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三重介质轴对称二维不定常渗流的精确解

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摘　要

本文求得了三重介质轴对称二维不定常渗流的线源解和有限封闭地层的面源解．它们不仅概括和发展了已有的均质和双重介质的主要结果，而且给出多重介质弹性渗流的基本特征．

一、微分方程组

文[1]求得了双重介质轴对称二维不定常渗流的线源解．本文研究三重介质不定常渗流的情况．

根据文[1,2]，弱可压缩液体在三重介质中不定常渗流遵循下述微分方程组：

$$\frac{1}{\bar r}\frac{\partial}{\partial \bar r}\left(\bar r\frac{\partial P_f}{\partial \bar r}\right)+\frac{\partial^2 P_f}{\partial \bar z^2}+\lambda_1(P_1-P_f)+\lambda_2(P_2-P_f)=\omega_f\frac{\partial P_f}{\partial \tau} \tag{1.1}$$

$$\lambda_1(P_f-P_1)=\omega_1\frac{\partial P_1}{\partial \tau} \tag{1.2}$$

$$\lambda_2(P_f-P_2)=\omega_2\frac{\partial P_2}{\partial \tau} \tag{1.3}$$

式中：

$$P_i=\frac{2\pi k_f h(p_0-p_i)}{\mu Q_0};\quad \lambda_j=\frac{\alpha_j k_j r_w^2}{k_f};\quad \tau=\frac{k_f\cdot t}{\mu\phi_t c_t r_w^2};$$

$$\omega_i=\frac{\phi_i c_i}{\phi_t c_t};\quad \phi_t c_t=\phi_f c_f+\phi_1 c_1+\phi_2 c_2;\quad \bar r=\frac{r}{r_w};\quad \bar z=\frac{z}{r_w};$$

p_i 是系统压力；p_0 是起始压力；h 是地层厚度；b 是不完善井长度；μ 是液体粘度；α_j 是形状系数；ϕ_i 是孔隙度；k_i 是渗透率；c_i 是压缩系数；Q_0 是不完善井定流量（线汇 b 的强度）；r_w 是井半径；足标 f 指裂缝系统；足标1指岩块系统Ⅰ；足标2指岩块系统Ⅱ．

对于无限地层等强度线汇的情况，其边值条件为：

$$P_i(\bar r,\bar z,0)=0\qquad(i=f,1,2) \tag{1.4}$$

$$\lim_{\bar{r}\to 0}\left(\bar{r}\frac{\partial P_f}{\partial \bar{r}}\right)=\begin{cases}-\frac{\bar{h}}{\bar{b}}, & 0\leqslant \bar{z}\leqslant \bar{b}\\ 0, & \bar{b}\leqslant \bar{z}\leqslant \bar{h}\end{cases} \tag{1.5}$$

$$\lim_{\bar{r}\to\infty}P_f=0 \tag{1.6}$$

$$\frac{\partial P_f}{\partial \bar{z}}(\bar{r},0,\tau)=0 \tag{1.7}$$

$$\frac{\partial P_f}{\partial \bar{z}}(\bar{r},\bar{h},\tau)=0 \tag{1.8}$$

二、线 源 解

首先对问题(1.1)～(1.8)进行Laplace变换，考虑到初始条件(1.4)，式(1.1)～(1.3)和式(1.5)～(1.8)的Laplace变换为：

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial U_f}{\partial \bar{r}}\right)+\frac{\partial^2 U_f}{\partial \bar{z}^2}+\lambda_1(U_1-U_f)+\lambda_2(U_2-U_f)=\omega_f sU_f \tag{2.1}$$

$$\lambda_1(U_f-U_1)=\omega_1 sU_1 \tag{2.2}$$

$$\lambda_2(U_f-U_2)=\omega_2 sU_2 \tag{2.3}$$

$$\lim_{\bar{r}\to 0}\left(\bar{r}\frac{\partial U_f}{\partial \bar{r}}\right)=\begin{cases}-\frac{\bar{h}}{\bar{b}}\cdot\frac{1}{s}, & 0\leqslant \bar{z}\leqslant \bar{b}\\ 0, & \bar{b}\leqslant \bar{z}\leqslant \bar{h}\end{cases} \tag{2.4}$$

$$\lim_{\bar{r}\to\infty}U_f(\bar{r},\bar{z},s)=0 \tag{2.5}$$

$$\frac{\partial U_f}{\partial \bar{z}}(\bar{r},0,s)=\frac{\partial U_f}{\partial \bar{z}}(\bar{r},\bar{h},s)=0 \tag{2.6}$$

式中：

$$U_i(\bar{r},\bar{z},s)=\int_0^\infty P_i(\bar{r},\bar{z},\tau)e^{-s\tau}\,d\tau$$

从方程(2.2)和(2.3)解出U_1和U_2代入式(2.4)得：裂缝系统中压力影象函数U_f应遵循的偏微分方程

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial U_f}{\partial \bar{r}}\right)+\frac{\partial^2 U_f}{\partial \bar{z}^2}=q^2(s)U_f \tag{2.7}$$

式中：

$$q^2(s)=s\left(\omega_f+\frac{\lambda_1}{s+\lambda_1/\omega_1}+\frac{\lambda_2}{s+\lambda_2/\omega_2}\right) \tag{2.8}$$

用分离变量法解式(2.7)，考虑到边界条件(2.5)和(2.6)，Poisson 方程 (2.7)的基本解是

$$K_0\left(\sqrt{q^2+\beta_n^2}\,\bar{r}\right)\cdot\cos\left(n\pi\frac{\bar{z}}{\bar{h}}\right) \tag{2.9}$$

式中：$K_0(\cdot)$ 是零阶第二类变形贝塞尔函数；$\beta_n^2=\frac{n^2\pi^2}{\bar{h}^2}$。

将间断内边界条件(2.4)展成Fourier级数

$$\left(\bar{r}\frac{\partial U_f}{\partial \bar{r}}\right)_{\bar{r}\to 0}=-\frac{1}{s}\left[1+\frac{2\bar{h}}{\pi\bar{b}}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(n\pi\frac{\bar{b}}{\bar{h}}\right)\cos\left(n\pi\frac{\bar{z}}{\bar{h}}\right)\right] \tag{2.10}$$

从而可得问题式(2.4)～(2.6)和式(2.7)的一般解

$$U_f(\bar{r},\bar{z},s)=\frac{1}{s}\left\{K_0(q\bar{r})+\frac{2h}{\pi\bar{b}}\sum_{n=1}^{\infty}\left[\frac{1}{n}\sin\left(n\pi\frac{b}{h}\right)\cos\left(n\pi\frac{\bar{z}}{h}\right)\cdot K_0\left(\sqrt{q^2+\beta_n^2}\,\bar{r}\right)\right]\right\} \tag{2.11}$$

下面求$U_f(\bar{r},\bar{z},s)$的原函数$P_f(\bar{r},\bar{z},\tau)$。

首先求

$$U_n(\bar{r},s)=\frac{1}{s}K_0(\sqrt{q^2(s)+\beta_n^2}\,\bar{r}) \tag{2.12}$$

的原函数$P_n(\bar{r},\tau)$。

已知[3]

$$K_0(\sqrt{q^2+\beta_n^2}\,\bar{r})=\int_0^{\infty}\frac{\rho\cdot J_0(\rho\bar{r})}{q^2(s)+\beta_n^2+\rho^2}d\rho \tag{2.13}$$

式中：$J_0(\cdot)$为零阶第一类贝塞尔函数。

根据Laplace变换反演定理得

$$P_n(\bar{r},\tau)=\int_0^{\infty}\bar{P}_n(\rho,\tau)\rho\cdot J_0(\rho\bar{r})d\rho \tag{2.14}$$

式中：

$$\bar{P}_n(\rho,\tau)=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\frac{e^{s\tau}-1}{s}\cdot\frac{ds}{q^2(s)+\beta_n^2+\rho^2} \tag{2.15}$$

式(2.15)的线积分等于函数

$$f(s)=\frac{1}{q^2(s)+\beta_n^2+\rho^2} \tag{2.16}$$

的极点的留数的$2\pi i$倍。

令式(2.16)中分母等于零得三次代数方程

$$s^3+a_{n,1}s^2+a_{n,2}s+a_{n,3}=0 \tag{2.17}$$

式中：

$$a_{n,1}=\frac{\lambda_1}{\omega_1}+\frac{\lambda_2}{\omega_2}+\frac{\lambda_1+\lambda_2}{\omega_f}+\frac{\beta_n^2+\rho^2}{\omega_f} \tag{2.17a}$$

$$a_{n,2}=\frac{\lambda_1\lambda_2}{\omega_f\omega_1\omega_2}+\left(\frac{\lambda_1}{\omega_1}+\frac{\lambda_2}{\omega_2}\right)\frac{\beta_n^2+\rho^2}{\omega_f} \tag{2.17b}$$

$$a_{n,3}=\frac{\lambda_1\lambda_2}{\omega_f\omega_1\omega_2}(\beta_n^2+\rho^2) \tag{2.17c}$$

式(2.17)的三个不同大小的负根$-s_1(n)$，$-s_2(n)$，$-s_3(n)$就是式(2.15)的极点，从而得

$$\bar{P}_n(\rho,\tau)=\sum_{j=1}^{3}\frac{1-e^{-s_j(n)\tau}}{s_j(n)\chi_j(n)} \tag{2.18}$$

式中：

$$\chi_j(n)=\omega_f+\frac{\lambda_1}{\lambda_1/\omega_1-s_j(n)}+\frac{\lambda_1 s_j(n)}{(\lambda_1/\omega_1-s_j(n))^2}+\frac{\lambda_2}{\lambda_2/\omega_2-s_j(n)}+\frac{\lambda_2 s_j(n)}{(\lambda_2/\omega_2-s_j(n))^2}$$

从上可得裂缝系统中压力分布公式

$$P_f(\bar r,\bar z,\tau)=P_0(\bar r,\tau)+\frac{2h}{\pi\bar b}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(n\pi\frac{\bar b}{\bar h}\right)\cdot\cos\left(n\pi\frac{\bar z}{\bar h}\right)\cdot P_n(\bar r,\tau) \tag{2.19}$$

不完善井中平均压力变化公式

$$P_w(\tau)=\frac{1}{\bar b}\int_0^{\bar b}P_f(1,\bar z,\tau)d\bar z$$

$$=P_0(1,\tau)+\frac{2\bar h^2}{\pi^2\bar b^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin^2\left(n\pi\frac{\bar b}{\bar h}\right)P_n(1,\tau) \tag{2.20}$$

式(2.19)还可应用到裂缝性含水层中潜水不完善井的情况[3]。
当$b=h$时，式(2.19)简化为三重介质径向不定常渗流的公式[2,3]。

$$P_f=\int_0^{\infty}\bar P_0(\rho,\tau)\rho\cdot J_0(\rho\bar r)d\rho$$

三、讨　论

1．当$\lambda_2=0$，$\omega_2=0$时；三重介质简化为双重介质的情况[1]。这时，式(2.8)、(2.17)、(2.18)分别简化为：

$$q^2=s\left(\omega_f+\frac{\lambda_1}{s+\lambda_1/\omega_1}\right) \tag{2.8$_1$}$$

$$s^2+a'_{n,1}s+a'_{n,2}=0 \tag{2.17$_1$}$$

$$\bar P_n(\rho,\tau)=\sum_{j=1}^{2}\frac{1-e^{-s'_j(n)\tau}}{s'_j(n)\chi'_j(n)} \tag{2.18$_1$}$$

式中：

$$a'_{n,1}=\frac{\lambda_1}{\omega_1}+\frac{\lambda_1}{\omega_f}+\frac{\beta_n^2+\rho^2}{\omega_f}$$

$$a'_{n,2}=\frac{\lambda_1}{\omega_1}\cdot\frac{\beta_n^2+\rho^2}{\omega_f}$$

$$s'_j(n)=\frac{a'_{n,1}+(-1)^j\sqrt{(a'_{n,1})^2-4a'_{n,2}}}{2}$$

$$\chi'_j(n)=\omega_f+\frac{\lambda_1}{\lambda_1/\omega_1-s'_j(n)}+\frac{\lambda_1 s'_j(n)}{(\lambda_1/\omega_1-s'_j(n))^2}$$

上述双重介质的结果还可用到粘弹性含水层渗流和有延迟补给潜水渗流的情况[4,5]。

2． 当$\lambda_2=\lambda_1=0$，$\omega_2=\omega_1=0$时；三重介质简化为均质介质的情况[6]。这时，式(2.8)、(2.17)、(2.18)、(2.14)分别简化为

$$q^2=s \tag{2.8$_2$}$$

$$s+(\beta_n^2+\rho^2)=0 \tag{2.17$_2$}$$

$$\bar{P}_w(\rho,\tau)=\frac{1-e^{-(\beta_n^2+\rho^2)\tau}}{\beta_n^2+\rho^2} \tag{2.18$_2$}$$

$$P_n(\bar{r},\tau)=\int_0^\infty \frac{1-e^{-(\beta_n^2+\rho^2)\tau}}{\beta_n^2+\rho^2}\rho\cdot J_0(\rho\bar{r})d\rho$$

$$=\frac{1}{2}\int_{\bar{r}^2/4\tau}^\infty \frac{1}{y}\exp\left(-y-\frac{\beta_n^2\bar{r}^2}{4y}\right)dy \tag{2.14$_2$}$$

3． 对于有限封闭地层线源的情况；三次方程式(2.17)仍然成立，只需将连续变化的ρ改为离散的ρ_m，且ρ_m是下述方程的根

$$J_1(\rho_m R)=0$$

式中：$J(\cdot)$是一阶第一类贝塞尔方程；$R=r_e/r_w$；r_e为封闭边界半径。

式(2.18)不变，无限积分式(2.20)改为无限和

$$P_n(\bar{r},\tau)=\frac{2}{R^2}\sum_{m=0}^\infty \frac{J_0(\rho_m\bar{r})}{J_0^2(\rho_m R)}\bar{P}_n(s,\tau) \tag{2.20$_3$}$$

即得有限封闭地层线源解。

4． 对于有限封闭地层面源的情况：只需将式(2.20)改为

$$P_n(\bar{r},\tau)=\pi\sum_{m=0}^\infty \frac{\rho_m\cdot J_1^2(\rho_m R)\cdot B(\rho_m\bar{r})}{J_1^2(\rho_m R)-J_1^2(\rho_m)}\bar{P}_n(s,\tau) \tag{2.20$_4$}$$

式中：

$$B(\rho_m\bar{r})=Y_1(\rho_m)\cdot J_0(\rho_m\bar{r})-J_1(\rho_m)Y_0(\rho_m\bar{r})$$

ρ_m是下述方程的根

$$B'(\rho_m R)=Y_1(\rho_m)J_1(\rho_m R)-J_1(\rho_m)Y_1(\rho_m R)=0$$

$Y_0(\cdot)$，$Y_1(\cdot)$分别是零阶和一阶第二类贝塞尔函数。

其余照旧，即得有限封闭地层面源解。

5． 对于N重介质，式(2.8)和(2.17)分别推广为：

$$q^2(s)=s\left(\omega_f+\frac{\lambda_1}{\lambda_1/\omega_1+s}+\frac{\lambda_2}{\lambda_2/\omega_2+s}+\cdots+\frac{\lambda_{N-1}}{\lambda_{N-1}/\omega_{N-1}+s}\right) \tag{2.8$'$}$$

$$s^N+a_{n,1}s^{N-1}+\cdots+a_{n,N}=0 \tag{2.17$'$}$$

式中：λ_{N-1}，ω_{N-1}分别表岩块$(N-1)$的传质系数和容积比。

从而可得与三重介质结构相似的N重介质不定常渗流的解析解。

式(2.8)′和(2.17)′表N重介质弹性渗流的特性方程。

6． 当$s\gg1$时，从式(2.11)可得不完善井中平均无因次压力变化短时近似解

$$\bar{P}_w(\tau)\approx\frac{1}{2}\frac{h}{b}\left(\ln\tau+\ln\frac{4}{\omega_f}\right)$$

当$s \ll 1$时，从式(2.11)可得长时近似解

$$\bar{P}_w(\tau)\approx\frac{1}{2}\ln\tau+A$$

式中：

$$A=\frac{2\bar{h}^2}{\pi^2\bar{b}^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin^2\left(n\pi\frac{\bar{b}}{\bar{h}}\right)K_0(\beta_n)$$

比较上式可知：短时解的斜率比长时解的斜率大h/b倍；多重介质的影响主要反映在压力变化的中间阶段。

7. 将边界条件(1.8)改为

$$\frac{\partial P_f}{\partial\bar{z}}(\bar{r},\bar{h},\tau)=-\lambda P_f(\bar{r},\bar{h},\tau)$$

式中：λ为越流系数．类似上述解题步骤，可得有越流补给时，三重介质轴对称二维不定常渗流的线源解

$$P_f(\bar{r},\bar{z},\tau)=\frac{2\bar{h}}{\bar{b}}\sum_{n=0}^{\infty}\frac{\sin\left(\mu_n\frac{\bar{b}}{\bar{h}}\right)\cdot\cos\left(\mu_n\frac{\bar{z}}{\bar{h}}\right)}{\mu_n+\sin\mu_n\cdot\cos\mu_n}P_n(\bar{r},\tau) \tag{3.1}$$

式中μ_n根据下述特征方程

$$\mu_n\cdot\mathrm{tg}\,\mu_n=\lambda \tag{3.2}$$

求得．相应式(2.15)，(2.17)中β_n改为$\mu_n/\bar{h}$．当$\lambda=0$时，从上式得$\mu_n=n\pi$；从而式(3.1)简化为无越流时三重介质不定常渗流线源解式(2.19)，理应如此．

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Exact Solution of Unsteady Axisymmetrical Two-Dimensional Flow through Triple Porous Media

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Abstract

This paper seeks for the line source and cylindrical plane source solutions of unsteady axisymmetrical two-dimensional flow through infinite and finite reservoirs with triple porosity. They not only reveal the essential characters of fractured reservoirs but also generalize and develop the existing primai results of homogeneous and double porous media.

Reference [1] obtained the line source solution of unsteady axisymmetrical two-dimensional flow in infinite reservoir with double porosity, in this paper we study the problem of flow through triple porous media.