

悬臂柔轴的干摩擦失稳*

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摘 要

本文讨论柔轴高速自转时由于摩擦激发的弓状回转现象, 摩擦力作用面不与集聚质量面重合, 导出了回转频率和回转形态的计算公式, 分析了它的运动稳定性, 结果指出, 存在着稳定的弓状回转, 其频率大于转轴的临界转速.

柔轴的弓状回转现象, 已被日益重视和研究. 激发弓状回转的因素很多: 油膜、内阻和轴的各种不对称因素等等, 均已有大量文献讨论. 相比之下, 讨论由于摩擦激发的弓状回转现象的文献就少些. Den Hartog在他的名著[1]中对该现象有过精辟的定性分析. 他指出, 柔轴高速自转而引起的变形导致与固定周边发生接触而产生切向摩擦力, 该力可分解为一个力偶和一个作用于轴心上的力, 力偶可被外力矩平衡, 而力则激发出与自转方向相反的逆向回转.

要定量分析其回转频率和形态, 由于力学模型和数学上的一些困难, 以前大多对单自由度转子的简单数学模型进行讨论, 而且摩擦力亦假定是直接作用在集聚质量的旋转平面上^[2,3], 本文对该现象作了进一步的分析, 采用较复杂些的二自由度模型, 并且摩擦力作用面不再限于在集聚质量平面上. 本文推导了该模型的干摩擦弓状回转的频率和形态的计算公式, 并分析了每个可能回转运动的稳定性. 结果表明, 存在着稳定的弓状回转, 其频率大于转轴的临界转速.

一、计算模型、运动方程和解

所用模型为长 L 的轻质悬臂轴, 端部 B 处附一个集中质量 m (为简化计算不计入其转动惯量) 轴上离固支端距离为 l 处的 A 点附有一轻质薄圆盘, 它与外壳间隙为 γ , 视图1. 文献[4]曾用该模型近似讨论过干摩擦何时不激发弓状回转的问题, 本文则欲用该模型讨论它可能产生的弓状回转及其频率和形态的计算问题.

设 Ω 为轴的自转角速度, ω 为弓状回转频率, μ 为薄圆盘和外壳间的摩擦系数, c 为集聚

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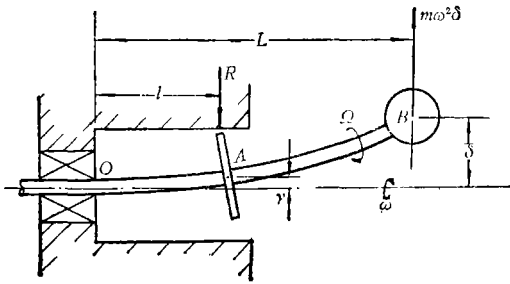


图 1

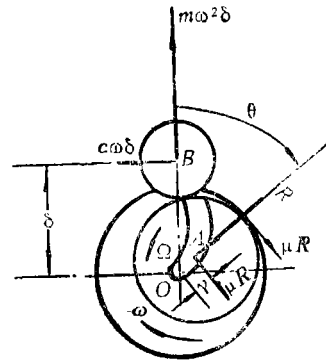


图 2

质量 B 处的外阻尼系数. f_{11} 为 A 处作用单位力而在 A 处引起的位移量. f_{22} 为 B 处作用单位力而在 B 处引起的位移量. f_{12} 为 B 处 (或 A 处) 作用单位力而在 A 处 (或 B 处) 引起的位移量.

当轴高速自转时, 由于某种原因 (如不平衡力) 使 A 处圆盘偏离而与外周接触, 圆盘在接触点处受到正压力 R 和切向干摩擦力 μR , 于是引起弓状回转. 轴在 B 点的离心力和阻尼力以及 A 处的正压力和摩擦力联合作用下, 引起空间弯曲变形, 并以该形态以 ω 作定常回转. 为描述其变形, 沿横向取一个固定的极坐标系, 设 B 点的径向位移 δ , A 点的径向位移应为间隙 γ , 该两径向位移间夹角 θ , 于是, 轴在 B 处的径向离心力为 $m\omega^2\delta$, 切向阻尼力为 $c\omega\delta$, 此外, A 处有径向正压力 R , 切向摩擦力 μR , 其方向见图 2. 该图为轴向投影图. 于是, 柔轴的变形方程为:

$$\delta = -f_{12}(R\cos\theta + \mu R\sin\theta) + f_{22}m\omega^2\delta \quad (1.1)$$

$$0 = -f_{12}(R\sin\theta - \mu R\cos\theta) - f_{22}c\omega\delta \quad (1.2)$$

$$\gamma = -f_{11}R + f_{12}(m\omega^2\delta\cos\theta - c\omega\delta\sin\theta) \quad (1.3)$$

$$0 = f_{11}\mu R - f_{12}(m\omega^2\delta\sin\theta + c\omega\delta\cos\theta) \quad (1.4)$$

这就是柔轴的弓状回转方程. 由此解出 $\omega, \delta, R, \theta$.

由该四式直接可得 $\mu R\gamma = c\omega\delta^2$, 即摩擦功等于阻尼功. 该公式下面不用.

$$\text{令: } \left. \begin{aligned} \bar{R} &= \frac{R}{m\omega^2\gamma} & \bar{\delta} &= \frac{\delta}{\gamma}, & \lambda &= \frac{\omega}{\omega_0} & \bar{c} &= \frac{c}{m\omega_0} \\ \bar{f}_{11} &= \frac{f_{11}}{f_{22}}, & \bar{f}_{12} &= \frac{f_{12}}{f_{22}}, & \omega_0^2 &= \frac{1}{mf_{22}} \end{aligned} \right\} \quad (1.5)$$

其中 ω_0 为该轴的临界转速. 于是 (1.1) — (1.4) 化为无量纲方程:

$$(1 - \lambda^2)\bar{\delta} + \bar{f}_{12}(\cos\theta + \mu\sin\theta)\bar{R} = 0 \quad (1.6)$$

$$\bar{c}\lambda\bar{\delta} + \bar{f}_{12}(\sin\theta - \mu\cos\theta)\bar{R} = 0 \quad (1.7)$$

$$\bar{f}_{12}(\lambda^2\cos\theta - \bar{c}\lambda\sin\theta)\bar{\delta} - \bar{f}_{11}\bar{R} = 1 \quad (1.8)$$

$$\bar{f}_{12}(\lambda^2\sin\theta + \bar{c}\lambda\cos\theta)\bar{\delta} - \mu\bar{f}_{11}\bar{R} = 0 \quad (1.9)$$

由 (1.8)、(1.9) 解出

$$\bar{R} = \frac{\lambda\sin\theta + \bar{c}\cos\theta}{\bar{f}_{11}[\lambda(\mu\cos\theta - \sin\theta) - \bar{c}(\cos\theta + \mu\sin\theta)]} \quad (1.10)$$

$$\bar{\delta} = \frac{\mu}{f_{12}\lambda[\lambda(\mu\cos\theta - \sin\theta) - \bar{c}(\cos\theta + \mu\sin\theta)]} \quad (1.11)$$

把(1.10)、(1.11)代入(1.7),得:

$$\bar{c}\mu\bar{f}_{11} + \bar{f}_{12}^2(\sin\theta - \mu\cos\theta)(\lambda\sin\theta + \bar{c}\cos\theta) = 0 \quad (1.12)$$

又若(1.6)、(1.7)对 \bar{R} 、 $\bar{\delta}$ 欲有非零解, 必须

$$\frac{\cos\theta + \mu\sin\theta}{\sin\theta - \mu\cos\theta} = \frac{1 - \lambda^2}{\bar{c}\lambda} \quad (1.13)$$

即:
$$\text{tg}\theta = \frac{(\lambda^2 - 1)\mu - \bar{c}\lambda}{\lambda^2 - 1 + \bar{c}\lambda\mu} \quad (1.14)$$

把(1.14)代回(1.10)、(1.11)和(1.12), 得:

$$\bar{R} = \frac{\mu\lambda(\lambda^2 - 1) - \bar{c} + \bar{c}^2\lambda\mu}{f_{11}\bar{c}(1 + \mu^2)} = \frac{\mu[(\lambda^2 - 1)^2 + \bar{c}^2\lambda^2]}{(1 + \mu^2)f_{12}\bar{c}\lambda} \quad (1.15)$$

$$\bar{\delta} = \frac{\mu}{f_{12}\bar{c}\lambda} \sqrt{\frac{\bar{c}^2\lambda^2 + (\lambda^2 - 1)^2}{1 + \mu^2}} \quad (1.16)$$

$$\mu\bar{f}_{11}[(\lambda^2 - 1)^2 + \bar{c}^2\lambda^2] - \lambda\bar{f}_{12}^2[\mu\lambda(\lambda^2 - 1) - \bar{c} + \bar{c}^2\lambda\mu] = 0 \quad (1.17)$$

其中(1.15)的第二等号是利用了(1.17)。(1.17)即为所求的特征方程, 给出 \bar{c} 、 μ 等值, 即可解出回转频率 λ 。再由(1.14)–(1.16)求出回转形态。

二、小 阻 尼 情 况

若 $\bar{c} = 0$, 由(1.17)得两根

$$\lambda_1 = 1, \quad \lambda_2 = \sqrt{\frac{f_{11}}{f_{11} - f_{12}^2}} \quad (2.1)$$

若 $0 < \bar{c} \ll 1$, 则令 $\lambda_1 = 1 + \varepsilon_1$ 和 $\lambda_2 = \sqrt{\frac{f_{11}}{f_{11} - f_{12}^2}} + \varepsilon_2$, 其中 ε_1 和 ε_2 均为 \bar{c} 阶小量, 代入(1.17),

保留 \bar{c} 的最低阶项, 得第一弓状回转频率和形态为

$$\lambda_1 \approx 1 + \frac{\bar{c}}{2\mu} \quad (2.2)$$

$$\theta_1 \approx 0, \quad \bar{R} \approx \frac{\bar{c}}{\mu f_{12}^2}, \quad \bar{\delta} \approx \frac{1}{f_{12}} \quad (2.3)$$

而第二弓状回转频率和形态为

$$\lambda_2 \approx \sqrt{\frac{f_{11}}{f_{11} - f_{12}^2}} - \frac{\bar{c}}{2\mu} \quad (2.4)$$

$$\left. \begin{aligned} \text{tg}\theta \approx \mu, \quad \bar{R} &\approx \frac{\mu}{\bar{c}(1 + \mu^2)} \sqrt{\frac{1}{f_{11}(f_{11} - f_{12}^2)}} \frac{f_{12}^2}{f_{11} - f_{12}^2} \\ \bar{\delta} &\approx \frac{\mu}{\bar{c}\sqrt{1 + \mu^2}} \frac{f_{12}}{\sqrt{f_{11}(f_{11} - f_{12}^2)}} \end{aligned} \right\} \quad (2.5)$$

它们为小阻尼时的显式解案。由(2.2)、(2.4)知,对本模型,回转频率都大于临界转速,此外,第一形态因 $\theta \approx 0$ 而近乎平面变形,第二形态为空间变形,且因 \bar{R} , $\bar{\delta}$ 呈 $\frac{1}{c}$ 量级,故变形量较大。

三、运动稳定性

需考察上述两定常弓状回转解的运动稳定性。现假设轴处于一般的运动状态下, B 点极坐标为 (ρ, ϕ) , A 点的为 $(\gamma, \phi + \theta)$, 这里 ρ, ϕ, θ 均为时间的函数。于是轴在 B 处受径向惯性力 $-m(\ddot{\rho} - \rho\dot{\phi}^2)$ 和切向惯性力 $-m(2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})$, 同时还受径向阻尼力 $-c\dot{\rho}$ 和切向阻尼力 $-c\rho\dot{\phi}$, 其中“ \cdot ”表示对时间求导。于是, 轴在 B 处所受的径向合力 N_B 和切向合力 T_B 分别为

$$N_B = m(\rho\dot{\phi}^2 - \ddot{\rho}) - c\dot{\rho} \quad (3.1)$$

$$T_B = -m(2\dot{\rho}\dot{\phi} + \rho\ddot{\phi}) - c\rho\dot{\phi} \quad (3.2)$$

此外, 轴在 A 处受径向正压力 R 和切向干摩擦力 μR 。方向见图3。它亦为轴向投影图。于是, 轴的运动方程为

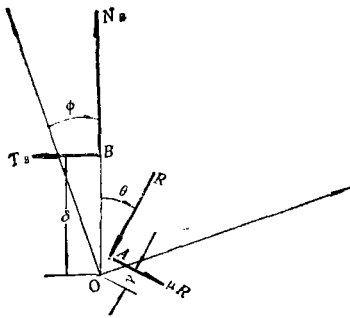


图 3

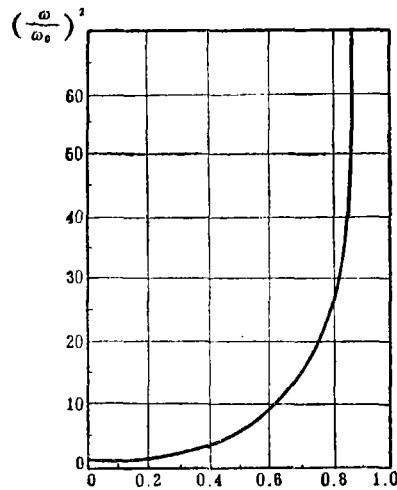


图 4

$$\gamma = -f_{11}R + f_{12}(N_B \cos \theta + T_B \sin \theta) \quad (3.3)$$

$$0 = f_{11}\mu R + f_{12}(-N_B \sin \theta + T_B \cos \theta) \quad (3.4)$$

$$\rho = -f_{12}(R \cos \theta + \mu R \sin \theta) + f_{22}N_B \quad (3.5)$$

$$0 = f_{12}(-R \sin \theta + \mu R \cos \theta) + f_{22}T_B \quad (3.6)$$

令 $\rho = \delta + \rho_1$, $\phi = \omega + \phi_1$, $\theta = \theta_0 + \theta_1$, $R = R_0 + R_1$, 其中 $\delta, \omega, \theta_0, R_0$ 用来表示定常回转解, $\rho_1, \phi_1, \theta_1, R_1$ 为一阶扰动量。代入(3.3)–(3.6), 保留扰动量的一阶项, 得如下无量纲扰动方程:

$$-\bar{f}_{11}\bar{R}_1 - \bar{f}_{12}\bar{\delta}\lambda(\lambda \sin \theta_0 + \bar{c} \cos \theta_0)\theta_1 + \bar{f}_{12} \cos \theta_0 \cdot G - \bar{f}_{12} \sin \theta_0 \cdot H = 0 \quad (3.7)$$

$$\mu \bar{f}_{11}\bar{R}_1 - \bar{f}_{12}\bar{\delta}\lambda(\lambda \cos \theta_0 - \bar{c} \sin \theta_0)\theta_1 - \bar{f}_{12} \sin \theta_0 \cdot G - \bar{f}_{12} \cos \theta_0 \cdot H = 0 \quad (3.8)$$

$$-(\cos\theta_0 + \mu\sin\theta_0)\bar{R}_1 + (\sin\theta_0 - \mu\cos\theta_0)\bar{R}_0\theta_1 = \frac{\bar{\rho}_1 - G}{\bar{f}_{12}} \quad (3.9)$$

$$-(\sin\theta_0 - \mu\cos\theta_0)\bar{R}_1 - (\cos\theta_0 + \mu\sin\theta_0)\bar{R}_0\theta_1 = H/\bar{f}_{13} \quad (3.10)$$

其中 $\bar{R}_1 = \frac{R_1}{m\omega^2\gamma}$, $\bar{\rho}_1 = \frac{\rho_1}{\gamma}$, $\tau = \omega_0 t$ (3.11)

$$G = \lambda^2\bar{\rho}_1 + 2\lambda\bar{\psi}_1 - \bar{\rho} - \bar{c}\bar{\beta}_1 \quad (3.12)$$

$$H = \bar{c}(\lambda\bar{\rho}_1 + \bar{\psi}_1) + \bar{\psi}_1 + 2\lambda\bar{\beta}_1 \quad (3.13)$$

$$\bar{\psi}_1 = \bar{\delta}\bar{\phi}_1 \quad (3.14)$$

λ , \bar{R}_0 , θ_0 , $\bar{\delta}$ 对应于无量纲的定常回转解。“ \cdot ”表示对无量纲时间 τ 的导数, 由 (3.9)、(3.10) 解出:

$$\begin{aligned} \bar{R}_1 &= \frac{-(\cos\theta_0 + \mu\sin\theta_0)(\bar{\rho}_1 - G) - (\sin\theta_0 - \mu\cos\theta_0)H}{(1 + \mu^2)\bar{f}_{12}} \\ &= -\frac{\sin\theta_0 \cdot \mu\cos\theta_0}{\bar{f}_{12}(1 + \mu^2)} \left[-\frac{1 - \lambda^2}{\bar{c}\lambda} (\bar{\rho}_1 - G) + H \right] \end{aligned} \quad (3.15)$$

$$\begin{aligned} \theta_1 &= \frac{-(\cos\theta_0 + \mu\sin\theta_0)H + (\sin\theta_0 - \mu\cos\theta_0)(\bar{\rho}_1 - G)}{(1 + \mu^2)\bar{R}_0\bar{f}_{12}} \\ &= \frac{\sin\theta_0 - \mu\cos\theta_0}{\bar{R}_0\bar{f}_{12}(1 + \mu^2)} \left[-\frac{1 - \lambda^2}{\bar{c}\lambda} H + (\bar{\rho}_1 - G) \right] \end{aligned} \quad (3.16)$$

上述两式的后一等号是利用了(1.13). 将该两式代入(3.7), (3.8), 利用(1.13)–(1.17), 得:

$$\left. \begin{aligned} &\bar{f}_{11}(\lambda^2 - 1)(\bar{\rho}_1 - G) - \bar{f}_{11}\bar{c}\lambda H + \mu\bar{f}_{11}(\lambda^2 - 1)H + \mu\bar{f}_{11}\bar{c}\lambda(\bar{\rho}_1 - G) \\ &+ \bar{f}_{12}^2(\lambda^2 - 1 + \bar{c}\mu\lambda)G - \bar{f}_{12}^2[-\bar{c}\lambda + \mu(\lambda^2 - 1)]H = 0 \\ &\mu\bar{f}_{11}[(\lambda^2 - 1)(\bar{\rho}_1 - G) - \bar{c}\lambda H] + \bar{f}_{12}^2[(\lambda^2 - 1)\mu - \bar{c}\lambda]G + \bar{f}_{12}^2(\lambda^2 - 1 \\ &+ \bar{c}\lambda\mu)H - \frac{\lambda\bar{f}_{12}^2[\lambda(\lambda^2 - 1) + \bar{c}^2\lambda + \bar{c}\mu]}{(\lambda^2 - 1)^2 + \bar{c}^2\lambda^2} [(\lambda^2 - 1)H + \bar{c}\lambda(\bar{\rho}_1 - G)] = 0 \end{aligned} \right\} \quad (3.17)$$

(3.17) 即为扰动方程,

小阻尼下第一弓状回转频率解为 (2.2), 代入 (3.17), 此时 $\lambda^2 - 1 = \frac{\bar{c}}{\mu}$, 每个方程中

仅保留 \bar{c} 的最低阶项 (第一式保留的是 \bar{c} 的一阶项, 第二式是零阶项) 得

$$\left. \begin{aligned} &\bar{f}_{11}(\bar{\rho}_1 - G) + \bar{f}_{12}^2 G = 0 \\ &H + \mu(\bar{\rho}_1 - G) = 0 \end{aligned} \right\} \quad (3.18)$$

将(3.12)、(3.13)代入, 仍保留 \bar{c} 的最低阶项, 得:

$$\left. \begin{aligned} &(\bar{f}_{11} - \bar{f}_{12}^2)\bar{\rho}_1 - 2(\bar{f}_{11} - \bar{f}_{12}^2)\bar{\psi}_1 + \bar{f}_{12}^2\bar{\rho}_1 = 0 \\ &\mu\bar{\rho}_1 + \bar{\psi}_1 + 2\bar{\beta}_1 - 2\mu\bar{\psi}_1 = 0 \end{aligned} \right\} \quad (3.19)$$

令 $\bar{\rho}_1 = \bar{\rho}_{10}e^{k\tau}$, $\psi_1 = \psi_{10}e^{k\tau}$ 代入 (3.19), 得 k 的特征方程:

$$(\bar{f}_{11} - \bar{f}_{12}^2)k^3 + (4\bar{f}_{11} - 3\bar{f}_{12}^2)k - 2\mu\bar{f}_{12}^2 = 0 \quad (3.20)$$

轴的弹性应变能总是正的, 故恒有 $\bar{f}_{11} - \bar{f}_{12}^2 > 0$, 但 (3.20) 的常数项为负, 由 Routh-Hurwitz 准则知, (3.20) 有正实数根, 由此证得第一弓状回转运动是不稳定的.

考虑第二弓状回转, 将 (2.4) 代入 (3.17), 保留 \bar{c} 的最低阶项 (零阶项), 得:

$$\left. \begin{aligned} \lambda_2^2 \bar{\rho}_1 - G + \mu H &= 0 \\ \mu \lambda_2^2 (1 - \lambda_2^2) \bar{\rho}_1 - \mu (1 - \lambda_2^2) G + (\lambda_2^2 - 1) H &= 0 \end{aligned} \right\} \quad (3.21)$$

其中 λ_2 为 (2.1) 式. 将 (3.12)、(3.13) 代入, 仍取最低阶项, 得:

$$\left. \begin{aligned} \bar{\rho}_1 + \mu \bar{\psi}_1 + 2\mu \lambda_2 \bar{\rho}_1 - 2\lambda_2 \bar{\psi}_1 &= 0 \\ -\mu \bar{\rho}_1 + \bar{\psi}_1 + 2\lambda_2 \bar{\rho}_1 + 2\lambda_2 \mu \bar{\psi}_1 &= 0 \end{aligned} \right\} \quad (3.22)$$

令 $\bar{\rho}_1 = \bar{\rho}_{10} e^{k\tau}$, $\bar{\psi}_1 = \psi_{10} e^{k\tau}$, 代入 (3.22), 得

$$k^2 + 4\lambda_2^2 = 0 \quad (3.23)$$

解出 $k = \pm 2\lambda_2 i$, 于是证得第二弓状回转是稳定的.

综上所述, 对本模型, 小阻尼时, 干摩擦实际激发的稳态弓状回转的频率近似为

$$\omega = \omega_0 \sqrt{\frac{f_{11} f_{22}}{f_{11} f_{22} - f_{12}^2}} \quad (3.24)$$

它显然大于 ω_0 , 比如, 取均匀悬臂轴, $f_{11} = \frac{l^3}{3EI}$, $f_{22} = \frac{L^3}{3EI}$, $f_{12} = \frac{l^2(3L-l)}{6EI}$, 代入 (3.24) 得

$$\frac{\omega^2}{\omega_0^2} = \frac{4}{4 - \xi(3 - \xi)^2} \quad (3.25)$$

其中 $\xi = \frac{l}{L}$, (3.25) 曲线见图 4. 除 $\xi = 0$ 外, 其回转频率都大于临界转速.

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Flexible Shaft Whirl Induced by Dry Friction and Its Stability

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Abstract

This paper is devoted to the study of the whirling phenomena of flexible rotors due to dry friction. The mechanical model used here is two-degree-of-freedom system in which the rubbing plane is not coincided with the rotating plane of the lumped mass. The characteristic equation of whirl speeds is derived. The whirling modes are obtained. The dynamic stability of each admissible whirling motion is also discussed. The results show that the whirl speeds are always higher than the critical speed of the shaft.