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具有不等厚表层的夹层旋转壳问题

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摘　　要

本文探讨了具有各向同性夹心及不等厚表层组成的三层旋转壳的小挠度问题，采用广义变分法，在考虑表层抗弯刚度的情况下，导出了基本方程，其特例即为文献[3]，[4]的结果.

根据所导出的基本方程，本文求解对称加载的夹层旋转壳问题，将其归结为只含一个广义位移的六阶微分方程.

本文的理论可应用于复合装甲及其它工程设计.

一、引　　言

近年来，由于夹层板壳结构具有重量轻、强度高、刚度大等特点，因而在航空、船舶制造、宇航等部门受到了愈来愈大的重视，钱伟长[9]早就对板壳理论作出了巨大贡献.

对于夹层结构的强度、刚度、稳定性及热效应、动应力等，国内外已进行了广泛的研究，不少人提出了各种计算模型．就线性理论而言，早期由 Reissner[5]提出的夹层板理论较为简单实用．他把表板看作薄膜，而忽略其抗弯刚度，并认为夹心只承受抗剪作用．对于许多工程问题，这种模型已具有足够的精度.

Hoff[7]提出的夹层板理论，不仅考虑夹心的横向剪切变形，还考虑了表板的抗弯刚度，从而解决了一些用 Reissner 理论不能求解的问题.

由于 Reissner 和 Hoff 的理论中基本方程的复杂性，难以求解实际问题，胡海昌[2]将 Reissner 理论的基本方程归结为求解两个位移函数的微分方程，从而为解决具体问题提供了方便的途径.

此后，由 Прусаков[6]、杜庆华[1]提出的夹层板理论指出，夹层板中除了反对称型的弯曲变形，还存在对称型弯曲．他们除了考虑表板的抗弯作用和夹心的抗剪作用外，还考虑了夹心的横向弹性变形.

近来，中国科学院北京力学所板壳组[3]对于夹层板壳问题出版了专著，阐明了有关夹层板壳的各种理论问题和实际问题，并提供了简化计算的方法.

最近，文献[4]将Hoff 的夹层板理论推广到了具有不等厚表板的夹层板．本文进一步研究了具有不等厚表层的夹层旋转壳问题，采用广义变分法，在考虑表层抗弯刚度的情况下导出了基本方程，并由此求解对称加载的夹层旋转壳问题，将它归结为只含一个广义位移的六阶微分方程.

二、基本方程的推导

图 1 表示具有不等厚表层的夹层旋转壳．夹心及表层壳都是由各向同性材料制成．夹心“c”较软；γ 为正的一侧为外表层，用“+”表示；γ 为负的一侧为内表层，用“−”表示；其厚度分别为 t_1，t_2；外载荷 $p^{\pm}$ 以沿 γ 方向作用为正．

本文采用正交曲线坐标，以 1，2，3分别表示三个坐标方向的应力应变分量．

本文采用下述假定：(1)内外表层壳作为薄壳处理；(2)由于夹心较软，故忽略夹心中平行于 $\alpha o\beta$ 坐标面的应力分量，即假定 $\sigma_1^c=\sigma_2^c=\tau_{12}^c=0$； (3)不计夹心的横向弹性，即认为 $\varepsilon_3^c=0$，又因 σ_3^c 甚小，故认为 $\sigma_3^c=0$；

上述假定通常不会影响工程应用中的精度要求．

1. 平衡方程的推导

设夹层旋转壳中面在 α，β 方向的曲率半径为 R_1，R_2，它们只是 α 的函数，其 Lamé 系数分别为 R_1，$R_2\sin\alpha$，Gauss 条件为

$$\frac{d}{d\alpha}(R_2\sin\alpha)=R_1\cos\alpha \tag{2.1}$$

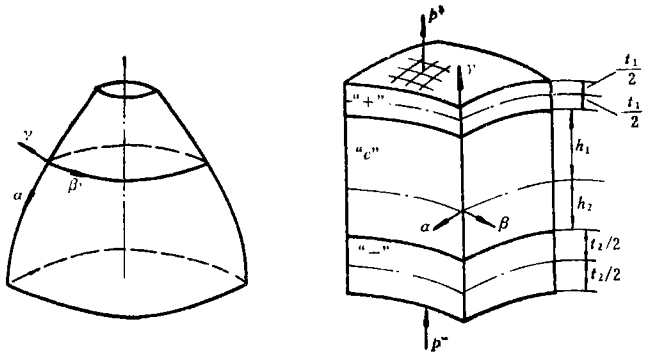

图 1

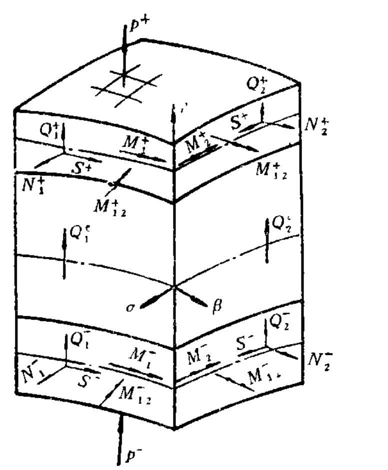

图 2

内外表层壳及夹心的内力如图 2 所示．内力-应变关系为：

$$\left.\begin{aligned}
&N_1^+=B_1(\varepsilon_1^++\mu\varepsilon_2^+), && N_2^+=B_1(\varepsilon_2^++\mu\varepsilon_1^+)\\
&S^+=\frac{1}{2}(1-\mu)B_1\varepsilon_{12}^+, && M_1^+=D_1(\chi_1^++\mu\chi_2^+)\\
&M_2^+=D_1(\chi_2^++\mu\chi_1^+), && M_{12}^+=D_1(1-\mu)\chi_{12}^+\\
&N_1^-=B_2(\varepsilon_1^-+\mu\varepsilon_2^-), && N_2^-=B_2(\varepsilon_2^-+\mu\varepsilon_1^-)\\
&S^-=\frac{1}{2}(1-\mu)B_2\varepsilon_{12}^-, && M_1^-=D_2(\chi_1^-+\mu\chi_2^-)\\
&M_2^-=D_2(\chi_2^-+\mu\chi_1^-), && M_{12}^-=D_2(1-\mu)\chi_{12}^-
\end{aligned}\right\} \tag{2.2a∼l}$$

其中

$$B_i=\frac{E_i t_i}{1-\mu^2};\qquad D_i=\frac{E_i t_i^3}{12(1-\mu^2)};$$

$(i=1,\ 2)$ μ 为泊松比

应变-位移关系是

$$\left.\begin{aligned}e_1^\pm&=\varepsilon_1^\pm+\gamma\chi_1^\pm\\e_2^\pm&=\varepsilon_2^\pm+\gamma\chi_2^\pm\\e_{12}^\pm&=\varepsilon_{12}^\pm+\gamma\chi_{12}^\pm\end{aligned}\right\}\qquad(2.3\text{a, b, c})$$

其中

$$\left.\begin{aligned}
\varepsilon_1^\pm&=\frac{1}{R_1}\left(\frac{\partial u^\pm}{\partial\alpha}+w^\pm\right)\\
\varepsilon_2^\pm&=\frac{1}{R_2}\left(\frac{1}{\sin\alpha}\frac{\partial v^\pm}{\partial\beta}+\operatorname{ctg}\alpha\cdot u^\pm+w^\pm\right)\\
\varepsilon_{12}^\pm&=\frac{1}{R_2\sin\alpha}\frac{\partial u^\pm}{\partial\beta}+\frac{R_2\sin\alpha}{R_1}\cdot\frac{\partial}{\partial\alpha}\left(\frac{v^\pm}{R_2\sin\alpha}\right)\\
\chi_1^\pm&=-\frac{1}{R_1}\frac{\partial}{\partial\alpha}\left(\frac{1}{R_1}\frac{\partial w^\pm}{\partial\alpha}-\frac{u^\pm}{R_1}\right)\\
\chi_2^\pm&=-\frac{1}{R_2^2\sin^2\alpha}\frac{\partial^2 w^\pm}{\partial\beta^2}-\frac{\operatorname{ctg}\alpha}{R_1R_2}\left(\frac{\partial w^\pm}{\partial\alpha}-u^\pm\right)+\frac{1}{R_2^2\sin\alpha}\cdot\frac{\partial v^\pm}{\partial\beta}\\
\chi_{12}^\pm&=-\frac{1}{R_1R_2\sin\alpha}\left(\frac{\partial^2 w^\pm}{\partial\alpha\partial\beta}-\frac{R_1\operatorname{ctg}\alpha}{R_2}\frac{\partial w^\pm}{\partial\beta}\right)\\
&\quad+\frac{1}{2}\cdot\frac{1}{R_1R_2\sin\alpha}\cdot\frac{\partial u^\pm}{\partial\beta}+\frac{1}{2}\cdot\frac{R_2\sin\alpha}{R_1}\frac{\partial}{\partial\alpha}\left(\frac{v^\pm}{R_2^2\sin\alpha}\right)
\end{aligned}\right\}\qquad(2.4\text{a}\sim\text{f})$$

对于夹层旋转壳有五个独立的广义位移：

$$\left.\begin{aligned}
u&=\frac{B_1u^++B_2u^-}{B_1+B_2}\\
v&=\frac{B_1v^++B_2v^-}{B_1+B_2}\\
\psi_\alpha&=\frac{1}{k}(u^+-u^-)\\
\psi_\beta&=\frac{1}{k}(v^+-v^-)\\
w&=w^+=w^-=w^c
\end{aligned}\right\}\qquad(2.5\text{a}\sim\text{e})$$

此处 $k=h_1+h_2+\frac{1}{2}(t_1+t_2)$

有十四个独立的广义力：

$$\left.\begin{aligned}
&N_i=N_i^++N_i^- \quad (i=1,2)\\
&M_i''=M_i^++M_i^-\\
&S=S^++S^-\\
&M_{12}''=M_{12}^++M_{12}^-\\
&M_1'=\frac{k}{B_1+B_2}\cdot(B_2N_1^+-B_1N_1^-)\\
&M_2'=\frac{k}{B_1+B_2}\cdot(B_2N_2^+-B_1N_2^-)\\
&M_{12}'=\frac{k}{B_1+B_2}\cdot(B_2S^+-B_1S^-)\\
&M_1^t=\frac{1}{2}(M_1^+-M_1^-)\\
&M_2^t=\frac{1}{2}(M_2^+-M_2^-)\\
&M_{12}^t=\frac{1}{2}(M_{12}^+-M_{12}^-)
\end{aligned}\right\} \tag{2.6a～j}$$

此外还有 Q_1^c 和 Q_2^c；

可导出的位移有u^c和v^c，可导出的内力有$Q_1^\pm$和$Q_2^\pm$。

在以下计算中设 $R_1\gg k$，$R_2\gg k$，故忽略$\frac{k}{R_1}$及$\frac{k}{R_2}$阶微量。

由假设(2.2)、(2.3)可知夹心微元的平衡方程为

$$\frac{\partial\tau_{13}^c}{\partial\gamma}=0;\qquad \frac{\partial\tau_{23}^c}{\partial\gamma}=0 \tag{2.7}$$

可见夹心中的剪应力沿 γ 方向均匀分布，即：

$$\tau_{13}^c=\frac{Q_1^c}{h_1+h_2};\qquad \tau_{23}^c=\frac{Q_2^c}{h_1+h_2} \tag{2.8}$$

夹心中的应力-应变关系为

$$\tau_{13}^c=G^ce_{13}^c;\qquad \tau_{23}^c=G^ce_{23}^c \tag{2.9}$$

此处 G^c 为夹心的剪切弹性模量。

夹心的应变-位移关系是：

$$\left.\begin{aligned}
e_{13}^c&=\frac{\partial u^c}{\partial\gamma}+\frac{1}{R_1}\frac{\partial w}{\partial\alpha}-\frac{u^c}{R_1}\\
e_{23}^c&=\frac{\partial v^c}{\partial\gamma}+\frac{1}{R_2\sin\alpha}\cdot\frac{\partial w}{\partial\beta}-\frac{v^c}{R_2}
\end{aligned}\right\} \tag{2.10a,b}$$

将(2.10)、(2.9)式代入(2.7)式可见：

$$\frac{\partial^2u^c}{\partial\gamma^2}=0;\qquad \frac{\partial^2v^c}{\partial\gamma^2}=0 \tag{2.11}$$

即 u^c，v^c 是 γ 的线性函数，如果注意到：

在 $\gamma=h_1$ 处: $u^{c+}=u^{+}+\frac{t_1}{2R_1}\left(\frac{\partial w}{\partial \alpha}-u^{+}\right)$

在 $\gamma=-h_2$ 处: $u^{c-}=u^{-}-\frac{t_2}{2R_1}\left(\frac{\partial w}{\partial \alpha}-u^{-}\right)$ (2.12a, b)

那么就可以求得夹心的位移:

$$u^c=u^{+}+\frac{t_1}{2R_1}\left(\frac{\partial w}{\partial \alpha}-u^{+}\right)-\left(\frac{h_1}{h_1+h_2}-\frac{\gamma}{h_1+h_2}\right)\cdot\left[\frac{(t_1+t_2)}{2R_1}\frac{\partial w}{\partial \alpha}+(u^{+}-u^{-})-\frac{t_1u^{+}+t_2u^{-}}{2R_1}\right] \quad (2.13a)$$

同理:

$$v^c=v^{+}+\frac{t_2}{2R_2\sin\alpha}\frac{\partial w}{\partial \beta}-\frac{t_2}{2R_2}v^{+}-\left(\frac{h_1}{h_1+h_2}-\frac{\gamma}{h_1+h_2}\right)\cdot\left[\frac{(t_1+t_2)}{2R_2\sin\alpha}\frac{\partial w}{\partial \beta}+(v^{+}-v^{-})-\frac{t_1v^{+}+t_2v^{-}}{2R_2}\right] \quad (2.13b)$$

由 (2.5)、(2.10)、(2.13) 式可得

$$\left.\begin{aligned} e_{13}^c&=\frac{k}{h_1+h_2}\left(\psi_\alpha+\frac{1}{R_1}\frac{\partial w}{\partial \alpha}-\frac{u}{R_1}\right)\\ e_{23}^c&=\frac{k}{h_1+h_2}\left(\psi_\beta+\frac{1}{R_2\sin\alpha}\frac{\partial w}{\partial \beta}-\frac{v}{R_2}\right)\end{aligned}\right\} \quad (2.14a, b)$$

(2.14)式代入(2.9)、(2.8)式得

$$\left.\begin{aligned} Q_1^c&=G^c\cdot k\cdot\left(\psi_\alpha+\frac{1}{R_1}\frac{\partial w}{\partial \alpha}-\frac{u}{R_1}\right)\\ Q_2^c&=G^c\cdot k\cdot\left(\psi_\beta+\frac{1}{R_2\sin\alpha}\frac{\partial w}{\partial \beta}-\frac{v}{R_2}\right)\end{aligned}\right\} \quad (2.15a, b)$$

夹层旋转壳的广义变分表达式为

$$\delta V=\delta V^{+}+\delta V^{-}+\delta V^{c}=0 \quad (2.16)$$

其中

$$\begin{aligned}\delta V^{\pm}=&\int_{\alpha_1}^{\alpha_2}\int_{\beta_1}^{\beta_2}[N_1^{\pm}\delta\varepsilon_1^{\pm}+N_2^{\pm}\delta e_2^{\pm}+S^{\pm}\delta e_{12}^{\pm}+M_1^{\pm}\delta\chi_1^{\pm}\\ &+M_2^{\pm}\delta\chi_2^{\pm}+2M_{12}^{\pm}\delta\chi_{12}^{\pm}-p^{\pm}\delta w]R_1R_2\sin\alpha\, d\beta d\alpha\\ &-\int_{\alpha_1}^{\alpha_2}\left\{\left[\bar{N}_2^{\pm}\delta v^{+}+\bar{S}^{\pm}\delta u^{+}-\frac{\bar{M}_2^{\pm}}{R_2\sin\alpha}\delta\left(\frac{\partial w}{\partial \beta}\right)\right.\right.\\ &\left.\left.+\bar{Q}_2^{\pm}\delta w\right]R_1\right\}_{\beta_1}^{\beta_2}d\alpha-\int_{\beta_1}^{\beta_2}\left\{\left[\bar{N}_1^{\pm}\delta u^{+}+\bar{S}^{\pm}\delta v^{\pm}\right.\right.\\ &\left.\left.-\frac{\bar{M}_1^{\pm}}{R_1}\delta\left(\frac{\partial w}{\partial \alpha}\right)+\bar{Q}_1^{\pm}\delta w\right]R_2\sin\alpha\right\}_{\alpha_1}^{\alpha_2}d\beta\end{aligned} \quad (2.17)$$

$$\begin{aligned}\delta V^{c}=&\int_{\alpha_1}^{\alpha_2}\int_{\beta_1}^{\beta_2}(Q_1^c\delta e_{13}^c+Q_2^c\delta e_{23}^c)R_1R_2\sin\alpha\, d\beta\, d\alpha\\ &-\int_{\alpha_1}^{\alpha_2}\bar{Q}_2^c\delta wR_1\Big|_{\beta_1}^{\beta_2}d\alpha-\int_{\beta_1}^{\beta_2}\bar{Q}_1^c\delta wR_2\sin\alpha\Big|_{\alpha_1}^{\alpha_2}d\beta\end{aligned} \quad (2.18)$$

此处$\bar{N}_i^{\pm}$，$\bar{Q}_i^{\pm}$，$\bar{M}_i^{\pm}$，$(i=1,\ 2)$及$\bar{S}^{\pm}$，$\bar{Q}_1^c$，$\bar{Q}_2^c$等表示边界力。

将(2.4)、(2.14)、(2.5)、(2.6)式代入(2.17)、(2.18)式经过变分运算后再代入(2.16)式可得到

$$
\begin{aligned}
\delta V=&-\int_{\alpha_1}^{\alpha_2}\int_{\beta_1}^{\beta_2}\Big\{\Big[\frac{\partial}{\partial\alpha}(N_1R_2\sin\alpha)-N_2R_1\cos\alpha+R_1\frac{\partial S}{\partial\beta}\\
&+\frac{1}{R_1}\frac{\partial}{\partial\alpha}(M_1''R_2\sin\alpha)-M_2''\cos\alpha+\frac{\partial M_{12}''}{\partial\beta}\\
&+\frac{k}{h_1+h_2}\cdot Q_1^cR_2\sin\alpha\Big]\delta u+\Big[R_1\frac{\partial N_2}{\partial\beta}+2SR_1\cos\alpha\\
&+\frac{\partial S}{\partial\alpha}\cdot R_2\sin\alpha+\frac{R_1}{R_2}\frac{\partial M_2''}{\partial\beta}+\frac{1}{R_2^2\sin\alpha}\cdot\frac{\partial}{\partial\alpha}(M_{12}''R_2^2\sin^2\alpha)\\
&+\frac{k}{h_1+h_2}Q_2^cR_1\sin\alpha\Big]\delta v+\Big[\frac{\partial}{\partial\alpha}(M_1'R_2\sin\alpha)\\
&-M_2'R_1\cos\alpha+R_1\frac{\partial M_{12}'}{\partial\beta}-\frac{k}{h_1+h_2}Q_1^cR_1R_2\sin\alpha\Big]\delta\psi_\alpha\\
&+\Big[R_1\frac{\partial M_2'}{\partial\beta}+2M_{12}'R_1\cos\alpha+\frac{\partial M_{12}'}{\partial\alpha}\cdot R_2\sin\alpha\\
&-\frac{k}{h_1+h_2}Q_2^cR_1R_2\sin\alpha\Big]\delta\psi_\beta\\
&+\Big[\frac{\partial}{\partial\alpha}\Big(\frac{1}{R_1}\frac{\partial}{\partial\alpha}(M_1''R_2\sin\alpha)\Big)-N_1R_2\sin\alpha-N_2R_1\sin\alpha\\
&-\frac{\partial}{\partial\alpha}(M_2''\cos\alpha)+\frac{R_1}{R_2\sin\alpha}\frac{\partial^2M_2''}{\partial\beta^2}+2\frac{\partial^2M_{12}''}{\partial\alpha\partial\beta}\\
&+\frac{2R_1\operatorname{ctg}\alpha}{R_2}\cdot\frac{\partial M_{12}''}{\partial\beta}+\frac{k}{h_1+h_2}\cdot\frac{\partial}{\partial\alpha}(Q_1^cR_2\sin\alpha)\\
&+\frac{k}{h_1+h_2}\cdot R_1\cdot\frac{\partial Q_2^c}{\partial\beta}+p\cdot R_1R_2\sin\alpha\Big]\delta w\Big\}d\beta\,d\alpha\\
&-\int_{\alpha_1}^{\alpha_2}\Big\{\Big[(\bar{N}_2-N_2)\delta v+(\bar{S}-S)\delta u-(\bar{M}_2''-M_2'')\cdot\frac{1}{R_2\sin\alpha}\delta\Big(\frac{\partial w}{\partial\beta}\Big)\\
&+\Big(\bar{Q}_2-\frac{1}{R_2\sin\alpha}\frac{\partial M_2''}{\partial\beta}-\frac{2\operatorname{ctg}\alpha}{R_2}M_{12}''-\frac{2}{R_1}\frac{\partial M_{12}''}{\partial\alpha}\\
&-\frac{k}{h_1+h_2}Q_2^c\Big)\delta w+(\bar{M}_2'-M_2')\delta\psi_\beta+(\bar{M}_{12}'-M_{12}')\delta\psi_\alpha\Big]R_1\Big\}_{\beta_1}^{\beta_2}d\alpha\\
&-\int_{\beta_1}^{\beta_2}\Big\{\Big[(\bar{N}_1-N_1)\delta u+(\bar{S}-S)\delta v-(\bar{M}_1''-M_1'')\cdot\frac{1}{R_1}\delta\Big(\frac{\partial w}{\partial\alpha}\Big)\\
&+(\bar{M}_1'-M_1')\delta\psi_\alpha+(\bar{M}_{12}'-M_{12}')\delta\psi_\beta
\end{aligned}
$$

$$+\left(\overline{Q}_1-\frac{1}{R_1R_2\sin\alpha}\cdot\frac{\partial}{\partial\alpha}(M''_1R_2\sin\alpha)+\frac{\operatorname{ctg}\alpha}{R_2}M''_2-\frac{2}{R_2\sin\alpha}\cdot\frac{\partial M''_{12}}{\partial\beta}-\frac{k}{h_1+h_2}Q^c_1\right)\cdot\delta w\Big]R_2\sin\alpha\Big\}_{\alpha_1}^{\alpha_2}d\beta$$

$$-\Big[\Big[2M''_{12}\delta w\Big]_{\alpha_1}^{\alpha_2}\Big]_{\beta_1}^{\beta_2}=0 \tag{2.19}$$

此处

$$p=p^++p^-;\quad \overline{Q}_i=\overline{Q}^+_i+\overline{Q}^-_i+\overline{Q}^c_i\qquad(i=1,\ 2)$$

由(2.19)式可得平衡微分方程：

$$\left.\begin{aligned}
&\frac{\partial}{\partial\alpha}(N_1R_2\sin\alpha)-N_2R_1\cos\alpha+R_1\frac{\partial S}{\partial\beta}+\frac{1}{R_1}\cdot\frac{\partial}{\partial\alpha}(M''_1R_2\sin\alpha)\\
&\qquad-M''_2\cos\alpha+\frac{\partial M''_{12}}{\partial\beta}+\frac{k}{h_1+h_2}Q^c_1R_2\sin\alpha=0\\
&R_1\frac{\partial N_2}{\partial\beta}+2SR_1\cos\alpha+\frac{\partial S}{\partial\alpha}\cdot R_2\sin\alpha+\frac{R_1}{R_2}\cdot\frac{\partial M''_2}{\partial\beta}\\
&\qquad+\frac{1}{R_2^2\sin\alpha}\cdot\frac{\partial}{\partial\alpha}(M''_{12}R_2^2\sin^2\alpha)+\frac{k}{h_1+h_2}Q^c_2R_1\sin\alpha=0\\
&\frac{\partial}{\partial\alpha}(M'_1R_2\sin\alpha)-M'_2R_1\cos\alpha+R_1\frac{\partial M'_{12}}{\partial\beta}-\frac{k}{h_1+h_2}Q^c_1R_1R_2\sin\alpha=0\\
&R_1\frac{\partial M'_2}{\partial\beta}+2M'_{12}R_1\cos\alpha+\frac{\partial M'_{12}}{\partial\alpha}R_2\sin\alpha-\frac{k}{h_1+h_2}Q^c_2R_1R_2\sin\alpha=0\\
&\frac{\partial}{\partial\alpha}\left[\frac{1}{R_1}\frac{\partial}{\partial\alpha}(M''_1R_2\sin\alpha)\right]-N_1R_2\sin\alpha-N_2R_1\sin\alpha\\
&\qquad-\frac{\partial}{\partial\alpha}(M''_2\cos\alpha)+\frac{R_1}{R_2\sin\alpha}\frac{\partial^2M''_2}{\partial\beta^2}+2\frac{\partial^2M''_{12}}{\partial\alpha\partial\beta}\\
&\qquad+\frac{2R_1\operatorname{ctg}\alpha}{R_2}\cdot\frac{\partial M''_{12}}{\partial\beta}+\frac{k}{h_1+h_2}\left[\frac{\partial}{\partial\alpha}(Q^c_1R_2\sin\alpha)\right.\\
&\qquad\left.+R_1\frac{\partial Q^c_2}{\partial\beta}\right]+pR_1R_2\sin\alpha=0
\end{aligned}\right\}\tag{2.20a～e}$$

边界条件为

在 $\alpha=\alpha_1$，$\alpha=\alpha_2$ 上：

$$\left.\begin{aligned}
&N_1=\overline{N}_1\quad 或\ \delta u=0\\
&S=\overline{S}\quad 或\ \delta v=0\\
&M''_1=\overline{M}''_1\quad 或\ \delta\left(\frac{\partial w}{\partial\alpha}\right)=0\\
&M'_1=\overline{M}'_1\quad 或\ \delta\psi_\alpha=0\\
&M'_{12}=\overline{M}'_{12}\quad 或\ \delta\psi_\beta=0\\
&\frac{1}{R_1R_2\sin\alpha}\frac{\partial}{\partial\alpha}(M''_1R_2\sin\alpha)-\frac{\operatorname{ctg}\alpha}{R_2}M''_2+\frac{2}{R_2\sin\alpha}\frac{\partial M''_{12}}{\partial\beta}\\
&\qquad+\frac{k}{h_1+h_2}Q^c_1=\overline{Q}_1\quad 或\ \delta w=0
\end{aligned}\right\}\tag{2.21a～f}$$

在 $\beta=\beta_1$，$\beta=\beta_2$上：

$$\left.\begin{aligned}&N_2=\bar{N}_2 \quad 或\ \delta v=0\\&S=\bar{S} \quad 或\ \delta u=0\\&M''_2=\bar{M}''_2 \quad 或\ \delta\left(\frac{\partial w}{\partial\beta}\right)=0\\&M'_2=\bar{M}'_2 \quad 或\ \delta\psi_\beta=0\\&M'_{12}=\bar{M}'_{12} \quad 或\ \delta\psi_\alpha=0\\&\frac{1}{R_2\sin\alpha}\cdot\frac{\partial M''_2}{\partial\beta}+\frac{2\operatorname{ctg}\alpha}{R_2}M''_{12}+\frac{2}{R_1}\frac{\partial M''_{12}}{\partial\alpha}+\frac{k}{h_1+h_2}Q^c_2=\bar{Q}_2\\&或\ \delta w=0\end{aligned}\right\}\tag{2.22a～f}$$

由内外表层壳本身的微元平衡知：

$$\left.\begin{aligned}&\frac{\partial}{\partial\alpha}(\sigma^\pm_1R_2\sin\alpha)+R_1\frac{\partial\tau^\pm_{12}}{\partial\beta}+R_1R_2\sin\alpha\cdot\frac{\partial\tau^\pm_{13}}{\partial\gamma}-R_1\cos\alpha\cdot\sigma^\pm_2=0\\&R_1\frac{\partial\sigma^\pm_2}{\partial\beta}+\frac{\partial}{\partial\alpha}(\tau^\pm_{12}R_2\sin\alpha)+R_1R_2\sin\alpha\cdot\frac{\partial\tau^\pm_{23}}{\partial\gamma}+R_1\cos\alpha\cdot\tau^\pm_{12}=0\end{aligned}\right\}\tag{2.23a, b}$$

若注意到

$$\left.\begin{aligned}&在\ \gamma=h_1+t_1\ 处:\quad \tau^+_{13}=\tau^+_{23}=0\\&在\ \gamma=-(h_2+t_2)\ 处:\quad \tau^-_{13}=\tau^-_{23}=0\end{aligned}\right\}\tag{2.24a, b}$$

而且由 (2.6) 式可知

$$\left.\begin{aligned}&N^+_i=\frac{B_1}{B_1+B_2}N_i+\frac{1}{k}\cdot M'_i \qquad (i=1,2)\\&M^+_i=\frac{M''_i}{2}+M'_i\\&S^+=\frac{B_1}{B_1+B_2}\cdot S+\frac{1}{k}\cdot M'_{12}\\&M^+_{12}=\frac{M''_{12}}{2}+M'_{12}\end{aligned}\right\}\tag{2.25a, b, c, d}$$

从而

$$\left.\begin{aligned}\sigma^+_1=&\frac{1}{(B_1+B_2)t_1}\left(B_1N_1+\frac{B_1+B_2}{k}M'_1\right)\\&+\frac{12}{t_1^3}\left(\frac{M''_1}{2}+M'_1\right)\cdot\left(\gamma-h_1-\frac{t_1}{2}\right)\\\sigma^+_2=&\frac{1}{(B_1+B_2)t_1}\left(B_1N_2+\frac{B_1+B_2}{k}M'_2\right)\\&+\frac{12}{t_1^3}\left(\frac{M''_2}{2}+M'_2\right)\left(\gamma-h_1-\frac{t_1}{2}\right)\\\tau^+_{12}=&\frac{1}{(B_1+B_2)t_1}\left(B_1S+\frac{B_1+B_2}{k}M'_{12}\right)\\&+\frac{12}{t_1^3}\left(\frac{M''_{12}}{2}+M'_{12}\right)\left(\gamma-h_1-\frac{t_1}{2}\right)\end{aligned}\right\}\tag{2.26a, b, c}$$

于是由 (2.23)、(2.24)、(2.26) 式可求得 Q_1^+ 如下:

$$\int_{h_1}^{h_1+t_1}\left[\frac{\partial}{\partial\alpha}(\sigma_1^+R_2\sin\alpha)+R_1\frac{\partial\tau_{12}^+}{\partial\beta}\right]\left(\gamma-h_1-\frac{t_1}{2}\right)d\gamma$$

$$=\frac{\partial}{\partial\alpha}\left[\left(\frac{M_1''}{2}+M_1^f\right)R_2\sin\alpha\right]+R_1\frac{\partial}{\partial\beta}\left(\frac{M_{12}''}{2}+M_{12}^f\right)$$

$$=\int_{h_1}^{h_1+t_1}\left(\sigma_2^+R_1\cos\alpha-\frac{\partial\tau_{13}^+}{\partial\gamma}R_1R_2\sin\alpha\right)\left(\gamma-h_1-\frac{t_1}{2}\right)d\gamma$$

$$=R_1\cos\alpha\left(\frac{M_2''}{2}+M_2^f\right)+R_1R_2\sin\alpha\cdot\left[\left(h_1+\frac{t_1}{2}\right)\tau_{13}^+\Big|_{h_1}^{h_1+t_1}\right.$$

$$\left.-\gamma\tau_{13}^+\Big|_{h_1}^{h_1+t_1}+\int_{h_1}^{h_1+t_1}\tau_{13}^+d\gamma\right]$$

$$=R_1\cos\alpha\left(\frac{M_2''}{2}+M_2^f\right)+R_1R_2\sin\alpha\cdot\left(-\frac{t_1}{2}\quad\frac{Q_1^c}{h_1+h_2}+Q_1^+\right)$$

因此

$$Q_1^+=\frac{1}{R_1R_2\sin\alpha}\left\{\frac{\partial}{\partial\alpha}\left[\left(\frac{M_1''}{2}+M_1^f\right)R_2\sin\alpha\right]\right.$$

$$\left.+R_1\frac{\partial}{\partial\beta}\left(\frac{M_{12}''}{2}+M_{12}^f\right)-R_1\cos\alpha\left(\frac{M_2''}{2}+M_2^f\right)\right\}+\frac{t_1}{2}\cdot\frac{Q_1^c}{(h_1+h_2)}$$

同理

$$Q_1^-=\frac{1}{R_1R_2\sin\alpha}\left\{\frac{\partial}{\partial\alpha}\left[\left(\frac{M_1''}{2}-M_1^f\right)R_2\sin\alpha\right]+R_1\frac{\partial}{\partial\beta}\left(\frac{M_{12}''}{2}-M_{12}^f\right)\right.$$

$$\left.-R_1\cos\alpha\left(\frac{M_2''}{2}-M_2^f\right)\right\}+\frac{t_2}{2}\cdot\frac{Q_1^c}{(h_1+h_2)}$$

从而

$$Q_1^++Q_1^-=\frac{1}{R_1R_2\sin\alpha}\left[\frac{\partial}{\partial\alpha}(M_1''R_2\sin\alpha)+R_1\frac{\partial M_{12}''}{\partial\beta}-R_1\cos\alpha\cdot M_2''\right]+\frac{(t_1+t_2)}{2}\quad\frac{Q_1^c}{(h_1+h_2)}$$

同理

$$Q_2^++Q_2^-=\frac{1}{R_1R_2\sin\alpha}\left[R_1\frac{\partial M_2''}{\partial\beta}+R_1\cos\alpha\cdot M_{12}''+\frac{\partial}{\partial\alpha}(M_{12}''R_2\sin\alpha)\right]+\frac{(t_1+t_2)}{2}\cdot\frac{Q_2^c}{(h_1+h_2)}\qquad(2.27\text{a, b})$$

若令

$$\left.\begin{aligned}&M_i=M_i'+M_i''\\&M_{12}=M_{12}'+M_{12}''\qquad(i=1,2)\\&Q_i=Q_i^++Q_i^-+Q_i^c\end{aligned}\right\}\qquad(2.28\text{a, b, c})$$

将(2.27)、(2.28)式代入(2.20)式即可得到:

$$\left.\begin{aligned}
&\frac{\partial}{\partial\alpha}(N_1R_2\sin\alpha)-N_2R_1\cos\alpha+R_1\frac{\partial S}{\partial\beta}+Q_1R_2\sin\alpha=0\\
&\frac{\partial S}{\partial\alpha}R_2\sin\alpha+2S\cdot R_1\cos\alpha+R_1\frac{\partial N_2}{\partial\beta}+Q_2R_1\sin\alpha=0\\
&\frac{\partial}{\partial\alpha}(M_1R_2\sin\alpha)-M_2R_1\cos\alpha+R_1\frac{\partial M_{12}}{\partial\beta}-Q_1R_1R_2\sin\alpha=0\\
&\frac{\partial M_{12}}{\partial\alpha}R_2\sin\alpha+2M_{12}R_1\cos\alpha+R_1\frac{\partial M_2}{\partial\beta}-Q_2R_1R_2\sin\alpha=0\\
&\frac{\partial}{\partial\alpha}(Q_1R_2\sin\alpha)+R_1\frac{\partial Q_2}{\partial\beta}-N_1R_2\sin\alpha-N_2R_1\sin\alpha+pR_1R_2\sin\alpha=0
\end{aligned}\right\}\quad(2.29\text{a}\sim\text{e})$$

可见夹层旋转壳与单层旋转壳的平衡方程形式完全相同.

今令

$$M_1=\int_{h_1}^{h_1+t_1}\sigma_1^+\gamma d\gamma+\int_{-(h_2+t_2)}^{-h_2}\sigma_1^-\gamma d\gamma$$

$$=\left(h_1+\frac{t_1}{2}\right)N_1^+-\left(h_2+\frac{t_2}{2}\right)N_1^-+M_1^++M_1^-$$

将 (2.6)、(2.28) 式代入上式则有

$$B_1\left(h_1+\frac{t_1}{2}\right)=B_2\left(h_2+\frac{t_2}{2}\right)\tag{2.30}$$

(2.30)式确定了 $\alpha o\beta$ 坐标面的位置.

2. 广义力-广义位移关系

由(2.6)、(2.2)、(2.4)、(2.5)式可得

$$\begin{aligned}
N_1&=B_1\cdot(\varepsilon_1^++\mu\varepsilon_2^+)+B_2(\varepsilon_1^-+\mu\varepsilon_2^-)\\
&=(B_1+B_2)\cdot\left[\frac{1}{R_1}\cdot\left(\frac{\partial u}{\partial\alpha}+w\right)\right.\\
&\quad\left.+\frac{\mu}{R_2}\left(\frac{1}{\sin\alpha}\frac{\partial v}{\partial\beta}+\text{ctg}\,\alpha\cdot u+w\right)\right]
\end{aligned}$$

同理:

$$N_2=(B_1+B_2)\cdot\left[\frac{1}{R_2}\left(\frac{1}{\sin\alpha}\cdot\frac{\partial v}{\partial\beta}+\text{ctg}\,\alpha\cdot u+w\right)+\frac{\mu}{R_1}\left(\frac{\partial u}{\partial\alpha}+w\right)\right]$$

$$S=\frac{1}{2}(1-\mu)(B_1+B_2)\left[\frac{1}{R_2\sin\alpha}\cdot\frac{\partial u}{\partial\beta}+\frac{R_2\sin\alpha}{R_1}\cdot\frac{\partial}{\partial\alpha}\left(\frac{v}{R_2\sin\alpha}\right)\right]$$

$$M_1'=\frac{B_1B_2k^2}{B_1+B_2}\left[\frac{1}{R_1}\cdot\frac{\partial\psi_\alpha}{\partial\alpha}+\frac{\mu}{R_2}\left(\frac{1}{\sin\alpha}\cdot\frac{\partial\psi_\beta}{\partial\beta}+\text{ctg}\,\alpha\cdot\psi_\alpha\right)\right]$$

$$M_2'=\frac{B_1B_2k^2}{B_1+B_2}\left[\frac{1}{R_2}\left(\frac{1}{\sin\alpha}\cdot\frac{\partial\psi_\beta}{\partial\beta}+\text{ctg}\,\alpha\cdot\psi_\alpha\right)+\frac{\mu}{R_1}\cdot\frac{\partial\psi_\alpha}{\partial\alpha}\right]$$

$$M_{12}'=\frac{1}{2}(1-\mu)\cdot\frac{B_1B_2k^2}{B_1+B_2}\left[\frac{1}{R_2\sin\alpha}\cdot\frac{\partial\psi_\alpha}{\partial\beta}+\frac{R_2\sin\alpha}{R_1}\cdot\frac{\partial}{\partial\alpha}\left(\frac{\psi_\beta}{R_2\sin\alpha}\right)\right]$$

$$
\left.
\begin{aligned}
M_1^{v} &= -(D_1+D_2)\cdot\left[\frac{1}{R_1}\cdot\frac{\partial}{\partial\alpha}\left(\frac{1}{R_1}\cdot\frac{\partial w}{\partial\alpha}-\frac{u}{R_1}\right)+\mu\left(\frac{1}{R_2^2\sin^2\alpha}\cdot\frac{\partial^2 w}{\partial\beta^2}\right.\right.\\
&\quad\left.\left.+\frac{\operatorname{ctg}\alpha}{R_1R_2}\cdot\frac{\partial w}{\partial\alpha}-\frac{\operatorname{ctg}\alpha}{R_1R_2}u-\frac{1}{R_2^2\sin\alpha}\cdot\frac{\partial v}{\partial\beta}\right)\right]\\
M_2^{v} &= -(D_1+D_2)\cdot\left[\frac{1}{R_2^2\sin^2\alpha}\cdot\frac{\partial^2 w}{\partial\beta^2}+\frac{\operatorname{ctg}\alpha}{R_1R_2}\left(\frac{\partial w}{\partial\alpha}-u\right)\right.\\
&\quad\left.-\frac{1}{R_2^2\sin\alpha}\frac{\partial v}{\partial\beta}+\frac{\mu}{R_1}\cdot\frac{\partial}{\partial\alpha}\left(\frac{1}{R_1}\frac{\partial w}{\partial\alpha}-\frac{u}{R_1}\right)\right]\\
M_{12}^{v} &= -(1-\mu)(D_1+D_2)\left[\frac{1}{R_1R_2\sin\alpha}\cdot\left(\frac{\partial^2 w}{\partial\alpha\partial\beta}-\frac{R_1\operatorname{ctg}\alpha}{R_2}\cdot\frac{\partial w}{\partial\beta}\right)\right.\\
&\quad\left.-\frac{1}{2R_1R_2\sin\alpha}\cdot\frac{\partial u}{\partial\beta}-\frac{R_2\sin\alpha}{2R_1}\cdot\frac{\partial}{\partial\alpha}\left(\frac{v}{R_2^2\sin\alpha}\right)\right]\\
M_1^{f} &= -\frac{1}{2}\cdot(D_1-D_2)\cdot\left[\frac{1}{R_1}\frac{\partial}{\partial\alpha}\left(\frac{1}{R_1}\cdot\frac{\partial w}{\partial\alpha}-\frac{u}{R_1}\right)+\mu\left(\frac{1}{R_2^2\sin^2\alpha}\right.\right.\\
&\quad\left.\left.\cdot\frac{\partial^2 w}{\partial\beta^2}+\frac{\operatorname{ctg}\alpha}{R_1R_2}\cdot\frac{\partial w}{\partial\alpha}-\frac{\operatorname{ctg}\alpha}{R_1R_2}\cdot u-\frac{1}{R_2^2\sin\alpha}\cdot\frac{\partial v}{\partial\beta}\right)\right]\\
M_2^{f} &= -\frac{1}{2}(D_1-D_2)\cdot\left[\frac{1}{R_2^2\sin^2\alpha}\cdot\frac{\partial^2 w}{\partial\beta^2}+\frac{\operatorname{ctg}\alpha}{R_1R_2}\left(\frac{\partial w}{\partial\alpha}-u\right)\right.\\
&\quad\left.-\frac{1}{R_2^2\sin\alpha}\cdot\frac{\partial v}{\partial\beta}+\frac{\mu}{R_1}\cdot\frac{\partial}{\partial\alpha}\left(\frac{1}{R_1}\cdot\frac{\partial w}{\partial\alpha}-\frac{u}{R_1}\right)\right]\\
M_{12}^{f} &= -\frac{1}{2}(1-\mu)(D_1-D_2)\cdot\left[\frac{1}{R_1R_2\sin\alpha}\left(\frac{\partial^2 w}{\partial\alpha\partial\beta}-\frac{R_1\operatorname{ctg}\alpha}{R_2}\cdot\frac{\partial w}{\partial\beta}\right)\right.\\
&\quad\left.-\frac{1}{2R_1R_2\sin\alpha}\cdot\frac{\partial u}{\partial\beta}-\frac{R_2\sin\alpha}{2R_1}\cdot\frac{\partial}{\partial\alpha}\left(\frac{v}{R_2^2\sin\alpha}\right)\right]\\
Q_1^{c} &= G^{c}\cdot k\cdot\left(\psi_\alpha+\frac{1}{R_1}\cdot\frac{\partial w}{\partial\alpha}-\frac{u}{R_1}\right)\\
Q_2^{c} &= G^{c}\cdot k\cdot\left(\psi_\beta+\frac{1}{R_2\sin\alpha}\cdot\frac{\partial w}{\partial\beta}-\frac{v}{R_2}\right)
\end{aligned}
\right\}\quad(2.31\text{a}\sim\text{n})
$$

三、几 个 特 例

1. 具有不等厚表板的夹层板

如果令 $R_1=\infty$，$R_2=\infty$，并采用直角坐标系，则(2.29)式将化为：

$$
\left.
\begin{aligned}
&\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=0\\
&\frac{\partial M_y}{\partial y}+\frac{\partial M_{xy}}{\partial x}-Q_y=0\\
&\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+p=0
\end{aligned}
\right\}\quad(3.1\text{a,b,c})
$$

而(2.31)式化为:

$$
\left.
\begin{aligned}
M_x &= \frac{B_1B_2k^2}{B_1+B_2}\cdot\left(\frac{\partial\psi_x}{\partial x}+\mu\frac{\partial\psi_y}{\partial y}\right)-(D_1+D_2)\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right)\\
M_y &= \frac{B_1B_2k^2}{B_1+B_2}\left(\frac{\partial\psi_y}{\partial y}+\mu\frac{\partial\psi_x}{\partial x}\right)-(D_1+D_2)\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)\\
M_{xy} &= \frac{1}{2}(1-\mu)\cdot\left[\frac{B_1B_2k^2}{B_1+B_2}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)-2(D_1+D_2)\cdot\frac{\partial^2 w}{\partial x\partial y}\right]\\
Q_x &= -(D_1+D_2)\cdot\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+\frac{G^c\cdot k^2}{h_1+h_2}\cdot\left(\psi_x+\frac{\partial w}{\partial x}\right)\\
Q_y &= -(D_1+D_2)\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+\frac{G^c\cdot k^2}{h_1+h_2}\left(\psi_y+\frac{\partial w}{\partial y}\right)
\end{aligned}
\right\}\qquad(3.2\text{a}\sim\text{e})
$$

若注意到当内外表板为同一材料制成时有

$$\frac{B_1B_2k^2}{B_1+B_2}=\frac{E}{1-\mu^2}\cdot\left[t_1\left(h_1+\frac{t_1}{2}\right)^2+t_2\left(h_2+\frac{t_2}{2}\right)^2\right]$$

则由本文就可得到与文献[4]完全相同的结果.

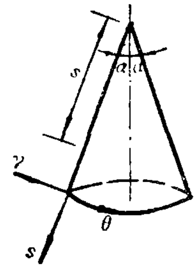

图　3

2.　对称加载的夹层圆锥壳

取坐标系如图3所示，此时有 $R_1=\infty$, $R_2=s\cdot\text{tg}\,\alpha$ 则(2.29)式成为:

$$
\left.
\begin{aligned}
&\frac{d}{ds}(N_s\cdot s)-N_\theta=0\\
&\frac{d}{ds}(M_s\cdot s)-M_\theta-Q_s\cdot s=0\\
&\frac{d}{ds}(Q_s\cdot s)-N_\theta\cdot\text{ctg}\,\alpha+p\cdot s=0
\end{aligned}
\right\}\qquad(3.3\text{a,b,c})
$$

当内外表层壳厚度相同且由同一材料制成时，$t_1=t_2=t$; $E_1=E_2=E$; 则(2.31)式化为:

$$
\left.
\begin{aligned}
N_s &= \frac{2Et}{(1-\mu^2)}\cdot\left[\frac{du}{ds}+\frac{\mu}{s}(u-w\cdot\text{ctg}\,\alpha)\right]\\
N_\theta &= \frac{2Et}{(1-\mu^2)}\left[\frac{1}{s}\cdot(u-w\,\text{ctg}\,\alpha)+\mu\cdot\frac{du}{ds}\right]\\
M_s' &= \frac{Etk^2}{2(1-\mu^2)}\left[\frac{d\psi_s}{ds}+\frac{\mu}{s}\cdot\psi_s\right]\\
M_\theta' &= \frac{Etk^2}{2(1-\mu^2)}\cdot\left[\frac{\psi_s}{s}+\mu\frac{d\psi_s}{ds}\right]\\
M_s'' &= -\frac{Et^3}{6(1-\mu^2)}\cdot\left(\frac{d^2w}{ds^2}+\frac{\mu}{s}\frac{dw}{ds}\right)\\
M_\theta'' &= -\frac{Et^3}{6(1-\mu^2)}\left(\frac{1}{s}\frac{dw}{ds}+\mu\frac{d^2w}{ds^2}\right)\\
Q_s' &= G^c\cdot k\cdot\left(\psi_s+\frac{dw}{ds}\right)
\end{aligned}
\right\}\qquad(3.4\text{a}\sim\text{g})
$$

从而得到与文献[3]一致的结果。

3. 对称加载时夹层旋转壳的弯曲问题

当对称加载时广义内力 $S=M'_{12}=M''_{12}=Q_2=0$，广义位移 $v=0$．且其余的广义力，广义位移都仅是 α 的函数。此时由(2.29)及(2.20c)式可得平衡方程为：

$$\left.\begin{aligned}&\frac{1}{R_1}\cdot\frac{d}{d\alpha}(N_1R_2\sin\alpha)-N_2\cos\alpha+\frac{R_2\sin\alpha}{R_1}\cdot Q_1=0\\&\frac{N_1}{R_1}+\frac{N_2}{R_2}-\frac{1}{R_1R_2\sin\alpha}\cdot\frac{d}{d\alpha}(Q_1R_2\sin\alpha)+p=0\\&\frac{1}{R_1}\cdot\frac{d}{d\alpha}(M_1R_2\sin\alpha)-M_2\cos\alpha-Q_1R_2\sin\alpha=0\\&\frac{1}{R_1}\cdot\frac{d}{d\alpha}(M'_1R_2\sin\alpha)-M'_2\cos\alpha-\frac{k}{h_1+h_2}\cdot Q^c_1R_2\sin\alpha=0\end{aligned}\right\}\qquad(3.5a,b,c,d)$$

1) 薄膜解：(下标加“0”)

此时(3.5)式成为：

$$\left.\begin{aligned}&\frac{1}{R_1}\cdot\frac{d}{d\alpha}(N_{10}R_2\sin\alpha)-N_{20}\cos\alpha=0\\&\frac{N_{10}}{R_1}+\frac{N_{20}}{R_2}+p=0\end{aligned}\right\}\qquad(3.6a,b)$$

由(3.6b)知

$$N_{20}=-R_2p-\frac{R_2}{R_1}N_{10}\qquad(3.7)$$

(3.7)代入(3.6a)式得：

$$\frac{1}{R_1}\cdot\frac{dN_{10}}{d\alpha}+\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\text{ctg}\,\alpha\cdot N_{10}=-p\cdot\text{ctg}\,\alpha\qquad(3.8)$$

若引入应力函数$U(\alpha)$使得：

$$N_{10}=U(\alpha)/R_2\sin^2\alpha\qquad(3.9)$$

(3.9)代入(3.8)式：

$$\frac{dU}{d\alpha}=-p\cdot R_1R_2\sin\alpha\cos\alpha$$

因此可解得

$$U(\alpha)=-\int_\alpha pR_1R_2\sin\alpha\cos\alpha\,d\alpha+C'_1$$

$$N_{10}=-\frac{1}{R_2\sin^2\alpha}\cdot\int_\alpha pR_1R_2\sin\alpha\cos\alpha\,d\alpha+C_1$$

其中 C_1 为常数，可由边界条件确定

2) 边界效应解：

令外载荷 $p=0$，引入应力函数 $F(\alpha)$使得：

$$\left.\begin{aligned}&N_2=\frac{1}{R_1}\cdot\frac{dF}{d\alpha}\\&N_1R_2\sin\alpha=F\cos\alpha\\&Q_1R_2=F\end{aligned}\right\}\qquad(3.10a,b,c)$$

此时

将(3.10)式代入(3.5a,b)式可见满足此二平衡方程.

在对称加载时，广义内力-广义应变-广义位移关系可由(2.31)式求得为:

$$\left.\begin{aligned}&N_1=(B_1+B_2)(\varepsilon_1+\mu\varepsilon_2)\\&N_2=(B_1+B_2)(\varepsilon_2+\mu\varepsilon_1)\\&M_1'=D\cdot\left(\frac{1}{R_1}\cdot\frac{d\psi_\alpha}{d\alpha}+\mu\cdot\frac{\operatorname{ctg}\alpha}{R_2}\cdot\psi_\alpha\right)\\&M_2'=D\left(\frac{\operatorname{ctg}\alpha}{R_2}\psi_\alpha+\frac{\mu}{R_1}\cdot\frac{d\psi_\alpha}{d\alpha}\right)\\&M_1''=-D_f(\chi_1+\mu\chi_2)\\&M_2''=-D_f(\chi_2+\mu\chi_1)\\&Q_1^c=G^c\cdot k\cdot\left(\psi_\alpha+\frac{1}{R_1}\cdot\frac{dw}{d\alpha}-\frac{u}{R_1}\right)\end{aligned}\right\}\qquad(3.11a\sim g)$$

其中

$$\left.\begin{aligned}&D=\frac{B_1B_2k^2}{B_1+B_2}\\&D_f=D_1+D_2\\&\varepsilon_1=\frac{1}{R_1}\cdot\frac{du}{d\alpha}+\frac{w}{R_1}\\&\varepsilon_2=\frac{\operatorname{ctg}\alpha}{R_2}u+\frac{w}{R_2}\\&\chi_1=\frac{1}{R_1}\cdot\frac{d}{d\alpha}\left(\frac{1}{R_1}\cdot\frac{dw}{d\alpha}-\frac{u}{R_1}\right)\\&\chi_2=\frac{\operatorname{ctg}\alpha}{R_2}\left(\frac{1}{R_1}\cdot\frac{dw}{d\alpha}-\frac{u}{R_1}\right)\end{aligned}\right\}\qquad(3.12a\sim f)$$

若引入位移函数$\varphi(\alpha)=\frac{1}{R_1}\cdot\frac{dw}{d\alpha}-\frac{u}{R_1}$，则可得到变形协调方程为:

$$\frac{1}{R_1}\frac{d}{d\alpha}(\varepsilon_2R_2\sin\alpha)-\varepsilon_1\cos\alpha-\varphi\cdot\sin\alpha=0\qquad(3.13)$$

由(3.10)、(3.11)式得

$$\left.\begin{aligned}&\varepsilon_1=\frac{1}{B}(N_1-\mu N_2)=\frac{1}{B}\cdot\left(\frac{\operatorname{ctg}\alpha}{R_2}\cdot F-\frac{\mu}{R_1}\cdot\frac{dF}{d\alpha}\right)\\&\varepsilon_2=\frac{1}{B}(N_2-\mu N_1)=\frac{1}{B}\left(\frac{1}{R_1}\cdot\frac{dF}{d\alpha}-\mu\cdot\frac{\operatorname{ctg}\alpha}{R_2}\cdot F\right)\end{aligned}\right\}\qquad(3.14a,b)$$

此处 $B=(B_1+B_2)(1-\mu^2)$

(3.14)式代入(3.13)式可得协调方程

$$L(F)+\frac{\mu}{R_1R_2}\cdot F=B\cdot\frac{\varphi}{R_2} \tag{3.15}$$

此处微分算子 $L(\ \ )$ 为：

$$L(\ \ )=\frac{1}{R_1}\cdot\frac{d}{d\alpha}\left[\frac{1}{R_1}\cdot\frac{d(\ \ )}{d\alpha}\right]+\frac{\operatorname{ctg}\alpha}{R_1R_2}\cdot\frac{d(\ \ )}{d\alpha}-\frac{\operatorname{ctg}^2\alpha}{R_2^2}(\ \ )$$

将(3.11c，d)、(3.11g)式代入(3.5d)式得：

$$L(\psi_\alpha)-\frac{\mu}{R_1R_2}\psi_\alpha-\frac{C}{D}(\psi_\alpha+\varphi)=0 \tag{3.16}$$

此处

$$C=\frac{G^c\cdot k^2}{h_1+h_2}\text{为剪切刚度.}$$

将(3.11c)～(3.11f)代入(3.5c)式得

$$D\left[L(\psi_\alpha)-\frac{\mu}{R_1R_2}\psi_\alpha\right]-D_f\left[L(\varphi)-\frac{\mu}{R_1R_2}\varphi\right]=\frac{F}{R_2} \tag{3.17}$$

最后将(3.16)、(3.17)式代入(3.15)式即可得到一个只含 ψ_α 的六阶微分方程：

$$C\cdot[L(R_2L)]\psi_\alpha-D_f[L(R_2L^2)-\frac{C}{D}\cdot L(R_2L)]\psi_\alpha-\frac{B}{R_2}\left(L-\frac{C}{D}\right)\psi_\alpha=0 \tag{3.18}$$

由此可结合相应的边界条件求解．将薄膜解与边界效应解迭加即为完全解．

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On the Problems of Sandwich Shells Having the Form of a Surface of Revolution and Face Layers of Non-Equal Thicknesses

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Abstract

Fundamental equations which govern the behavior of an elastic sandwich shell having the form of a surface of revolution and face layers of non-equal thicknesse are derived, with the solution of refs. [3] and [4] as special examples.

The problems of the shell under the action of symmetrical loads are reduced to the solution of a displacement-function ψ_α, where ψ_α satisfies a differential equation of sixth order.