1^2 n_1 t_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} (2\pi)^3 \int_0^1 \left[2 \left(\frac{\eta^2}{8\nu t_0} \right)^2 - 4(r^2 - x_1^2) \left(\frac{\eta^2}{8\nu t_0} \right)^3 \right. \\ &+ \left. (r^4 - 2r^2 x_1^2 + x_1^4) \left(\frac{\eta^2}{8\nu t_0} \right)^4 \right] \exp\left(\frac{-\eta^2 r^2}{8\nu t_0} \right) d\eta \\ &- \frac{4}{15} B^2 \frac{n t_0}{8\nu t_0} \frac{U t_0}{h} \left(\frac{\pi}{2\nu t_0} \right)^{3/2} (2\pi)^3 \int_0^1 \left[2(r^2 x_1 x_2 - x_1^3 x_2) \left(\frac{\eta^2}{8\nu t_0} \right)^2 \right. \\ &+ \left. 4x_1 x_2 \left(\frac{\eta^2}{8\nu t_0} \right) \right] \exp\left(\frac{-\eta^2 r^2}{8\nu t_0} \right) d\eta \\ &- \frac{8}{15} B^2 \frac{n t_0}{(8\nu t_0)^2} \frac{2\pi\sqrt{\pi}}{\sqrt{2\nu t_0}} (2\pi)^3 \frac{U t_0}{h} \int_0^1 \int_0^1 \beta^4 \left[16 \left(\frac{\eta^2 \beta^2}{8\nu t_0} \right)^2 x_1^3 x_2 \right. \\ &- \left. 24 \left(\frac{\eta^2 \beta^2}{8\nu t_0} \right) x_1 x_2 \right] \exp\left(-\frac{\eta^2 \beta^2 r^2}{8\nu t_0} \right) d\beta d\eta \\ &- \frac{\lambda_1^2 A_1^2 n_1 t_0}{8\nu t_0} \frac{4U t_0}{h} \left(\frac{\pi}{2\nu t_0} \right)^{3/2} (2\pi)^3 \int_0^1 \left[-8(r^4 x_1 x_2 - 2r^2 x_1^3 x_2 + x_1^5 x_2) \cdot \left(\frac{\eta^2}{8\nu t_0} \right)^4 \right. \\ &+ \left. 32x_1^3 x_2 \left(\frac{\eta^2}{8\nu t_0} \right)^3 \right] \exp\left(\frac{-\eta^2 r^2}{8\nu t_0} \right) d\eta \\ &- \frac{8\lambda_1^2 A_1^2 n_1 t_0}{(8\nu t_0)^2} \frac{U t_0}{h} \frac{2\pi\sqrt{\pi}}{\sqrt{2\nu t_0}} (2\pi)^3 \int_0^1 \int_0^1 \beta^4 \left[64x_1^3 x_2 \left(\frac{\beta^2 \eta^2}{8\nu t_0} \right)^4 \right. \\ &- \left. 320x_1^3 x_2 \left(\frac{\beta^2 \eta^2}{8\nu t_0} \right)^3 + 240x_1 x_2 \left(\frac{\beta^2 \eta^2}{8\nu t_0} \right)^2 \right] \exp\left(-\frac{\beta^2 \eta^2 r^2}{8\nu t_0} \right) d\eta d\beta \\ \overline{W_1^i W_2^i} &= \frac{8}{15} B^2 n t_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} (2\pi)^3 \int_0^1 \left[\frac{\eta^2}{8\nu t_0} - (r^2 - x_2^2) \left(\frac{\eta^2}{8\nu t_0} \right)^2 \right] \\ &\cdot \exp\left(\frac{-\eta^2 r^2}{8\nu t_0} \right) d\eta \\ &+ 8\lambda_1^2 A_1^2 n_1 t_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} (2\pi)^3 \int_0^1 \left[\left(\frac{\eta^2}{8\nu t_0} \right)^2 - 2(x_1^2 + x_2^2) \left(\frac{\eta^2}{8\nu t_0} \right)^3 \right. \\ &+ \left. 4x_1^2 x_2^2 \left(\frac{\eta^2}{8\nu t_0} \right)^4 \right] \exp\left(\frac{-\eta^2 r^2}{8\nu t_0} \right) d\eta \\ &- \frac{4}{15} B^2 \frac{n t_0}{8\nu t_0} \frac{U t_0}{h} \left(\frac{\pi}{2\nu t_0} \right)^{3/2} (2\pi)^3 \int_0^1 \left[2(r^2 x_1 x_2 - x_1 x_2^3) \left(\frac{\eta^2}{8\nu t_0} \right)^2 \right. \\ &+ \left. 4x_1 x_2 \left(\frac{\eta^2}{8\nu t_0} \right) \right] \exp\left(\frac{-\eta^2 r^2}{8\nu t_0} \right) d\eta \end{aligned}$$

$$\begin{aligned}
& -\frac{8B^2}{15} \frac{nt_0}{(8\nu t_0)^2} \frac{Ut_0}{h} \frac{2\pi\sqrt{\pi}}{\sqrt{2\nu t_0}} (2\pi)^3 \int_0^1 \int_0^1 \beta^4 \left[16x_1 x_2 \left(\frac{\eta^2 \beta^2}{8\nu t_0} \right)^2 \right. \\
& \left. - 24x_1 x_2 \left(\frac{\eta^2 \beta^2}{8\nu t_0} \right) \right] \exp\left(-\frac{\beta^2 \eta^2 r^2}{8\nu t_0} \right) d\beta d\eta \\
& + \frac{4\lambda_1^2 A_1^2 n_1 t_0}{8\nu t_0} \frac{Ut_0}{h} \left(\frac{\pi}{2\nu t_0} \right)^{3/2} (2\pi)^3 \int_0^1 \left[8x_1^3 x_2^3 \left(\frac{\eta^2}{8\nu t_0} \right)^4 + (4x_1^3 x_2 \right. \\
& \left. - 12x_1 x_2^3) \left(\frac{\eta^2}{8\nu t_0} \right)^3 - 6x_1 x_2 \left(\frac{\eta^2}{8\nu t_0} \right)^2 \right] \exp\left(-\frac{\eta^2 r^2}{8\nu t_0} \right) d\eta \\
& - \frac{4\lambda_1^2 A_1^2 n_1 t_0}{(8\nu t_0)^2} \frac{Ut_0}{h} \frac{4\pi\sqrt{\pi}}{\sqrt{2\nu t_0}} (2\pi)^3 \int_0^1 \int_0^1 \beta^4 \left[64 \left(\frac{\beta^2 \eta^2}{8\nu t_0} \right)^4 x_1^3 x_2^3 \right. \\
& \left. - 96(x_1^3 x_2 + x_1 x_2^3) \left(\frac{\beta^2 \eta^2}{8\nu t_0} \right)^3 + 144x_1 x_2 \left(\frac{\beta^2 \eta^2}{8\nu t_0} \right)^2 \right] \exp\left(-\frac{\beta^2 \eta^2 r^2}{8\nu t_0} \right) d\eta d\beta \\
\overline{W_1 W_2} = & \frac{8B^2}{15} nt_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} (2\pi)^3 \int_0^1 \left[\left(\frac{\eta^2}{8\nu t_0} \right) - (r^2 - x_2^2) \left(\frac{\eta^2}{8\nu t_0} \right)^2 \right] \exp\left(-\frac{\eta^2 r^2}{8\nu t_0} \right) d\eta \\
& + 8\lambda_1^2 A_1^2 n_1 t_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} (2\pi)^3 \int_0^1 \left[\left(\frac{\eta^2}{8\nu t_0} \right)^2 - 2(x_1^2 + x_2^2) \left(\frac{\eta^2}{8\nu t_0} \right) \right. \\
& \left. + 4x_1^2 x_2^2 \left(\eta^2 / 8\nu t_0 \right)^4 \right] \exp\left(-\frac{\eta^2 r^2}{8\nu t_0} \right) d\eta \\
& - \frac{nB^2 t_0}{15} \frac{4Ut_0}{h} \frac{1}{8\nu t_0} \left(\frac{\pi}{2\nu t_0} \right)^{3/2} (2\pi)^3 \int_0^1 \left[2(r^2 x_1 x_2 - x_2^3 x_1 x_2) \left(\frac{\eta^2}{8\nu t_0} \right)^2 \right. \\
& \left. - 2x_1 x_2 \left(\frac{\eta^2}{8\nu t_0} \right) \right] \exp\left(-\frac{\eta^2 r^2}{8\nu t_0} \right) d\eta \\
& - \frac{2nB^2 t_0}{15} \frac{4Ut_0}{h} \frac{1}{(8\nu t_0)^2} \frac{2\pi\sqrt{\pi}}{\sqrt{2\nu t_0}} (2\pi)^3 \int_0^1 \int_0^1 \beta^4 \left[16 \left(\frac{\eta^2 \beta^2}{8\nu t_0} \right)^2 x_1 x_2 x_2^3 \right. \\
& \left. - 8 \left(\frac{\eta^2 \beta^2}{8\nu t_0} \right) x_1 x_2 \right] \exp\left(-\frac{\beta^2 \eta^2 r^2}{8\nu t_0} \right) d\beta d\eta \\
& + \frac{2\lambda_1^2 A_1^2 n_1 t_0}{8\nu t_0} \frac{2Ut_0}{h} \left(\frac{\pi}{2\nu t_0} \right)^{3/2} (2\pi)^3 \int_0^1 \left[8x_1^3 x_2 x_2^3 \left(\frac{\eta^2}{8\nu t_0} \right)^4 \right. \\
& \left. - 4(3x_1 x_2 x_2^3 + x_1^3 x_2) \left(\frac{\eta^2}{8\nu t_0} \right)^3 + 6x_1 x_2 \left(\frac{\eta^2}{8\nu t_0} \right)^2 \right] \exp\left(-\frac{\eta^2 r^2}{8\nu t_0} \right) d\eta \\
& - \frac{2\lambda_1^2 A_1^2 n_1 t_0}{(8\nu t_0)^2} \frac{4Ut_0}{h} \frac{2\pi\sqrt{\pi}}{\sqrt{2\nu t_0}} (2\pi)^3 \int_0^1 \int_0^1 \beta^4 \left[64 \left(\frac{\beta^2 \eta^2}{8\nu t_0} \right)^4 x_1^3 x_2 x_2^3 \right. \\
& \left. - 32(3x_1 x_2 x_2^3 + x_1^3 x_2) \left(\frac{\beta^2 \eta^2}{8\nu t_0} \right)^3 + 48x_1 x_2 \left(\frac{\beta^2 \eta^2}{8\nu t_0} \right)^2 \right] \exp\left(-\frac{\beta^2 \eta^2 r^2}{8\nu t_0} \right) d\eta d\beta \\
\overline{W_1 W_2} = & \frac{2B^2}{15} nt_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} (2\pi)^3 \int_0^1 \exp\left(-\frac{\eta^2 r^2}{8\nu t_0} \right) \left[4x_1 x_2 \left(\frac{\eta^2}{8\nu t_0} \right) \right] d\eta \\
& + 2\lambda_1^2 A_1^2 n_1 t_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} (2\pi)^3 \int_0^1 \exp\left(-\frac{\eta^2 r^2}{8\nu t_0} \right) \left[16(x_1^3 x_2 \right.
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
 & -r^2 x_1 x_3 \left(\frac{\eta^2}{8\nu t_0} \right)^4 + 32 x_1 x_3 \left(\frac{\eta^2}{8\nu t_0} \right)^3 \Big] d\eta \\
 & - \frac{2}{15} B^2 n t_0 \frac{U t_0}{h} \frac{1}{8\nu t_0} \left(\frac{\pi}{2\nu t_0} \right)^{3/2} (2\pi)^3 \int_0^1 \left[6 x_2 x_3 \left(\frac{\eta^2}{8\nu t_0} \right) \right. \\
 & \left. - 4 x_1^2 x_2 x_3 \left(\frac{\eta^2}{8\nu t_0} \right)^2 \right] \exp\left(-\frac{\eta^2 r^2}{8\nu t_0} \right) d\eta \\
 & - \frac{8 B^2}{15} n t_0 \frac{U t_0}{h} \left(\frac{1}{8\nu t_0} \right)^2 \frac{2\pi\sqrt{\pi}}{\sqrt{2\nu t_0}} (2\pi)^3 \int_0^1 \int_0^1 \beta^4 \exp\left(-\frac{\eta^2 \beta^2 r^2}{8\nu t_0} \right) \\
 & \cdot \left[16 x_1^2 x_2 x_3 \left(\frac{\eta^2 \beta^2}{8\nu t_0} \right)^2 - 8 x_2 x_3 \left(\frac{\eta^2 \beta^2}{8\nu t_0} \right) \right] d\eta d\beta \\
 & + \frac{2\lambda_1^2 A_1^2 n_1 t_0}{8\nu t_0} \frac{U t_0}{h} \left(\frac{\pi}{2\nu t_0} \right)^{3/2} (2\pi)^3 \int_0^1 \exp\left(-\frac{\eta^2 r^2}{8\nu t_0} \right) \left[16 (x_1^2 x_2 x_3 \right. \\
 & \left. - r^2 x_1^2 x_2 x_3) \left(\frac{\eta^2}{8\nu t_0} \right)^4 + (8 r^2 x_2 x_3 - 40 x_1^2 x_2 x_3) \left(\frac{\eta^2}{8\nu t_0} \right)^3 \right. \\
 & \left. + 16 x_2 x_3 \left(\frac{\eta^2}{8\nu t_0} \right)^2 \right] d\eta \\
 & - \frac{8\lambda_1^2 A_1^2 n_1 t_0}{(8\nu t_0)^2} \frac{U t_0}{h} \frac{2\pi\sqrt{\pi}}{\sqrt{2\nu t_0}} (2\pi)^3 \int_0^1 \int_0^1 \beta^4 \exp\left(-\frac{\beta^2 \eta^2 r^2}{8\nu t_0} \right) \\
 & \cdot \left[64 \left(\frac{\beta^2 \eta^2}{8\nu t_0} \right)^4 x_1^2 x_2 x_3 - 192 \left(\frac{\beta^2 \eta^2}{8\nu t_0} \right)^3 x_1^2 x_2 x_3 + 48 x_2 x_3 \left(\frac{\beta^2 \eta^2}{8\nu t_0} \right) \right] d\beta d\eta \\
 \overline{W_1 W_2} = & \frac{2 B^2}{15} n t_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} (2\pi)^3 \int_0^1 4 x_1 x_2 \left(\frac{\eta^2}{8\nu t_0} \right)^2 \exp\left(-\frac{\eta^2 r^2}{8\nu t_0} \right) d\eta \\
 & + 2\lambda_1^2 A_1^2 n_1 t_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} (2\pi)^3 \int_0^1 \exp\left(-\frac{\eta^2 r^2}{8\nu t_0} \right) \left[16 (x_1^2 x_2 \right. \\
 & \left. - r^2 x_1 x_2) \left(\frac{\eta^2}{8\nu t_0} \right)^4 + 32 x_1 x_2 \left(\frac{\eta^2}{8\nu t_0} \right)^3 \right] d\eta \\
 & - \frac{2 B^2}{15} n t_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} \frac{U t_0}{h} \frac{1}{8\nu t_0} (2\pi)^3 \int_0^1 \exp\left(-\frac{\eta^2 r^2}{8\nu t_0} \right) \left[-4 x_1^2 x_2^2 \left(\frac{\eta^2}{8\nu t_0} \right)^2 \right. \\
 & \left. + (2 x_1^2 + 2 x_2^2 - 4 x_3^2) \left(\frac{\eta^2}{8\nu t_0} \right) + 1 \right] d\eta \\
 & - \frac{8 B^2}{15} n t_0 \frac{2\pi\sqrt{\pi}}{\sqrt{2\nu t_0}} \frac{U t_0}{h} \frac{1}{(8\nu t_0)^2} (2\pi)^3 \int_0^1 \int_0^1 \beta^4 \exp\left(-\frac{\beta^2 \eta^2 r^2}{8\nu t_0} \right) \\
 & \cdot \left[16 x_1^2 x_2^2 \left(\frac{\eta^2 \beta^2}{8\nu t_0} \right)^2 - 8 (x_1^2 + x_2^2) \left(\frac{\eta^2 \beta^2}{8\nu t_0} \right) + 4 \right] d\beta d\eta \\
 & - 2\lambda_1^2 A_1^2 n_1 t_0 \frac{U t_0}{h} \left(\frac{\pi}{2\nu t_0} \right)^{3/2} \frac{1}{8\nu t_0} (2\pi)^3 \int_0^1 \exp\left(-\frac{\eta^2 r^2}{8\nu t_0} \right) \left[16 (x_1^2 x_2^2 r^2 \right. \\
 & \left. - x_1^2 x_2^2) \left(\frac{\eta^2}{8\nu t_0} \right)^4 + (40 x_1^2 x_2^2 - 8 x_1^2 + 8 x_2^2 r^2 - 8 x_2^2 r^2) \left(\frac{\eta^2}{8\nu t_0} \right)^3 \right.
 \end{aligned}$$

$$\begin{aligned}
& + (-16x_2^2 - 4r^2 - 36x_1^2) \left(\frac{\eta^2}{8\nu t_0} \right)^2 + 20 \left(\frac{\eta^2}{8\nu t_0} \right) \Big] d\eta \\
& - \frac{8\lambda_1^2 A_1^2 n_1 t_0}{(8\nu t_0)^2} \frac{U t_0}{h} \frac{2\pi\sqrt{\pi}}{\sqrt{2\nu t_0}} (2\pi)^3 \int_0^1 \int_0^1 \beta^4 \exp\left(-\frac{\beta^2 \eta^2 r^2}{8\nu t_0}\right) \left[64x_1^2 x_2^2 \left(\frac{\beta^2 \eta^2}{8\nu t_0} \right)^4 \right. \\
& - 32(x_1^2 + 6x_1^2 x_2^2) \left(\frac{\beta^2 \eta^2}{8\nu t_0} \right)^3 + 48(2x_1^2 + x_2^2) \left(\frac{\eta^2 \beta^2}{8\nu t_0} \right)^2 \\
& \left. - 24 \left(\frac{\eta^2 \beta^2}{8\nu t_0} \right) \right] d\beta d\eta \\
\overline{W_2' W_3''} = & \frac{2B^2}{15} n t_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} (2\pi)^3 \int_0^1 \left[4x_3 x_2 \left(\frac{\eta^2}{8\nu t_0} \right)^2 \right] \exp\left(-\frac{\eta^2 r^2}{8\nu t_0}\right) d\eta \\
& + 2\lambda_1^2 A_1^2 n_1 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} (2\pi)^3 \int_0^1 \exp\left(-\frac{\eta^2 r^2}{8\nu t_0}\right) \left[16x_1^2 x_2 x_3 \left(\frac{\eta^2}{8\nu t_0} \right)^4 \right. \\
& \left. - 8x_2 x_3 \left(\frac{\eta^2}{8\nu t_0} \right)^3 \right] d\eta \\
& + \frac{2B^2}{15} n t_0 \frac{U t_0}{h} \left(\frac{\pi}{2\nu t_0} \right)^{3/2} \frac{1}{8\nu t_0} (2\pi)^3 \int_0^1 \exp\left(-\frac{\eta^2 r^2}{8\nu t_0}\right) \left[4x_2^2 x_1 x_3 \left(\frac{\eta^2}{8\nu t_0} \right)^2 \right. \\
& \left. - 6x_1 x_3 \left(\frac{\eta^2}{8\nu t_0} \right) \right] d\eta \\
& - \frac{8B^2}{15} n t_0 \frac{U t_0}{h} \frac{2\pi\sqrt{\pi}}{\sqrt{2\nu t_0}} \frac{1}{(8\nu t_0)^2} (2\pi)^3 \int_0^1 \int_0^1 \left[16x_1 x_2^2 x_3 \left(\frac{\eta^2 \beta^2}{8\nu t_0} \right)^4 \right. \\
& \left. - 8x_1 x_3 \left(\frac{\eta^2 \beta^2}{8\nu t_0} \right) \right] \beta^4 \exp\left(-\frac{\beta^2 \eta^2 r^2}{8\nu t_0}\right) d\beta d\eta \\
& + 2\lambda_1^2 A_1^2 n_1 t_0 \frac{U t_0}{h} \left(\frac{\pi}{2\nu t_0} \right)^{3/2} \frac{1}{8\nu t_0} (2\pi)^3 \int_0^1 \exp\left(-\frac{\eta^2 r^2}{8\nu t_0}\right) \left[16x_1^3 x_2^2 x_3 \left(\frac{\eta^2}{8\nu t_0} \right)^4 \right. \\
& \left. + 8(x_1^3 x_3 - 3x_1 x_2^2 x_3) \left(\frac{\eta^2}{8\nu t_0} \right)^3 - 12x_1 x_3 \left(\frac{\eta^2}{8\nu t_0} \right)^2 \right] d\eta \\
& - 8\lambda_1^2 A_1^2 n_1 t_0 \frac{U t_0}{h} \frac{2\pi\sqrt{\pi}}{\sqrt{2\nu t_0}} \frac{1}{(8\nu t_0)^2} (2\pi)^3 \int_0^1 \int_0^1 \beta^4 \exp\left(-\frac{\beta^2 \eta^2 r^2}{8\nu t_0}\right) \left[64 \left(\frac{\beta^2 \eta^2}{8\nu t_0} \right)^4 x_1^3 x_2^2 x_3 \right. \\
& \left. - 32 \left(\frac{\beta^2 \eta^2}{8\nu t_0} \right)^3 (3x_1 x_2^2 x_3 + x_1^3 x_3) + 48 \left(\frac{\beta^2 \eta^2}{8\nu t_0} \right)^2 x_1 x_3 \right] d\beta d\eta
\end{aligned}$$

从以上的表达式，我们可以看到，速度梯度的一次项并不影响到 x 方向， y 方向， z 方向的关联系数 $f\left(\frac{r}{\lambda}\right)$ 。原因很简单，因为关联系数 $f\left(\frac{r}{\lambda}\right)$ 的定义是速度分量方向和坐标方向一致。而速度梯度的一次项要不等于零，必须要有两个互相垂直的坐标出现。理由是，平均速度梯度占有一个指标1，另一个指标2，而在 f 函数中出现的都是偶数个指标1（或2，或3），因此剩下的必须为1和2的奇数指标，而这是与 f 的定义相矛盾的。对于函数 $g\left(\frac{r}{\lambda}\right)$ 也是类似。但这仅仅是对原来坐标 x ， y ， z 三个方向而言。如果我们更换坐标系。令

$$x_1^* = \frac{1}{\sqrt{2}}(x_1 + x_2), \quad x_2^* = \frac{1}{\sqrt{2}}(x_2 - x_1), \quad x_3^* = x_3$$

或

$$x_1 = \frac{1}{\sqrt{2}}(x_1^* - x_2^*), \quad x_2 = \frac{1}{\sqrt{2}}(x_1^* + x_2^*), \quad x_3 = x_3^*$$

则

$$\overline{W_1^* W_1^{*'}} = \frac{1}{2} \overline{W_1' W_1''} + \overline{W_1' W_2''} + \frac{1}{2} \overline{W_2' W_2''}$$

当我们的距离取在 x_1^* 方向, 则平均速度梯度的一次项, 不仅在 $f(\frac{r}{\lambda})$ 各向异性部分不等于

零, 即使对各向同性部分, 也并不为零. 具体说来, 对 $f(\frac{r}{\lambda})$ 来说在 x_1^* 方向 λ^2 将减小, 而

$f(\frac{r}{\lambda})$ 的函数形状将变得有所扩展. 而在 x_2^* 方向, λ^2 将有所增大, 而 $f(\frac{r}{\lambda})$ 函数的形状将缩窄. 当各向异性部分不存在的时候, Taylor 微尺度 λ^2 随方向的变化情形如下

$$\lambda_x^2 = \lambda_y^2 = \lambda_z^2 = \frac{20}{3} \nu t_0$$

$$\lambda_{x^*}^2 = \frac{\left(\frac{8}{3} - \frac{26}{5} \frac{U t_0}{h}\right) \nu t_0}{2\left(\frac{1}{5} - \frac{17}{84} \frac{U t_0}{h}\right)} = \frac{20}{3} \nu t_0 \left(1 - \frac{197}{210} \frac{U t_0}{h}\right)$$

$$\lambda_{y^*}^2 = \frac{\left(\frac{8}{3} + \frac{26}{5} \frac{U t_0}{h}\right) \nu t_0}{2\left(\frac{1}{5} + \frac{17}{84} \frac{U t_0}{h}\right)} = \frac{20}{3} \nu t_0 \left(1 + \frac{197}{210} \frac{U t_0}{h}\right)$$

这些结果是以前没有人得到过的.

因为缺乏平均速度梯度为常数时槽流或管流的湍流关联函数的实验数据, 所以只能和网格后端流的实验数据进行比较. 结果见图 1 和图 2. 图 1 曲线为各向异性部分不存在时的 f 函

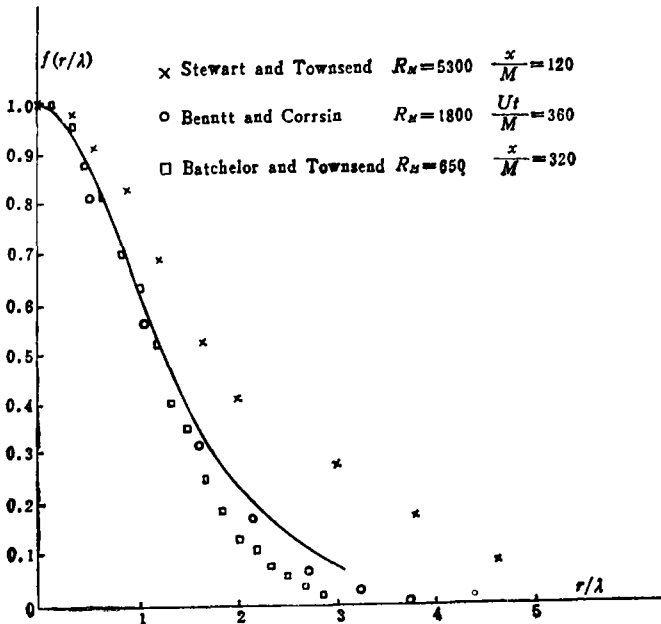


图 1

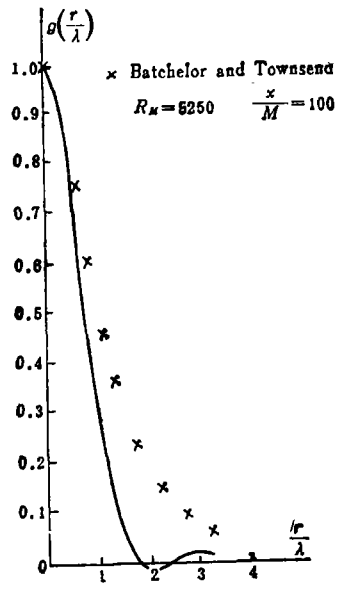


图 2

数, 从图中实验点可以看出, f 函数曲线在前期实验点和后期实验点之间. 图 2 为 g 函数的曲线, 实验点为前期衰变时的实验点.

从表达式(4.4)可以看出, 在没有平均速度梯度时, 所有速度关联函数远处的性质都随 $\frac{1}{r^3}$ 趋于零. 在有平均速度梯度存在以后, 关联函数远处的性质将随 $\frac{1}{r}$ 趋于零, 比原来的速度关联函数趋于零的速度要慢得多了.

五、Reynolds 应力和湍流粘滞系数

从(4.4)的第五个式子, 我们可以得到Reynolds剪切应力的表达式

$$\begin{aligned} \tau_{12} = & -\rho \overline{W_1 W_2} = (2\pi)^3 \rho \left[\frac{2B^2}{15} n t_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} \frac{U t_0}{h} \frac{1}{8\nu t_0} \right. \\ & + \frac{8B^2}{15} n t_0 \left(\frac{\pi}{2\nu t_0} \right)^{1/2} 2\pi \frac{U t_0}{h} \frac{1}{(8\nu t_0)^2} \cdot \frac{4}{5} \\ & + 2\lambda_1^2 A_1^2 n_1 t_0 \frac{U t_0}{h} \left(\frac{\pi}{2\nu t_0} \right)^{3/2} \frac{1}{(8\nu t_0)^2} \cdot \frac{20}{3} \\ & \left. - \frac{8\lambda_1^2 A_1^2 n_1 t_0}{(8\nu t_0)^3} \frac{U t_0}{h} \frac{2\pi\sqrt{\pi}}{\sqrt{2\nu t_0}} \cdot \frac{8}{7} \right] = (2\pi)^3 \rho \left[\frac{B^2}{15} n t_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} \frac{U t_0}{h} \frac{1}{8\nu t_0} \cdot \frac{26}{5} \right. \\ & \left. + \lambda_1^2 A_1^2 n_1 t_0 \frac{U t_0}{h} \left(\frac{\pi}{2\nu t_0} \right)^{3/2} \frac{1}{(8\nu t_0)^2} \cdot \frac{184}{21} \right] \end{aligned} \quad (5.1)$$

因为我们的理论是线性化理论, 所以得到Reynolds剪应力和速度梯度 $\frac{U}{h}$ 成正比的结果. 这个结果和其他方面得到的结果一致^[8-10]. 所以这时湍流粘滞系数 μ_T 为

$$\mu_T = (2\pi)^3 \rho \left[\frac{26}{5} \cdot \frac{B^2}{15} n t_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} \frac{1}{8\nu t_0} + \frac{184}{21} \lambda_1^2 A_1^2 n_1 t_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} \frac{1}{(8\nu t_0)^2} \right] \quad (5.2)$$

其它两个剪切分量

$$\tau_{32} = \tau_{31} = 0 \quad (5.3)$$

另外三个正应力各自为

$$\begin{aligned} -\rho \overline{W_1^2} = & -(2\pi)^3 \rho \left[\frac{8}{15} B^2 n t_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} \frac{1}{8\nu t_0} \cdot \frac{1}{3} \right. \\ & \left. + 32\lambda_1^2 A_1^2 n_1 t_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} \frac{1}{(8\nu t_0)^2} \cdot \frac{2}{5} \right] \\ = & -(2\pi)^3 \rho \left[\frac{B^2}{45} n t_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} \frac{1}{\nu t_0} + \frac{1}{5} \lambda_1^2 A_1^2 n_1 t_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} \frac{1}{(\nu t_0)^2} \right] \\ -\rho \overline{W_2^2} = & -(2\pi)^3 \rho \left[\frac{B^2}{45} n t_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} \frac{1}{\nu t_0} + \frac{1}{40} \lambda_1^2 A_1^2 n_1 t_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} \frac{1}{(\nu t_0)^2} \right] \\ -\rho \overline{W_3^2} = & -(2\pi)^3 \rho \left[\frac{B^2}{45} n t_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} \frac{1}{\nu t_0} + \frac{1}{40} \lambda_1^2 A_1^2 n_1 t_0 \left(\frac{\pi}{2\nu t_0} \right)^{3/2} \frac{1}{(\nu t_0)^2} \right] \end{aligned} \quad (5.4)$$

在通常的槽流或管流问题中, 在中心区域(平均速度梯度很小的区域)接近于各向同性湍流, 而且 y 方向和 z 方向脉动速度均方根值几乎相等, 而 x 方向的脉动速度较大一些^[11]. 从

(5.4) 式可以看出, 这是各向异性涡旋所贡献的值对 x 方向偏大造成的. 而且 (5.4) 式中也刚好有 $\overline{w_1^2} > \overline{w_2^2} = \overline{w_3^2}$.

至于近壁的湍流场, 则湍流场比较强. 湍流的能量由 Reynolds 应力和平均速度梯度相互作用得来. 这时是非线性项起重要作用, 而最后破裂所产生的小涡旋数 n 和 n_1 都和速度梯度发生关系. 因此实质上也是对平均速度梯度非线性的.

六、结论和讨论

(1) 本文所讨论的问题粗看起来完全是一个理想化问题, 实际上并不然. 它具有槽流或管流等问题的重要特点. 例如对槽流和管流来说, 中心压域的湍流能量是在边壁附近产生的涡旋的能量经过运流和扩散等过程把涡旋带到中心区域去的. 这和我们假定时间 t_0 时候产生 n 个后期涡旋很接近. 其次在管流和槽流的中心区域, 平均速度梯度比较小, 这一点和我们所讨论的问题也是比较接近的. 当然管流和槽流的中心区域, 平均速度梯度并不是常数, 和我们讨论的问题有一些不同.

(2) 我们得到的平均速度梯度对关联函数所产生的影响应该具有代表性的. 虽然我们用了平均速度梯度为常数的假定, 但我们知道湍流涡旋是比较小的, 而平均速度分布的范围是比较大的. 所以从小范围来说, 把平均速度梯度当作常数也是无可非议的. 所以我们得到的平均速度梯度对关联函数的影响应该具有普遍的意义.

(3) 我们求得的湍流运动粘滞系数, 也是有一定普遍意义的, 它可以用来衡量湍流扩散系数的大小. 虽然它不能精确代表湍流扩散系数的数值大小, 至少可以代表它的量级和变化性质. 虽然粗糙, 但总比河流泥沙问题常把表面扩散系数作为零要好一些.

(4) 我们这个湍流场的能量平衡. 当我们不考虑各向异性部分时候, 单位时间单位体积中输入的能量为

$$\begin{aligned} \frac{3}{2} \rho u_0^2 n &= \frac{3}{2} \rho \cdot \frac{1}{45} B^2 n \left(\frac{\pi}{2\nu t_0} \right)^{3/2} \frac{1}{\nu t_0} \\ &= \frac{1}{30} \rho B^2 n \left(\frac{\pi}{2\nu t_0} \right)^{3/2} \frac{1}{\nu t_0} \end{aligned}$$

单位时间单位体积内粘性消耗的能量为

$$\begin{aligned} 15\mu \frac{\bar{u}^2}{\lambda^2} &= 15\mu \frac{1}{45} B^2 n \left(\frac{\pi}{2\nu t_0} \right)^{3/2} \frac{1}{\nu t_0} \frac{1}{4\nu t_0} \int_{t_0}^{\infty} \left(\frac{t}{t_0} \right)^{-7/5} dt \\ &= \frac{1}{30} \rho B^2 n \left(\frac{\pi}{2\nu t_0} \right)^{3/2} \frac{1}{\nu t_0} \end{aligned}$$

两者刚好平衡.

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The Double Velocity Correlation Function of Homogeneous Turbulence with Constant Mean Velocity Gradient

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Abstract

In this article, as the velocity gradient is taken as a constant value, we obtain the solutions of the equation of fluctuation velocity after Fourier transformation. Under the condition of the mean velocity gradient being small, they represent the picture of eddies, of which the homogeneous turbulence (both isotropic and non-isotropic) of the final period is composed. By using the eddies of these types at different times, we may compose the steady turbulent field with the constant velocity gradient and this field may represent the turbulent field in the central part of the channel flow or pipe flow approximately. Then we may obtain the double velocity correlation function of this turbulent field, which involves both longitudinal correlation coefficient $f\left(\frac{r}{\lambda}\right)$ and the transversal correlation coefficient $g\left(\frac{r}{\lambda}\right)$. We compare theoretical coefficients with the experimental data of these coefficients at initial period and final period of isotropic homogeneous turbulence. And then we obtain the relationship between the turbulent double velocity correlation coefficient $f\left(\frac{r}{\lambda}\right)$ and the mean velocity gradient. Finally, we get the expressions of the Reynolds stress and the eddy viscosity coefficient.