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# 关于非线性狄立克雷问题的一致有效渐近解

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**摘　要**

本文研究最高阶导数项带小参数的二阶拟线性椭圆型方程的狄立克雷问题．在退化方程不存在奇点的情形下，当参数$\varepsilon$是充分小时，证明了解的存在性和唯一性，并在整个区域导出解的一致有效渐近近似式．

## 一、引　言

关于最高阶导数项带小参数的椭圆型方程的狄立克雷问题的研究，近十年来已进入非线性方程的领域．1970年，M. S. Berger和L. E. Fraenkel[1]首先研究了椭圆型方程$\varepsilon^2\Delta u+u-g^2(x)u^3=0$的齐次狄立克雷问题，证明当$\varepsilon$充分小时，狄立克雷问题的解是存在和唯一，并在整个区域导出解的一致有效渐近近似式．1973年，P. C. Fife又研究了方程$\varepsilon^2\sum\limits_{i,j=1}^{n}\dfrac{\partial}{\partial x_i}\Big[a_{i,j}(x,\varepsilon)\dfrac{\partial u}{\partial x_j}\Big]-h(x,u)=0$的非齐次狄立克雷问题．1976年，C. J. Holland[3]又考虑退化方程具有特征的半线性椭圆型方程：

$$\varepsilon\left(a(x,\ y)\frac{\partial^2 u}{\partial x^2}+2b\ (x,y)\frac{\partial^2 u}{\partial x\partial y}+c(x,y)\frac{\partial^2 u}{\partial y^2}\right)$$
$$+A(x,y)\frac{\partial u}{\partial x}+B(x,y)\frac{\partial u}{\partial y}+F(x,y,u)=0$$

的非齐次狄立克雷问题，但该文只在除去切点（指区域的边界与退化方程的特征的切点）的正则四边形 (regular quadrilateral) 上作出解的渐近近似式，未能导出解的一致有效渐近近似式，并且还需假设狄立克雷问题的解存在，和在正则四边形上关于$\varepsilon$是一致有界：$|u(x,\ y;\ \varepsilon)|\leqslant M$，$M$是与$\varepsilon$无关的常数．

本文应用边界层校正法重新研究上述问题．当$\varepsilon$充分小时，证明了解的存在性和唯一性，导出解的一致有效渐近近似式，得出与文[1]、[2]相当的结果，改进了Holland的工作．显然本文所用的方法，也适用于方程的系数依赖于$u$和$\varepsilon$的情形．

## 二、形式渐近解

考察二阶半线性椭圆型方程的狄立克雷问题:

$$N_\varepsilon[u] \equiv (\varepsilon L_2 - L_1)[u] - F(x, y, u) = 0 \qquad (x, y) \in \Omega \tag{2.0.1}$$

$$u|_{\partial\Omega} = \varphi(x, y)|_{\partial\Omega} \tag{2.0.2}$$

其中 $0 < \varepsilon \leqslant \varepsilon_0$, ($\varepsilon_0 \ll 1$), $\Omega$ 表示 $xy$ 平面上的有界凸域, $\partial\Omega$ 表示 $\Omega$ 的边界, 和

$$L_2[u] \equiv a(x, y)\frac{\partial^2 u}{\partial x^2} + 2b(x, y)\frac{\partial^2 u}{\partial x \partial y} + c(x, y)\frac{\partial^2 u}{\partial y^2}$$

$$+ d(x, y)\frac{\partial u}{\partial x} + e(x, y)\frac{\partial u}{\partial y} + f(x, y)u$$

$$(a\xi_1^2 + 2b\xi_1\xi_2 + c\xi_2^2 \geqslant m_0 > 0, \quad (x, y) \in \Omega, \ m_0 \text{是常数})$$

$$L_1[u] \equiv A(x, y)\frac{\partial u}{\partial x} + B(x, y)\frac{\partial u}{\partial y}$$

假设

($H1$). 方程(2.0.1)中的系数和边值函数 $\varphi(x, y)$ 在所定义的区域中是无限次可微, 边界$\partial\Omega$是无限次光滑;

($H2$). 非线性项 $F(x, y, u)$是 $x$, $y$, $u$ 在区域 $\Omega \times (-\infty < u < \infty)$ 中的无限次可微函数;

($H3$). $A^2(x, y) + B^2(x, y) \geqslant n_0 > 0$. $n_0$ 是常数. 在假设($H3$)下, 可作非奇异的自变量的变换使微分算子 $L_1$ 具有形式: $L_1 \equiv \frac{\partial}{\partial y}$ (仍用原来的字母表示变换后的相应的表达式), 所以不妨假设 $A \equiv 0$, $B \equiv 1$.

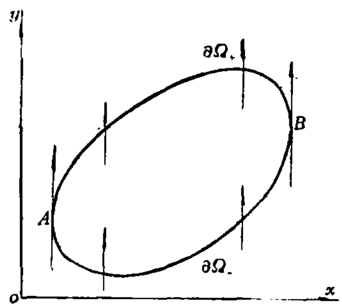


图1 边界$\partial\Omega_-$和$\partial\Omega_+$

**1. 外部解**

假设区域 $\Omega$ 的边界与退化方程的特征线 $x = x_0$、$x = x_1$, ($x_0 < x_1$)相切于 $A$、$B$ 两点. 以$\partial\Omega_-$表示边界被特征线族 $x = c$, ($x_0 < c < x_1$)按 $y$ 的正向穿入区域的部分, 设其方程是 $y = y_-(x)$; 以$\partial\Omega_+$表示被特征线族穿出区域的部分, 设其方程是$y = y_+(x)$.

假设狄立克雷问题(2.0.1)—(2.0.2)的解的 $m$ 阶渐近近似式是

$$W_m(x, y; \varepsilon) = \sum_{i=0}^{m} \varepsilon^i w_i(x, y) \tag{2.1.1}$$

代入方程(2.0.1), 并令$\varepsilon^i$, ($i = 0, 1, \cdots m$) 的系数为零, 得到关于 $w_i(x, y)$ 的递推方程:

$$\frac{\partial w_0}{\partial y} + F(x, y, w_0) = 0 \tag{2.1.2}$$

$$\frac{\partial w_i}{\partial y} + F_u(x, y, w_0)w_i = L_2[w_{i-1}] - \frac{1}{i!}h_i(x, y, w_0 \cdots, w_{i-1}) \tag{2.1.3}$$

$$(i = 1, 2, \cdots, m)$$

式中 $h_1\equiv 0$, $h_i\equiv \sum\limits_{\substack{l_1+2l_2+\cdots+pl_p=i\\(p<i)}} \dfrac{i!}{l_1!l_2!\cdots l_p!}\dfrac{\partial^{l_1+l_2+\cdots+l_p}F(x,y,w_0)}{\partial u^{l_1+l_2+\cdots+l_p}}w_1^{l_1}w_2^{l_2}\cdots w_p^{l_p}$

$(i=2,3,\cdots m)$.

因在方程(2.0.1)中，$\dfrac{\partial u}{\partial y}$的系数与$L_2$中的系数$a(x,y)$是异号，依据线性椭圆型方程的正则退化理论[4]知应在$\partial\Omega_-$上给出$w_i$的定解条件：

$$w_0\Big|_{\partial\Omega_-}=\varphi(x,y_-(x))\qquad (x_0<x<x_1) \tag{2.1.4}$$

$$w_i\Big|_{\partial\Omega_-}=0\qquad (x_0<x<x_1),\ (i=1,2,\cdots,m) \tag{2.1.5}$$

再假设

$(H4)$．退化边值问题：

$$\frac{\partial w_0}{\partial y}+F(x,y,w_0)=0\qquad w_0\Big|_{y=y_-(x)}=\varphi(x,y_-(x))$$

在区域$\bar{\Omega}\backslash[\partial\Omega\cap(x=x_0,\ x=x_1)]$存在无限次可微的解$w_0$.

将$w_0$代入(2.1.3)、(2.1.5)(取$i=1$)得到关于$w_1$的柯西问题，解得

$$w_1=\int_{y_-(x)}^{y}L_2[w_0]\exp\left[-\int_t^y F_u(x,s,w_0(x,s))ds\right]dt$$

因$y=y_-(x)$的导数在切点$A$、$B$具有奇性，所以$L_2[w_0]$在点$A$、$B$一般也具有奇性．在不包含切点$A$、$B$的区域内，可以根据(2.1.3)、(2.1.5)式逐步地解出$w_i$，$(i=1,2,\cdots,m)$，最后求得$W_m$.

显然如此作得的$W_m$在$\partial\Omega_+$上一般不满足边值条件(2.0.2)，并且在切点$A$、$B$具有奇性，即只可能在不包含$\partial\Omega_+$和$A$、$B$的外部区域中有效，称为狄立克雷问题的外部解(或解的外部展开式)．下面再在$\partial\Omega_+$和$A$、$B$的邻域构造边界层校正项.

**2. 常型边界层**

在边界$\partial\Omega$的邻域建立局部坐标系：过$\partial\Omega$的每点$P$作长为$\rho_0$的内法线，取$\rho_0$充分小使各内法线互不相交；对于$\partial\Omega$的邻域中的每点$M$，取$M$沿内法线到边界的距离作为它的$\rho$坐标$(0\leqslant\rho\leqslant\rho_0)$，取此内法线在边界上的原点$P$沿边界到切点$A$的弧长作为$M$点的$\theta$坐标，以顺时针方向为正．设切点$B$的$\theta$坐标为$\theta_B$．在局部坐标系$(\rho,\ \theta)$下，算子$N_\varepsilon$具有形式

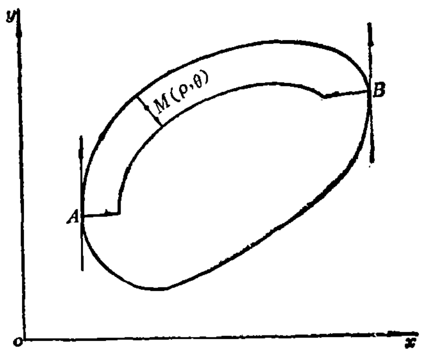

图2 局部坐标系.

$$\tilde{N}_\varepsilon[u]\equiv(\varepsilon\tilde{L}_2-\tilde{L}_1)[u]-\tilde{F}(\rho,\theta,u)$$

其中$\tilde{L}_2$、$\tilde{L}_1$和$\tilde{F}$分别表示$L_2$、$L_1$和$F$在局部坐标系中的表达式：

$$\tilde{L}_2\equiv\tilde{a}(\rho,\ \theta)\frac{\partial^2}{\partial\rho^2}+2\tilde{b}(\rho,\ \theta)\frac{\partial^2}{\partial\rho\partial\theta}+\tilde{c}(\rho,\ \theta)\frac{\partial^2}{\partial\theta^2}$$

$$+\tilde{d}(\rho,\ \theta)\frac{\partial}{\partial\rho}+\tilde{e}(\rho,\ \theta)\frac{\partial}{\partial\theta}+\tilde{f}(\rho,\theta)$$

$$\tilde{L}_1 \equiv \alpha(\rho,\theta)\frac{\partial}{\partial\rho}+\beta(\rho,\theta)\frac{\partial}{\partial\theta}$$

$$\widetilde{F}(\rho,\theta,u)\equiv F(x(\rho,\theta),y(\rho,\theta),u)$$

易知在$\partial\Omega_+$上成立

$$\alpha(\rho,\theta)|_{\rho=0}=\cos(\widehat{\rho,\ y})<0,\quad (0<\theta<\theta_B) \tag{2.2.1}$$

$$\alpha(0,0)=\alpha(0,\theta_B)=0$$

记 $D_{\rho_0}=\{(\rho,\theta)\,|\,0\leqslant\rho\leqslant\rho_0,\ 0<\theta<\theta_B\}$，在$D_{\rho_0}$中作伸展变换

$$\tau=\frac{\rho}{\varepsilon}$$

并将算子中的系数关于$\varepsilon$展开

$$\tilde{a}(\varepsilon\tau,\ \theta)=\sum_{i=0}^{m+1}a_i(\theta)\varepsilon^i\tau^i+\varepsilon^{m+2}\tau^{m+2}\bar{a}_{m+2}(\mu\varepsilon\tau,\ \theta)$$

其中 $a_i=\frac{1}{i!}\left.\frac{\partial^i\tilde{a}(\varepsilon\tau,\theta)}{\partial\rho^i}\right|_{\varepsilon=0}$, $\bar{a}_{m+2}=\frac{1}{(m+2)!}\left.\frac{\partial^{m+2}\tilde{a}(\varepsilon\tau,\theta)}{\partial\rho^{m+2}}\right|_{\varepsilon=\mu\varepsilon}$; $\cdots$

可将算子$\tilde{N}_\varepsilon$分解成

$$\tilde{N}_\varepsilon[u]\equiv\varepsilon^{-1}(B_0+\varepsilon B_1+\cdots+\varepsilon^{m+1}B_{m+1}+\varepsilon^{m+2}\bar{B}_{m+2})[u]-\widetilde{F}(\varepsilon\tau,\theta,u) \tag{2.2.2}$$

其中

$$B_0\equiv a_0(\theta)\frac{\partial^2}{\partial\tau^2}-\alpha_0(\theta)\frac{\partial}{\partial\tau}$$

$$B_i\equiv\left(a_i\tau^i\frac{\partial^2}{\partial\tau^2}-\alpha_i\tau^i\frac{\partial}{\partial\tau}\right)+\left(2b_{i-1}\tau^{i-1}\frac{\partial^2}{\partial\tau\partial\theta}+d_{i-1}\tau^{i-1}\frac{\partial}{\partial\tau}-\beta_{i-1}\tau^{i-1}\frac{\partial}{\partial\theta}\right)$$

$$+\left(c_{i-1}\tau^{i-2}\frac{\partial^2}{\partial\theta^2}+e_{i-2}\tau^{i-2}\frac{\partial}{\partial\theta}+f_{i-2}\tau^{i-2}\right),\quad (i=1,2,\cdots,m+1)$$

$$\bar{B}_{m+2}\equiv\left(\bar{a}_{m+2}\tau^{m+2}\frac{\partial^2}{\partial\tau^2}-\bar{\alpha}_{m+2}\tau^{m+2}\frac{\partial}{\partial\tau}\right)+\cdots+(f_m\tau^m+\cdots+\bar{f}_{m+2}\tau^{m+2})$$

今后都将负下标的量取作零，和假设$0\leqslant\mu\leqslant1$.

假设在$\partial\Omega_+$的邻域的边界层（校正）项的展开式是

$$V_m(\tau,\theta;\varepsilon)=\sum_{i=0}^{m+1}\varepsilon^iv_i(\tau,\theta) \tag{2.2.3}$$

定义

$$\tilde{U}_m=\widetilde{W}_m(\rho,\theta;\varepsilon)+V_m(\tau,\theta;\varepsilon) \tag{2.2.4}$$

其中$\widetilde{W}_m$表示$W_m$在局部坐标系$(\rho,\theta)$下的表达式．将(2.2.4)代入(2.2.2)式得

$$\tilde{N}_\varepsilon[\tilde{U}_m]=\tilde{N}_\varepsilon[\widetilde{W}_m]+\tilde{N}_\varepsilon[V_m]+\widetilde{F}(\varepsilon\tau,\theta,V_m)$$

$$+\widetilde{F}(\varepsilon\tau,\theta,\widetilde{W}_m)-\widetilde{F}(\varepsilon\tau,\theta,\widetilde{W}_m+V_m) \tag{2.2.5}$$

令$\varepsilon^i$，$(i=-1,0,\cdots,m)$的系数为零，得到关于$v_i$，$(i=0,1,\cdots,m+1)$的递推方程：

$$B_0[v_0]\equiv a_0(\theta)\frac{\partial^2v_0}{\partial\tau^2}-\alpha_0(\theta)\frac{\partial v_0}{\partial\tau}=0 \tag{2.2.6}$$

$$B_0[v_i]=-\sum_{p=1}^{i}B_p[v_{i-p}]+\frac{1}{(i-1)!}g_{i-1}(\tau,\theta,\tilde{w}_0,v_0,\cdots,\tilde{w}_{i-1},v_{i-1}) \tag{2.2.7}$$

$$(i=1,2,\cdots m+1)$$

其中

$$g_i=\frac{\partial^i\tilde{F}(\varepsilon\tau,\ \theta,\tilde{W}_m+V_m)}{\partial\varepsilon^i}\bigg|_{\varepsilon=0}-\frac{\partial^iF(\varepsilon\tau,\ \theta,\ \tilde{W}_m)}{\partial\varepsilon^i}\bigg|_{\varepsilon=0}$$

$$=\sum_{k+l_1+\cdots+pl_p=i}\frac{i!}{k!l_1!\cdots l_p!}\tau^k\Big[\frac{\partial^l\tilde{F}(0,\ \theta,\ \tilde{w}_0(0,\ \theta)+v_0(\tau,\ \theta))}{\partial\rho^k\partial u^{l_1+\cdots+l_p}}$$

$$\prod_{j=1}^{p}\left(\frac{\partial^j(\tilde{W}_m+V_m)}{\partial\varepsilon^j}\bigg|_{\varepsilon=0}\right)^{l_j}-\frac{\partial^l\tilde{F}(0,\ \theta,\ \tilde{w}_0(0,\ \theta))}{\partial\rho^k\partial u^{l_1+\cdots+l_p}}\prod_{j=1}^{p}\left(\frac{\partial^j\tilde{W}_m}{\partial\varepsilon^j}\bigg|_{\varepsilon=0}\right)^{l_j}\Big]$$

再给出定解条件：

$$v_0|_{\tau=0}=\varphi(\theta)-\tilde{w}_0(0,\ \theta),\quad \lim_{\tau\to\infty}v_0=0,\quad (0<\theta<\theta_B) \tag{2.2.8}$$

$$v_i|_{\tau=0}=-\tilde{w}_i(0,\ \theta),\quad \lim_{\tau\to\infty}v_i=0,\quad (0<\theta<\theta_B) \tag{2.2.9}$$

$$(i=1,\ 2,\ \cdots,\ m+1),$$

其中$\varphi(\theta)$表示$\varphi(x,\ y)$在$(\rho,\ \theta)$坐标系的表达式，$w_{m+1}\equiv 0$.

从(2.2.6)、(2.2.8)可以解得

$$v_0=(\varphi(\theta)-\tilde{w}_0(0,\ \theta))e^{-K(\theta)\tau}$$

其中 $K(\theta)\equiv-\dfrac{\alpha_0(\theta)}{a_0(\theta)}>0$. 将$v_0$再代入(2.2.7)、(2.2.9)式（取$i=1$），解得

$$v_1=\frac{1}{K(\theta)}\int_\tau^\infty f_1(t,\ \theta)dt-\left(\tilde{w}_1(0,\ \theta)+\frac{1}{K(\theta)}\int_0^\infty f_1(t,\ \theta)dt\right)e^{-K(\theta)\tau}$$

$$+\frac{1}{K(\theta)}e^{-K(\theta)\tau}\int_0^\tau f_1(t,\ \theta)e^{K(\theta)t}dt$$

其中$f_1(\tau,\ \theta)=-B_1[v_0]+\tilde{F}(0,\ \theta,\ \tilde{w}_0(0,\ \theta)+v_0(\tau,\ \theta))-\tilde{F}(0,\ \theta.\ \tilde{w}_0(0,\ \theta))$. 因对于确定的$\theta$值，$(0<\theta<\theta_B)$成立

$$|f_1(\tau,\ \theta)|\leqslant M_1(\theta)e^{\frac{-1}{2}K(\theta)\tau}$$

所以

$$|v_1|\leqslant M_2(\theta)e^{\frac{-1}{2}K(\theta)\tau}$$

其中$M_1(\theta)$、$M_2(\theta)$是$0<\eta_0\leqslant\theta\leqslant\theta_B-\eta_0$中的有界函数（$\eta_0$可以取得充分小），所以$v_1$是$D_{\rho_0}\cap(\eta_0\leqslant\theta\leqslant\theta_B-\eta_0)$中的边界层型函数，但在$\theta=0$，$\theta=\theta_B$因$\alpha_0(\theta)=0$所以$v_1$一般具有奇性．再将$v_0$、$v_1$代入(2.2.7)、(2.2.9)式(取$i=2$)可以类似地求得边界层型函数$v_2$等等，最后求得$V_m$. 为了将$V_m$延拓到整个区域可再引进无限次可微的截断函数：

$$X(t)=\begin{cases}1, & \text{当}\ t\leqslant\dfrac{1}{2}\\ 0, & \text{当}\ t\geqslant 1\end{cases}$$

定义函数：

$$U_m^{(1)}=W_m(x,\ y;\ \varepsilon)+V_m^{(1)}(x,\ y;\ \varepsilon)$$

其中

$$V_m^{(1)}=\begin{cases}X(\rho/\rho_0)V_m, & 当 0\leqslant\rho\leqslant\rho_0\\ 0, & 当(x,\ y)\in\bar{\Omega}\backslash D_{\rho_0}\end{cases}$$

有定理:

**定理1.** 在正则四边形 $G=\{(x,\ y)\,|\,x_0+\delta_0'\leqslant x\leqslant x_1-\delta_0',\ y_-(x)\leqslant y\leqslant y_+(x)\}$ 上，$\tilde{U}_m^{(1)}$ 是狄立克雷问题的形式渐近解，即成立

$$N_\varepsilon[\tilde{U}_m^{(1)}]=O(\varepsilon^{m+1}),\qquad (x,\ y)\in G;\ \ \tilde{U}_m^{(1)}|_{\partial\Omega\cap G}=\varphi|_{\partial\Omega\cap G}$$

其中 $\delta_0'$ 表示小于 $\dfrac{x_1-x_0}{2}$ 的任意正数.

证: 在区域 $G\cap D_{\frac{1}{2}\rho_0}$ 有

$$N_\varepsilon[\tilde{U}_m^{(1)}]=\tilde{N}_\varepsilon[\tilde{W}_m+V_m]=O(\varepsilon^{m+1})$$

在区域 $G\cap(\bar{\Omega}\backslash D_{\rho_0})$ 有 $N_\varepsilon[\mathrm{U}^{(1)}]=N_\varepsilon[W_m]=O(\varepsilon^{m+1})$

在区域 $G\cap(D_{\rho_0}\backslash D_{\frac{1}{2}\rho_0})$ 有

$$\begin{aligned}N_\varepsilon[U_m^{(1)}]&=\tilde{N}_\varepsilon[\tilde{W}_m+XV_m]=\tilde{N}_\varepsilon[\tilde{W}_m]+\tilde{N}_\varepsilon[XV_m]+\tilde{F}(\varepsilon\tau,\ \theta,\ XV_m)\\&\quad+\tilde{F}(\varepsilon\tau,\ \theta,\ \tilde{W}_m)-\tilde{F}(\varepsilon\tau,\ \theta,\ \tilde{W}_m+XV_m)\\&=O(\varepsilon^{m+1})+\varepsilon^{-1}(B_0+\varepsilon B_1+\cdots+\varepsilon^{m+2}\bar{B}_{m+2})[X(v_0+\varepsilon v_1+\cdots+\varepsilon^{m+1}v_{m+1})]\\&\quad-\tilde{F}_u(\varepsilon\tau,\ \theta,\ \tilde{W}_m+\mu XV_m)XV_m\end{aligned}$$

因在该区域 $XV_m=O(\varepsilon^N)$，$B_i[Xv_i]=O(\varepsilon^N)$，其中$N$是任意正整数，所以有 $N_\varepsilon[\tilde{U}_m^{(1)}]=O(\varepsilon^{m+1})$. 在边界 $\partial\Omega\cap G$ 上 $\tilde{U}_m^{(1)}$ 显然取值 $\varphi(\theta)$，定理证毕.

**3. 预备知识——$W_m$ 和 $V_m$ 在切点邻域的性质**

为了证明后面第 4 节中求得的函数在切点 $A(B)$ 具有边界层项的性质，先考察 $\tilde{W}_m$ 和 $V_m$ 在切点 $A(B)$ 邻域的性质. 先考察 $A$ 点的邻域，类似地可知在 $B$ 点邻域的性质.

在$A$点的邻域 $D_A=\{(\rho,\ \theta)\,|\,0\leqslant\rho\leqslant\rho_0,\ |\theta|\leqslant\delta_0\}$，作伸展变换

$$\xi=\frac{\rho}{\varepsilon_1^{\,2}},\qquad \eta=\frac{\theta}{\varepsilon_1}\tag{2.3.1}$$

$(\varepsilon_1=\varepsilon^{\frac{1}{3}},\delta_0$ 是适当小的正数)

在内部坐标系 $(\xi,\ \eta)$ 下，确定 $w_i$ 的柯西问题(2.1.2)-(2.1.5)具有形式

$$\hat{L}_1[\hat{w}_0]\equiv\varepsilon_1^{-2}\Big[\alpha(\varepsilon_1^2\xi,\ \varepsilon_1\eta)\frac{\partial\hat{w}_0}{\partial\xi}+\varepsilon_1\beta(\varepsilon_1^2\xi,\ \varepsilon_1\eta)\frac{\partial\hat{w}_0}{\partial\eta}=-\tilde{F}(\varepsilon_1^2\xi,\ \varepsilon_1\eta,\ \hat{w}_0)\tag{2.3.2}$$

$$\hat{w}_0|_{\xi=0}=\varphi\ (\varepsilon_1\eta),\qquad \left(-\frac{\delta_0}{\varepsilon_1}\leqslant\eta<0\right)\tag{2.3.3}$$

$$\begin{aligned}&\hat{L}_1[\hat{w}_i]+\tilde{F}_u(\varepsilon_1^2\xi,\ \varepsilon_1\eta,\ \hat{w}_0)\hat{w}_i\\&\quad=\hat{L}_2[\hat{w}_{i-1}]-\frac{1}{i!}\hat{h}_i(\varepsilon_1^2\xi,\ \varepsilon_1\eta,\ \hat{w}_0,\ \cdots,\ \hat{w}_{i-1})\end{aligned}\tag{2.3.4}$$

$$\hat{w}_i|_{\xi=0}=0,\qquad \left(-\frac{\delta_0}{\varepsilon_1}\leqslant\eta<0\right)\tag{2.3.5}$$

$(i=1,\ 2,\ \cdots,\ m)$

其中$\hat{L}_1$，$\hat{L}_2$，$\hat{w}_i$和$\hat{h}_i$分别表示$\tilde{L}_1$，$\tilde{L}_2$，$\tilde{w}_i$和$h_i$在$(\xi,\ \eta)$坐标系下的表达式．将系数和$\varphi(\varepsilon_1\eta)$关于$\varepsilon_1$展开

$$a(\varepsilon_1{}^2\xi,\ \varepsilon_1\eta)=a_0+\sum_{i=1}^{3m+2}\varepsilon_1{}^i a_i(\xi,\eta)+\varepsilon_1{}^{3m+3}\bar{a}_{3m+3}(\xi,\ \eta,\ \mu\varepsilon_1)$$

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$$\varphi(\varepsilon_1\eta)=\varphi(0)+\sum_{i=1}^{3m+1}\varepsilon_1{}^i\frac{1}{i!}\eta^i\varphi^{(i)}(0)+\varepsilon_1{}^{3m+2}\frac{1}{(3m+2)!}\eta^{3m+2}\bar{\varphi}^{(3m+2)}(\mu\varepsilon_1\eta)$$

其中$a_0=a(0,\ 0)\geqslant m_0>0$，$a_i(\xi,\ \eta)=\sum\limits_{2k+l=i}\frac{1}{(2k)!l!}\frac{\partial^{k+l}a_i(0,\ 0)}{\partial\rho^k\partial\theta^l}(2\xi)^k\eta^l\cdots$等等．为了讨论确定起见，假设$\frac{\partial\alpha(0,\ 0)}{\partial\theta}\neq0$，即边界与特征线在$A$点是一阶相切（关于高阶相切的情形可以类似地讨论）．由于在$A$点的邻域当$\theta$由负值增大到正值时$\cos(\widehat{\rho,y})$由正值减小到负值所以$\frac{\partial\alpha(0,\ 0)}{\partial\theta}<0$．记$b=-\frac{\partial\alpha(0,\ 0)}{\partial\theta}>0$，有$a_1(\xi,\eta)=-b\eta$；又$\beta_0(0)=\left.\frac{\partial\theta}{\partial y}\right|_A>0$，记$\beta_0=\beta_0(0)$．将算子$\hat{L}_1$和$\hat{L}_2$关于$\varepsilon_1$展开

$$\hat{L}_1\equiv\varepsilon_1^{-1}\left(\sum_{j=0}^{3m+1}\varepsilon_1{}^j\mathscr{L}_{1,j}+\varepsilon_1^{3m+2}\overline{\mathscr{L}}_{1,3m+2}\right)\tag{2.3.6}$$

$$\hat{L}_2\equiv\varepsilon_1^{-4}\left(\sum_{j=0}^{3m+1}\varepsilon_1{}^j\mathscr{L}_{2,j}+\varepsilon_1^{3m+2}\overline{\mathscr{L}}_{2,3m+2}\right)\tag{2.3.7}$$

其中

$$\left.\begin{aligned}&\mathscr{L}_{1,0}\equiv-b\eta\frac{\partial}{\partial\xi}+\beta_0\frac{\partial}{\partial\eta}\\&\mathscr{L}_{1,j}\equiv\alpha_{j+1}(\xi,\ \eta)\frac{\partial}{\partial\xi}+\beta_j(\xi,\ \eta)\frac{\partial}{\partial\eta}\quad(j=1,\ 2,\ \cdots,\ 3m+1)\\&\overline{\mathscr{L}}_{1,3m+2}\equiv\bar{\alpha}_{3m+3}\frac{\partial}{\partial\xi}+(\beta_{3m+2}+\varepsilon_1\bar{\beta}_{3m+3})\frac{\partial}{\partial\eta}\end{aligned}\right\}\tag{2.3.8}$$

$$\left.\begin{aligned}&\mathscr{L}_{2,0}\equiv a_0\frac{\partial^2}{\partial\xi^2}\\&\mathscr{L}_{2,j}\equiv a_j(\xi,\ \eta)\frac{\partial^2}{\partial\xi^2}+2b_{j-1}(\xi,\ \eta)\frac{\partial^2}{\partial\xi\partial\eta}+\cdots+f_{j-4}(\xi,\ \eta)\\&\qquad(j=1,\ 2,\ \cdots,\ 3m+1)\\&\overline{\mathscr{L}}_{2,3m+2}\equiv(a_{3m+2}+\varepsilon\bar{a}_{3m+3})\frac{\partial^2}{\partial\xi^2}+\cdots+(f_{3m-2}+\cdots+\varepsilon_1{}^5\bar{f}_{3m+3})\end{aligned}\right\}\tag{2.3.9}$$

先考察$\hat{w}_0$在$A$点邻域的性质．假设$\hat{w}_0$具有展开式：

$$\hat{w}_0=\sum_{j=0}^{3m+1}\varepsilon_1^j\hat{w}_{0,j}(\xi,\ \eta)+\varepsilon_1^{3m+2}r^{(0)}\tag{2.3.10}$$

代入方程(2.3.2)，令$\varepsilon_1^j$，$(j=-1,0,\cdots,3m)$的系数为零，得到关于$\hat{w}_{0,j}$的递推方程：

$$\mathscr{L}_{1,0}[\hat{w}_{0,0}]\equiv -b\eta\frac{\partial\hat{w}_{0,0}}{\partial\xi}+\beta_0\frac{\partial\hat{w}_{0,0}}{\partial\eta}=0,\quad \left(-\frac{\delta_0}{\varepsilon_1}\leqslant\eta<0\right) \tag{2.3.11}$$

$$\mathscr{L}_{1,0}[\hat{w}_{0,j}]\equiv\sum_{p=1}^{j}\mathscr{L}_{1,p}[\hat{w}_{0,j-p}]-\frac{1}{(j-1)!}\hat{h}_{0,j-1}(\xi,\eta,\hat{w}_{0,1},\cdots,w_{0,j-1}) \tag{2.3.12}$$

$$(j=1,2,\cdots,3m+1),\quad \left(-\frac{\delta_0}{\varepsilon_1}\leqslant\eta<0\right)$$

其中

$$\hat{h}_{0,j}=\frac{\partial^j\widetilde{F}(\varepsilon_1^2\xi,\varepsilon_1\eta,\hat{w}_0)}{\partial\varepsilon_1^j}\Bigg|_{\varepsilon_1=0}=\sum_{\substack{2k_1+k_2+l_1+\cdots+pl_p=j\\(p\leqslant j)}}\frac{j!}{(2k_1)!k_2!\cdots l_p!}$$

$$\frac{\partial^{k_1+k_2+\cdots+l_p}\widetilde{F}(0,0\hat{w}_{,0,0})}{\partial\rho^{k_1}\partial\theta^{k_2}\partial u^{l_1+\cdots+l_p}}(2\xi)^{k_1}\eta^{k_2}(\hat{w}_{0,1})^{l_1}\cdots(\hat{w}_{0,p})^{l_p}$$

又从(2.3.3)得到关于$\hat{w}_{0,j}$的边界条件

$$\hat{w}_{0,j}\Big|_{\xi=0}=\frac{1}{j!}\eta^j\varphi^{(j)}(0),\quad\left(-\frac{\delta_0}{\varepsilon_1}\leqslant\eta<0\right),(j=0,1,\cdots,3m+1) \tag{2.3.13}$$

作坐标变换

$$\sigma=\xi+\frac{b}{2\beta_0}\eta^2,\qquad t=\frac{\eta}{\beta_0} \tag{2.3.14}$$

算子$\mathscr{L}_{1,j}$、$\mathscr{L}_{2,j}(j=0,1,\cdots,3m+1)$变换成

$$\mathscr{L}_{1,0}\equiv\frac{\partial}{\partial t},$$

$$\mathscr{L}_{1,j}\equiv\alpha_{j+1}\frac{\partial}{\partial\sigma}+\beta_j\left(bt\frac{\partial}{\partial\sigma}+\frac{1}{\beta_0}\frac{\partial}{\partial t}\right),\quad(j=1,2,\cdots,3m+1)$$

$$\mathscr{L}_{2,0}\equiv a_0\frac{\partial^2}{\partial\sigma^2}$$

$$\mathscr{L}_{2,j}\equiv a_j\frac{\partial^2}{\partial\sigma^2}+2b_{j-1}\left(bt\frac{\partial^2}{\partial\sigma^2}+\frac{1}{\beta_0}\frac{\partial^2}{\partial\sigma\partial t}\right)+c_{j-2}\left(b^2t^2\frac{\partial^2}{\partial\sigma^2}+\frac{2bt}{\beta_0}\frac{\partial^2}{\partial\sigma\partial t}\right.$$

$$\left.+\frac{b}{\beta_0}\frac{\partial}{\partial\sigma}+\frac{1}{\beta_0^2}\frac{\partial^2}{\partial t^2}\right)+\cdots+f_{j-4},\quad(j=1,2,\cdots,3m+1)$$

边界条件(2.3.13)变换成

$$\hat{w}_{0,j}\Big|_{t=-\sqrt{2\sigma/b\beta_0}}=\frac{1}{j!}\beta_0^j(-\sqrt{2\sigma/b\beta_0})^j\varphi^{(j)}(0) \tag{2.3.15}$$

$$(j=0,1,\cdots,3m+1)$$

从关于$\hat{w}_{0,0}$的柯西问题(2.3.11)、(2.3.15)（取$j=0$）求得

$$\hat{w}_{0,0}=\varphi(0) \tag{2.3.16}$$

将$\hat{w}_{0,0}$代入关于$\hat{w}_{0,1}$的柯西问题(2.3.12)、(2.3.15)（取$j=1$）求得

$$\hat{w}_{0,1}=-\widetilde{F}(0,0,\varphi(0))(t+\sqrt{2\sigma/b\beta_0})-\beta_0\sqrt{2\sigma/b\beta_0}\varphi'(0) \tag{2.3.17}$$

是关于$\sqrt{\sigma}$和$t$的一次齐次多项式．若以$R_p$表示$\sqrt{\sigma}$和$t$的$p$次齐次多项式的集合，$R_p^{(i)}$记$R_p$中的元素$R_p^{(i)}=\sum\limits_{l+m=p} a_{l,m}^{(i)}(\sqrt{\sigma})^l t^m$，利用性质：

$$(i)\quad R_p^{(i)}\ R_q^{(i)}\in R_{p+q},\quad \frac{\partial R_p^{(i)}}{\partial\sigma}\in R_{p-2},\quad \frac{\partial R_p^{(i)}}{\partial t}\in R_{p-1}$$

$$(ii)\quad R_p^{(i)}=O\left(\sqrt{\sigma}^{\,p}\right)=O\left(\varepsilon_1^{-p}\Phi^{\frac{p}{2}}\right),\quad \Phi\equiv\rho+\frac{b}{2\beta_0}\theta^2$$

可以应用数学归纳法证知

$$\hat{w}_{0,j}=R_j^{(0)}=\sum_{l+m=j} a_{l,m}^{(0)}(\sqrt{\sigma})^p\ t^m,\ (j=0,1,\cdots,3m+1) \tag{2.3.18}$$

再从关于$r^{(0)}$的方程式和边界条件

$$\alpha(\rho,\theta)\frac{\partial r^{(0)}}{\partial\rho}+\beta(\rho,\theta)\frac{\partial r^{(0)}}{\partial\theta}=H^{(0)} \tag{2.3.19}$$

$$r^{(0)}\Big|_{\rho=0}=\varepsilon_1^{-(3m+2)}\frac{1}{(3m+2)!}\theta^{3m+2}\bar{\varphi}^{(3m+2)}(\mu\theta) \tag{2.3.20}$$

其中$H^{(0)}$在$(\xi,\eta)$坐标系的表达式是

$$\begin{aligned}H^{(0)}\equiv-\varepsilon_1^{-1}\Big[&(\mathscr{L}_{1,1}[\hat{w}_{0,3m+1}]+\cdots+\overline{\mathscr{L}}_{1,3m+2}[\hat{w}_{0,0}])+\varepsilon_1(\mathscr{L}_{1,2}[\hat{w}_{0,3m+1}]+\cdots\\&+\overline{\mathscr{L}}_{1,3m+2}[\hat{w}_{0,1}])+\cdots+\varepsilon_1^{3m+1}\ \overline{\mathscr{L}}_{1,3m+2}[\hat{w}_{0,3m+1}]\\&+\frac{1}{(3m+1)!}\ \frac{\partial^{3m+1}\widetilde{F}(\varepsilon_1^2\xi,\varepsilon_1\eta,\hat{w}_0)}{\partial\varepsilon_1{}^{3m+1}}\Big|_{\varepsilon_1=\mu\varepsilon_1}\Big]\end{aligned}$$

易知当$\Phi=O(\varepsilon_1)$时，考虑到$|y-y_-(x)|\leqslant 2|\theta|$和$\dfrac{dy_-(x)}{dx}=-\dfrac{\rho_x(x,y_-(x))}{\rho_y(x,y_-(x))}=O(|\theta|^{-1})$

可以证知
$$\frac{\partial^l r^{(0)}}{\partial x^{l_1}\partial y^{l_2}}=O\left(\varepsilon_1^{-(3m+2)}\Phi^{\frac{3m+2-2l_1-l_2}{2}}\right) \tag{2.3.21}$$

一般地，应用数学归纳法可以证知

$$\hat{w}_i=\varepsilon_1^{-3i}\left(\sum_{j=0}^{3m+1}\varepsilon_1^j\ \hat{w}_{i,j}(\xi,\ \eta)+\varepsilon_1^{3m+2}\ r^{(i)}\right),\quad (i=0,1,\cdots,m) \tag{2.3.22}$$

其中$\hat{w}_{i,0}=0$，$\hat{w}_{i,j}=R_{j-3i}^{(i)}$，$(j=1,2,\cdots,3m+1)$和

$$\frac{\partial^l r^{(i)}}{\partial x^{l_1}\partial y^{l_2}}=O\left(\varepsilon_1^{-(3m+2-3i)}\ \Phi^{\frac{3m+2-3i-2l_1-l_2}{2}}\right) \tag{2.3.23}$$

下而再考察$v_i(\tau,\theta)$，$(i=0,1,\cdots,m+1)$在$A$点邻域的性质．在$(\xi,\ \eta)$坐标系下关于$\hat{v}_i$的柯西问题(2.2.6)－(2.2.9)具有形式

$$\left.\begin{aligned}&\hat{B}_0[\hat{v}_0]\equiv a_0(\varepsilon_1\eta)\varepsilon_1^2\frac{\partial^2\hat{v}_0}{\partial\xi^2}-\alpha_0(\varepsilon_1\eta)\varepsilon_1\frac{\partial\hat{v}_0}{\partial\xi}=0\\&\hat{v}_0|_{\xi=0}=\varphi(\varepsilon_1\eta)-\hat{w}_0(0,\ \varepsilon_1\eta),\ \lim_{\xi\to\infty}\hat{v}_0=0,\ \left(0<\eta\leqslant\frac{\delta_0}{\varepsilon_1}\right)\end{aligned}\right\} \tag{2.3.24}$$

$$\left.\begin{aligned}&\hat{B}_0[\hat{v}_i]=-\sum_{p=1}^{i}\hat{B}_p[\hat{v}_{i-p}]+\frac{1}{(i-1)!}\hat{g}_{i-1}\\&\hat{v}_i|_{\xi=0}=-\hat{w}_i(0,\varepsilon_1\eta),\ \lim_{\xi\to\infty}\hat{v}_i=0,\ \left(0<\eta\leqslant\frac{\delta_0}{\varepsilon_1}\right)\end{aligned}\right\}\tag{2.3.25}$$

$$(i=1,2,\cdots,m+1)$$

其中$\hat{B}_i$、$\hat{v}_i$和$\hat{g}_{i-1}$分别表示$B_i$、$v_i$和$g_{i-1}$在$(\xi,\ \eta)$坐标系下的表达式:

$$\hat{B}_i\equiv a_i(\varepsilon_1\eta)\xi^i\varepsilon_1^{-i+2}\frac{\partial^2}{\partial\xi^2}+2b_{i-1}(\varepsilon_1\eta)\xi^{i-1}\varepsilon_1^{-i+1}\frac{\partial^2}{\partial\xi\partial\eta}+c_{i-2}(\varepsilon_1\eta)\xi^{i-2}\varepsilon_1^{-i}\frac{\partial^2}{\partial\eta^2}$$

$$+d_{i-1}(\varepsilon_1\eta)\xi^{i-1}\varepsilon_1^{-i+2}\frac{\partial}{\partial\xi}+e_{i-2}(\varepsilon_1\eta)\xi^{i-2}\varepsilon_1^{-i+1}\frac{\partial}{\partial\eta}+f_{i-2}(\varepsilon_1\eta)\xi^{i-2}\varepsilon_1^{-i+2}$$

$$-\alpha_i(\varepsilon_1\eta)\xi^i\varepsilon_1^{-i+1}\frac{\partial}{\partial\xi}-\beta_{i-1}(\varepsilon_1\eta)\xi^{i-1}\varepsilon_1^{-i}\frac{\partial}{\partial\eta},\quad(i=1,2,\cdots,m+1)$$

$$\bar{B}_{m+2}\equiv\bar{a}_{m+2}(\mu\varepsilon_1^2\xi,\ \varepsilon_1\eta)\xi^{m+2}\varepsilon_1^{-m}\frac{\partial^2}{\partial\xi^2}+\cdots+\bar{f}_{m+2}(\mu\varepsilon_1^2\xi,\varepsilon_1\eta)\xi^{m+2}\varepsilon_1^{-m-2}$$

将算子中的系数关于$\varepsilon_1$展开

$$a_i(\varepsilon_1\eta)=\sum_{j=0}^{3m+2}\varepsilon_1^j\eta^ja_{i,j}+\varepsilon_1^{3m+3}\eta^{3m+3}\bar{a}_{i,3m+3}(\mu\varepsilon_1\eta)$$

其中 $a_{i,j}=\dfrac{1}{j!}\dfrac{d^ja_i(0)}{d\varepsilon^j}$ 等等，将算子分解成

$$\hat{B}_0\equiv\varepsilon_1^2\left(\sum_{j=0}^{3m+1}\varepsilon_1^jB_{0,j}+\varepsilon_1^{3m+2}\bar{B}_{0,3m+2}\right)$$

$$\hat{B}_i\equiv\varepsilon_1^{-i}\left(\sum_{j=0}^{3m+1}\varepsilon_1^jB_{i,j}+\varepsilon_1^{3m+2}\bar{B}_{i,3m+2}\right),\quad(i=1,2,\cdots,m+1)$$

其中

$$B_{0,0}\equiv a_0\frac{\partial^2}{\partial\xi^2}+b\eta\frac{\partial}{\partial\xi}$$

$$B_{0,j}\equiv a_{0,j}\eta^j\frac{\partial^2}{\partial\xi^2}-\alpha_{0,j+1}\eta^{j+1}\frac{\partial}{\partial\xi},\quad(j=1,2,\cdots,3m+1)$$

$$\bar{B}_{0,3m+2}\equiv(a_{0,3m+2}\eta^{3m+2}+\varepsilon_1\bar{a}_{0,3m+3}(\mu\varepsilon_1\eta)\eta^{3m+3})\frac{\partial^2}{\partial\xi^2}-\bar{\alpha}_{0,3m+3}(\mu\varepsilon_1\eta)\eta^{3m+3}\frac{\partial}{\partial\xi}$$

$$B_{i,0}\equiv-\beta_{i-1,0}\xi^{i-1}\frac{\partial}{\partial\eta}+c_{i-2,0}\xi^{i-2}\frac{\partial^2}{\partial\eta^2}$$

$$B_{i,j}\equiv-\beta_{i-1,j}\xi^{i-1}\eta^j\frac{\partial}{\partial\eta}+c_{i-2,j}\xi^{i-2}\eta^j\frac{\partial^2}{\partial\eta^2}+2b_{i-1,j-1}\xi^{i-1}\eta^{j-1}\frac{\partial^2}{\partial\xi\partial\eta}$$

$$+e_{i-2,j-1}\xi^{i-2}\eta^{j-1}\frac{\partial}{\partial\eta}-\alpha_{i,j-1}\xi^i\eta^{j-1}\frac{\partial}{\partial\xi}+a_{i,j-2}\xi^i\eta^{j-2}\frac{\partial^2}{\partial\xi^2}$$

$$+d_{i-1,j-2}\xi^{i-1}\eta^{j-2}\frac{\partial}{\partial\xi}+f_{i-2,j-2}\xi^{i-2}\eta^{j-2},\quad (j=1,\ 2,\ \cdots,\ 3m+1)$$

$$\bar{B}_{i,3m+2}\equiv(a_{i,3m}\xi^i\eta^{3m}+\cdots+\varepsilon_1{}^3\bar{a}_{i,3m+3}\xi^i\eta^{3m+3})\frac{\partial^2}{\partial\xi^2}+\cdots$$

$$+(f_{i-2,3m}+\cdots+\varepsilon_1{}^3\bar{f}_{i-2,3m+3}\xi^{i-2}\eta^{3m+3})$$

假设$v_0$在$A$点的邻域具有展开式

$$\hat{v}_0=\sum_{j=0}^{3m+1}\varepsilon_1^j v_{0,j}(\xi,\ \eta)+\varepsilon_1^{3m+2}\psi^{(0)}$$

代入(2.3.24)式，比较$\varepsilon_1$的同次幂的系数，得到确定$v_{0,j}$，$(j=0,\ 1,\ \cdots,\ 3m+1)$的柯西问题

$$\left.\begin{aligned}&a_0\frac{\partial^2\hat{v}_{0,0}}{\partial\xi^2}+b\eta\frac{\partial\hat{v}_{0,0}}{\partial\xi}=0\\&\hat{v}_{0,0}|_{\xi=0}=0,\qquad \lim_{\xi\to\infty}\hat{v}_{0,0}=0,\ \left(0<\eta\leqslant\frac{\delta_0}{\varepsilon_1}\right)\end{aligned}\right\}\tag{2.3.26}$$

$$\left.\begin{aligned}&a_0\frac{\partial^2\hat{v}_{0,j}}{\partial\xi^2}+b\eta\frac{\partial v_{0,j}}{\partial\xi}=-\sum_{p=1}^{j}B_{0,p}[\hat{v}_{0,j-p}]\\&\hat{v}_{0,j}|_{\xi=0}=\frac{1}{j!}\eta^j\varphi^{(j)}(0)-\hat{w}_{0,j}(0,\ \eta),\quad \lim_{\xi\to\infty}\hat{v}_{0,j}=0,\ \left(0<\eta\leqslant\frac{\delta_0}{\varepsilon_1}\right)\\&\qquad(j=1,\ 2,\cdots,\ 3m+1)\end{aligned}\right\}\tag{2.3.27}$$

从(2.3.26)解得

$$\hat{v}_{0,0}=0$$

代入(2.3.27)式（取$j=1$），考虑到(2.3.17)式，解得

$$\hat{v}_{0,1}=e^{\frac{-b}{a_0}\xi\eta}\left[2\eta\varphi'(0)+\frac{2\eta}{\beta_0}\widetilde{F}(0,\ 0,\ \varphi(0))\right]\tag{2.3.28}$$

记$S_{p,n}$为下面形式的函数所构成的函数集合

$$S_{p,n}^{(i)}=e^{\frac{-nb}{a_0}\xi\eta}\sum_{l-m=p}b_{l,m}^{(i)}\eta^l\xi^m,\qquad (n\text{是正整数})$$

$S_{p,n}^{(i)}$ 表示$S_{p,n}$中的元素则$\hat{v}_{0,1}\in S_{1,1}$. 利用性质

(i) $S_{p_1,n_1}^{(i)}S_{p_2,n_2}^{(i)}\in S_{p_1+p_2,n_1+n_2}$，$\dfrac{\partial S_{p,n}^{(i)}}{\partial\xi}\in S_{p+1,n}$，$\dfrac{\partial S_{p,n}^{(i)}}{\partial\eta}\in S_{p-1,n}$

(ii) $S_{p,n}^{(i)}=O\left(\eta^p e^{\frac{-nb}{2a_0}\xi\eta}\right)=O\left(\varepsilon_1^{-p}\theta^p e^{\frac{-b}{2a_0}\tau\theta}\right)$

可以应用数学归纳法证知$\hat{v}_{0,j}\in S_{j,1}$，$(j=1,\ 2,\ \cdots,\ 3m+1)$.

又$\psi^{(0)}$确定于方程和边界条件：

$$a_0(\theta)\frac{\partial^2\psi^{(0)}}{\partial\tau^2}-\alpha_0(\theta)\frac{\partial\psi^{(0)}}{\partial\tau}=Q^{(0)}$$

$$\psi^{(0)}|_{\xi=0}=q^{(0)},\quad \left(0<\eta\leqslant\frac{\delta_0}{\varepsilon_1}\right)$$

其中$Q^{(0)}$、$q^{(0)}$在$(\xi, \eta)$坐标系的表达式是

$$Q^{(0)} \equiv -\varepsilon_1^2(B_{0,1}[\hat{v}_{0,3m+1}]+\cdots+\bar{B}_{0,3m+2}[\hat{v}_{0,0}])+\varepsilon_1(B_{0,2}[\hat{v}_{0,3m+1}]+\cdots+\bar{B}_{0,3m+2}[\hat{v}_{0,1}])+\cdots+\varepsilon_1^{3m+1}\bar{B}_{0,3m+2}[\hat{v}_{0,3m+1}]$$

$$q^{(0)} \equiv -\frac{1}{(3m+2)!}\eta^{3m+2}\varphi^{(3m+2)}(\mu\varepsilon_1\eta)-r^{(0)}|_{\rho=0}$$

易知

$$\psi^{(0)}=\frac{1}{K(\theta)}\int_\tau^\infty Q^{(0)}d\tau+\left(q^{(0)}-\frac{1}{K(\theta)}\int_0^\infty Q^{(0)}d\tau\right)e^{-K(\theta)\tau}+\frac{e^{-K(\theta)\tau}}{K(\theta)}\int_0^\tau Q^{(0)}e^{K(\theta)\tau}d\tau$$

其中 $K(\theta)=\dfrac{-a_0(\theta)}{a_0(\theta)}$. 考虑到当$0<\theta<\dfrac{b}{2m}$时$\dfrac{1}{K(\theta)}\leqslant\dfrac{2M}{b\theta}$，其中$m=\max\left|\dfrac{d^2a_0(\theta)}{d\theta^2}\right|$，$M=\max|a_0(\theta)|$，可以证知当 $\theta=O(\varepsilon_1)$时

$$\frac{\partial^l\psi^{(0)}}{\partial\tau^{l_1}\partial\theta^{l_2}}=O(\varepsilon_1^{-(3m+2)}\theta^{3m+2-l_2}e^{\frac{-b}{2a_0}\tau\theta})$$

一般地，在确定$\hat{v}_i$的柯西问题(2.3.25)中，将$\hat{g}_{i-1}$关于$\varepsilon_1$展开，可以应用数学归纳法证知

$$\hat{v}_i=\varepsilon_1^{-3i}\left(\sum_{j=0}^{3m+1}\varepsilon_1^j\hat{v}_{0,j}(\xi, \eta)+\varepsilon_1^{3m+2}\psi^{(i)}\right), \quad (i=1, 2, \cdots, m+1)$$

其中$\hat{v}_{i,0}=0$，和

$$\hat{v}_{i,j}=\sum_{p=1}^{i-1}\left(S^{(i)}_{j-3i,p}+\cdots+S^{(i)}_{j-3i-2r+2,p}\right)$$

$(j=1, 2, \cdots, 3m+1;\ r=\left[\dfrac{j+2}{2}\right]$当$j\leqslant 2i-1$，$[\cdots]$表示取整数部分；$r=i$ 当 $j>2i-1)$

$$\frac{\partial^l\psi^{(i)}}{\partial\tau^{l_1}\partial\theta^{l_2}}=O(\varepsilon_1^{-(3m+2-3i)}\theta^{3m+2-3i-l_2}e^{\frac{-b}{2a_0}\tau\theta})$$

## 4. 抛物型边界层（Parobolic boundary layer）

先构造$A$点邻域的边界层项. 引进内部坐标系$(\xi, \eta)$，将算子$\hat{N}_\varepsilon$中的系数关于$\varepsilon_1$展开，将$\hat{N}_\varepsilon$分解成

$$\hat{N}_\varepsilon[u]\equiv\varepsilon_1^{-1}\left(\sum_{i=0}^{3m+1}\varepsilon^iE_i+\varepsilon_1^{3m+2}\bar{E}_{3m+2}\right)[u]-\tilde{F}(\varepsilon_1^2\xi, \varepsilon_1\eta, u) \tag{2.4.1}$$

其中

$$E_0\equiv a_0\frac{\partial^2}{\partial\xi^2}+b\eta\frac{\partial}{\partial\xi}-\beta_0\frac{\partial}{\partial\eta}$$

$$E_i\equiv a_i(\xi, \eta)\frac{\partial^2}{\partial\xi^2}+2b_{i-1}(\xi, \eta)\frac{\partial^2}{\partial\xi\partial\eta}+c_{i-2}(\xi, \eta)\frac{\partial^2}{\partial\eta^2}+d_{i-2}(\xi, \eta)\frac{\partial}{\partial\xi}$$

$$+e_{i-3}(\xi,\ \eta)\frac{\partial}{\partial\eta}+f_{i-4}(\xi,\ \eta)-\alpha_{i+1}(\xi,\ \eta)\frac{\partial}{\partial\xi}-\beta_i(\xi,\ \eta)\frac{\partial}{\partial\eta}$$

$$(i=1,\ 2,\ \cdots,\ 3m+1)$$

$$\bar{E}_{3m+2}\equiv(a_{3m+2}+\varepsilon_1\bar{a}_{3m+3})\frac{\partial^2}{\partial\xi^2}+2(b_{3m+1}+\varepsilon_1 b_{3m+2}+\varepsilon_1^2\bar{b}_{3m+3})\frac{\partial^2}{\partial\xi\partial\eta}$$

$$+\cdots+(f_{3m-2}+\cdots+\varepsilon_1^5\bar{f}_{3m+3})-\bar{\alpha}_{3m+3}\frac{\partial}{\partial\xi}$$

$$-(\beta_{3m+2}+\varepsilon_1\bar{\beta}_{3m+3})\frac{\partial}{\partial\eta}$$

假设狄立克雷问题(2.0.1)—(2.0.2)的解在$A$点邻域的展开式是

$$Y_{3m}^{(A)}(\xi,\eta;\ \varepsilon_1)=\sum_{i=0}^{3m+1}\varepsilon_1^i y_i(\xi,\ \eta) \tag{2.4.2}$$

代入(2.4.1)式，令$\varepsilon_1^i$，$(i=-1,\ 0,\ \cdots,\ 3m)$的系数为零，得到关于$y_i$，$(i=0,1,\ \cdots,\ 3m+1)$的递推方程：

$$E_0[y_0]\equiv a_0\frac{\partial^2 y_0}{\partial\xi^2}+b\eta\frac{\partial y_0}{\partial\xi}-\beta_0\frac{\partial y_0}{\partial\eta}=0 \tag{2.4.3}$$

$$E_0[y_i]=-\sum_{p=1}^{i}E_p[y_{i-p}]+\frac{1}{(i-1)!}H_{i-1}(\xi,\ \eta,\ y_0,\ \cdots,\ y_{i-1}) \tag{2.4.4}$$

$$(i=1,\ 2,\ \cdots,\ 3m+1)$$

其中

$$H_{i-1}=\frac{\partial^{i-1}\widetilde{F}(\varepsilon_1^2\xi,\ \varepsilon_1\eta,\ Y_{3m})}{\partial\varepsilon_1^{i-1}}\bigg|_{\varepsilon_1=0}=\sum_{\substack{2k_1+k_2+l_1+\cdots+pl_p=i-1\\(p\leqslant i-1)}}\frac{(i-1)!}{(2k_1)!k_2!\cdots l_p!}$$

$$\cdot\frac{\partial^l\widetilde{F}(0,\ 0,\ y_0)}{\partial\rho^{k_1}\partial\theta^{k_2}\partial u^{l_1+\cdots+l_p}}(2\xi)^{k_1}\eta^{k_2}y_1^{l_1}\cdots y_p^{l_p}$$

再给出定解条件

$$y_i|_{\xi=0}=\frac{1}{i!}\ \eta^i\varphi^{(i)}(0),\qquad |\eta|\leqslant\frac{\delta_0}{\varepsilon_1}, \tag{2.4.5}$$

$$(i=0,\ 1,\ \cdots,\ 3m+1)$$

从(2.4.3)、(2.4.5)（取$i=0$）求得

$$y_0=\varphi(0) \tag{2.4.6}$$

代入(2.4.4)、(2.4.5)（取$i=1$）可以得到确定$y_1$的柯西问题．由于尚无法求解，下面我们应用文献[7]的方法作出其展开式，和研究解在$A$点邻域的性质．

以$D$表示区域$D=\{(\xi,\ \eta)\,|\,\xi\geqslant 0,\ \eta\leqslant -1\}$，以$K_m(D)$表示在$D$是无限次可微和成立估计式

$$\frac{\partial^l v}{\partial\xi^{l_1}\partial\eta^{l_2}}=O\left[\left(\frac{2\beta_0}{b}\xi+\eta^2\right)^{\frac{-m+l_2+2l_1}{2}}\right] \tag{2.4.7}$$

的函数集合，有下面的引理

**引理1** ([ 7 ]的引理 6). 设$F(\xi, \eta)\in K_r(D)$,$r$是正整数, 则存在$v(\xi, \eta)\in K_{r-1}(D)$使成立

$$E_0[v]=F(\xi, \eta) \qquad v|_{\xi=0}=0 \tag{2.4.8}$$

**定理2**. 递推方程(2.4.3)、(2.4.4)当$\xi>0$时, 分别存在无限次可微的解$y_i$, $(i=1,2,\cdots,3m+1)$满足边界条件(2.4.5), 并且当$\eta\leqslant-1$时具有展开式:

$$y_i=\sum_{j=0}^{N} y_{i,j}(\sigma, t)+\beta_N^{(i)} \tag{2.4.9}$$

其中$y_{i,j}\in R_{i-3j}$, $(j=0,1,\cdots,N)$, $\beta_N^{(i)}\in K_{3N+2-6(i-1)}$, $N$是正整数.

证: 先引进$(\sigma,t)$坐标系将柯西问题(2.4.4)—(2.4.5)改写成

$$(\mathscr{L}_{2,0}-\mathscr{L}_{1,0})[y_i]=-\sum_{p=1}^{i}(\mathscr{L}_{2,p}-\mathscr{L}_{1,p})[y_{i-p}]+\frac{1}{(i-1)!}H_{i-1} \tag{2.4.10}$$

$$y_i\Big|_{t=-\sqrt{2\sigma/b\beta_0}}=\frac{1}{i!}\beta_0^i(-\sqrt{2\sigma/b\beta_0})^i\varphi^{(i)}(0) \tag{2.4.11}$$

$$(i=1,2,\cdots,N)$$

再应用数学归纳法来证明定理. 假设$y_1$当$\eta\leqslant-1$时具有展开式:

$$y_1=\sum_{j=0}^{N} y_{1,j}(\sigma, t)+\beta_N^{(1)} \tag{2.4.12}$$

代入(2.4.10)、(2.4.11) (取$i=1$), 并取$y_{1,j}$, $(j=1,2,\cdots,N)$是下面柯西问题的解

$$\mathscr{L}_{1,0}[y_{1,0}]\equiv\frac{\partial y_{1,0}}{\partial t}=-\widetilde{F}(0,0,\varphi(0)), \quad y_{1,0}\Big|_{t=-\sqrt{2\sigma/b\beta_0}}=-\sqrt{2\sigma/b\beta_0}\varphi'(0)$$

$$\mathscr{L}_{1,0}[y_{1,j}]=\mathscr{L}_{2,0}[y_{1,j-1}], \quad y_{1,j}\Big|_{t=-\sqrt{2\sigma/b\beta_0}}=0,$$

$$(j=1,2,\cdots,N)$$

得到关于$\beta_N^{(1)}$的柯西问题:

$$a_0\frac{\partial^2\beta_N^{(1)}}{\partial\xi^2}+b\eta\frac{\partial\beta_N^{(1)}}{\partial\xi}-\beta_0\frac{\partial\beta_N^{(1)}}{\partial\eta}=-\mathscr{L}_{2,0}[y_{1,N}], \quad \beta_N^{(1)}\Big|_{\xi=0}=0$$

因$\mathscr{L}_{2,0}[y_{1,N}]\in K_{3N+3}$, 所以由引理1知$\beta_N^{(1)}\in K_{3N+2}$. 又当$\eta>-1$时, 则$y_1$由抛物型方程(2.4.4)和边界条件:

$$y_1|_{\xi=0}=\eta\varphi'(0), \quad y_1|_{\eta=-1}=\left(\sum_{j=0}^{N} y_{1,j}+\beta_N^{(1)}\right)\Big|_{\eta=-1}$$

确定. 定理对于$y_1$成立. 假设定理对于$i=1,2,\cdots,k-1$成立, 可以类似地证明定理对于$i=k$也成立. 定理证毕.

[注]. 当$\eta$为有限值,而$\xi\to\infty$时, 展开式(2.4.9)仍然成立.

下面再考察$y_i$, $(i=1,2,\cdots,3m+1)$当$\eta\to\infty$时的性质.

**引理2** ([5]的引理7). 设无限次可微函数$v(\xi, \eta)$在区域$G_1=\{(\xi,\eta)|\xi>0, \eta>1\}$满足方程

$$E_0[v]\equiv a_0\frac{\partial^2 v}{\partial\xi^2}+b\eta\frac{\partial v}{\partial\xi}-\beta_0\frac{\partial v}{\partial\eta}=\Psi(\xi,\eta)$$

其中 $\Psi=O\Big[\Big(\frac{2\beta_0}{b}\xi+\eta^2\Big)^{\frac{-m}{2}}+\eta^{-1}e^{-\gamma\xi\eta}\Big]$, $0<\gamma<1$, $m$ 和 $l$ 是满足不等式 $m\leqslant l+3$ 的充分大的正数，若

$$v|_{\xi=0}=O(\eta^{1-m}),\quad v|_{\eta=1}=O\Big[\Big(\frac{2\beta_0}{b}\xi+\eta^2\Big)^{\frac{1-m}{2}}\Big]$$

则在 $G_1$ 成立

$$v=O\Big[\Big(\frac{2\beta_0}{b}\xi+\eta^2\Big)^{\frac{1-m}{2}}\Big] \tag{2.4.13}$$

**定理3.** 柯西问题(2.4.4)-(2.4.5)的解 $y_i$，$(i=1,2,\cdots,3m+1)$ 在区域 $G_1$ 具有展开式

$$y_i=\sum_{j=0}^{N}y_{i,j}(\sigma,t)+\sum_{j=0}^{N}\tilde{y}_{i,j}(\xi,\eta)+\beta_N^{(i)} \tag{2.4.14}$$

其中 $y_{i,j}=\hat{\omega}_{j,i}$, $(j=0,1,\cdots,m)$, $y_{i,j}\in R_{i-3j}$, $(j=m+1,\cdots,N)$; $\tilde{y}_{i,j}=v_{j,i}$, $(j=0,1,\cdots,m+1)$,

$\tilde{y}_{i,j}\in\sum_{p=1}^{i-1}S_{i-3j,p}$, $(j=m+2,\cdots,N)$; 和

$$\frac{\partial^l\beta_N^{(i)}}{\partial\xi^{l_1}\partial\eta^{l_2}}=O\Big[\Big(\frac{2\beta_0}{b}\xi+\eta^2\Big)^{\frac{-3(N+1)+1+6(i-1)+p_0}{2}}\Big] \tag{2.4.15}$$

$p_0$ 是只与 $l_1$ 和 $l_2$ 有关的常数.

证：应用数学归纳法来证明．假设 $y_1$ 具有展开式

$$y_1=\sum_{j=0}^{N}y_{1,j}(\sigma,t)+\sum_{j=0}^{N}\tilde{y}_{1,j}(\xi,\eta)+\beta_N^{(1)}$$

代入(2.4.4)式（取 $i=1$），考虑到

$$E_i\equiv\mathscr{L}_{2,i}-\mathscr{L}_{1,i}\equiv B_{0,i}+\sum_{p=0}^{[\frac{i}{2}]}B_{p+1,i-2p},\ (i=0,1,\cdots,3m+1)$$

有

$$(\mathscr{L}_{2,0}-\mathscr{L}_{1,0})\Big[\sum_{j=0}^{N}y_{1,j}\Big]+(B_{0,0}-B_{1,0})\Big[\sum_{j=0}^{N}\tilde{y}_{1,j}\Big]+E_0[\beta_N^{(1)}]=\tilde{F}(0,0,\varphi(0))$$

取 $y_{1,j}$, $(j=0,1,\cdots,N)$ 分别是下面柯西问题的解

$$\left.\begin{aligned}&-\mathscr{L}_{1,0}[y_{1,0}]\equiv-\frac{\partial y_{1,0}}{\partial t}=\tilde{F}(0,0,\varphi(0))\\&y_{1,0}\Big|_{t=\sqrt{2\sigma/b\beta_0}}=\hat{\omega}_{0,1}\Big|_{t=\sqrt{2\sigma/b\beta_0}}\end{aligned}\right\}$$

$$\left.\begin{aligned}&-\mathscr{L}_{1,0}[y_{1,j}]=-\mathscr{L}_{2,0}[y_{1,j-1}]\\&y_{1,j}\Big|_{t=\sqrt{2\sigma/b\beta_0}}=\hat{\omega}_{j,1}\Big|_{t=\sqrt{2\sigma/b\beta_0}}\end{aligned}\right\},\quad(j=0,1,\cdots,N)$$

其中令 $\hat{\omega}_{j,1}\equiv0$，当 $j>m$．因 $y_{1,j}$ 与 $\hat{\omega}_{j,1}$, $(j=0,1,\cdots,m)$ 确定于同一柯西问题，所以 $y_{1,j}=\hat{\omega}_{j,1}$;

当$j>m$时显然$y_{1,j}\in R_{1-3j}$. 再取$\tilde{y}_{1,j}$,$(j=0,1,\cdots,N)$分别是下面柯西问题的解

$$\left.\begin{aligned}&B_{0,0}[\tilde{y}_{1,0}]\equiv a_0\frac{\partial\tilde{y}_{1,0}}{\partial\xi^2}+b\eta\frac{\partial\tilde{y}_{1,0}}{\partial\xi}=0\\&\tilde{y}_{1,0}\Big|_{\xi=0}=\eta\varphi'(0)-\hat{w}_{0,1}(0,\eta)\end{aligned}\right\}$$

$$\left.\begin{aligned}&B_{0,0}[\tilde{y}_{1,j}]=-B_{1,0}[\tilde{y}_{1,j-1}]\\&\tilde{y}_{1,j}\Big|_{\xi=0}=-\hat{w}_{j,1}(0,\eta)\end{aligned}\right\}(j=0,1,\cdots,N)$$

因$\tilde{y}_{1,j}$与$v_{j,1}$, $(j=0,1,\cdots,m)$确定于同一柯西问题, 所以$\tilde{y}_{1,j}=v_{j,1}$; 当$j>m$时显然

$$\tilde{y}_{1,j}\in\sum_{p=1}^{j-1}S_{1-3j,p}$$

这时$\beta_N^{(1)}$确定于柯西问题:

$$E_0[\beta_N^{(1)}]=-\mathscr{L}_{2,0}[y_{1,N}]-B_{1,0}[\tilde{y}_{1,N}]\equiv\Psi^{(1)}(\xi,\eta)$$

$$\beta_N^{(1)}\Big|_{\xi=0}=0,\ \beta_N^{(1)}\Big|_{\eta=1}=\Big(y_1-\sum_{j=0}^{N}y_{1,j}-\sum_{j=0}^{N}\tilde{y}_{1,j}\Big)\Big|_{\eta=1}\equiv h^{(1)}(\xi)$$

因 $\Psi^{(1)}=O\Big[\Big(\frac{2\beta_0}{b}\xi+\eta^2\Big)^{\frac{-3N-3}{2}}+\eta^{-3N}e^{-\gamma\eta\xi}\Big]$,其中$\gamma=1$,当$\frac{b}{2a_0}$是正整数;$\gamma=\frac{b}{2a_0}-\Big[\frac{b}{2a_0}\Big]$当$\frac{b}{2a_0}$不是正整数; 又$h^{(1)}=O\Big[\Big(\frac{2\beta_0}{b}\xi+\eta^2\Big)^{\frac{-3N-2}{2}}\Big]$所以由引理 2 知

$$\beta_N^{(1)}=O\Big[\Big(\frac{2\beta_0}{b}\xi+\eta^2\Big)^{\frac{-3(N+1)+1}{2}}\Big]$$

再根据抛物型方程解的先验估计式知([5]的引理8)

$$\frac{\partial^l\beta_N^{(1)}}{\partial\xi^{l_1}\partial\eta^{l_2}}=O\Big[\Big(\frac{2\beta_0}{b}\xi+\eta^2\Big)^{\frac{-3(N+1)+1+p_0}{2}}\Big]$$

$p_0$是只与$l_1$和$l_2$有关的常数. 所以定理当$i=1$时成立. 假设定理当$i<k$成立时, 可以类似地证明当$i=k$时也成立, 定理证毕.

## 5. 形式渐近解

将切点$A$的邻域$D_A$区分成三部分: $D_{A,1}=\{(\rho,\theta)\,|\,0\leqslant\rho\leqslant\varepsilon_1,\ |\theta|\leqslant\varepsilon_1^{\frac{1}{2}}\}$,$D_{A,2}=\Big\{(\rho,\theta)\,|\,0\leqslant\rho\leqslant\frac{1}{2}\varepsilon_1,\ |\theta|\leqslant\frac{1}{2}\varepsilon_1^{\frac{1}{2}}\Big\}$, $D_{A,3}=D_{A,1}\backslash D_{A,2}$. 类似地将切点$B$的邻域$D_B$也区分成三部分:

$D_{B,1}=\Big\{(\rho,\theta)\,|\,0\leqslant\rho\leqslant\varepsilon_1\,|\theta-\theta_B|\leqslant\varepsilon_1^{\frac{1}{2}}\Big\}$, $D_{B,2}=\Big\{(\rho,\theta)\,|\,0\leqslant\rho\leqslant\frac{1}{2}\varepsilon_1,\ |\theta-\theta_B|\leqslant\frac{1}{2}\varepsilon_1^{\frac{1}{2}}\Big\}$, $D_{B,3}=D_{B,1}\backslash D_{B,2}$.

定义函数

$$\begin{aligned}U_m=&W_m(x,y;\varepsilon)[1-\Lambda_A(x,y)][1-\Lambda_B(x,y)]\\&+V_m(x,\eta;\varepsilon)[1-\Lambda_A(x,y)][1-\Lambda_B(x,y)]\\&+Y_{3m}^{(A)}(\xi,\eta;\varepsilon_1)\Lambda_A(x,y)+Y_{3m}^{(B)}(\xi,\eta;\varepsilon_1)\Lambda_B(x,y)\end{aligned}\qquad(7.5.1)$$

其中

$$\Lambda_A=\begin{cases} X(\varepsilon_1\xi)X(\varepsilon_1^{\frac{1}{2}}\eta)X(-\varepsilon_1^{\frac{1}{2}}\eta), & \text{当}(x,y)\in D_A \\ 0, \text{当}(x,y)\in\bar{\Omega}\backslash D_A \end{cases}$$

$$\Lambda_B=\begin{cases} X(\varepsilon_1\xi)X(\varepsilon_1^{\frac{1}{2}}\eta')X(-\varepsilon_1^{\frac{1}{2}}\eta'), & \text{当}(x,y)\in D_B \\ 0, \text{当}(x,y)\in\bar{\Omega}\backslash D_B \end{cases}$$

$$\widetilde{\Lambda}_A=\begin{cases} X(\varepsilon_1^{\frac{1}{2}}\eta), & \text{当}(x,y)\in D_A \\ 0, \text{当}(x,y)\in\bar{\Omega}\backslash D_A \end{cases}$$

$$\widetilde{\Lambda}_B=\begin{cases} X(\varepsilon_1^{\frac{1}{2}}\eta'), & \text{当}(x,y)\in D_B \\ 0, \text{当}(x,y)\in\bar{\Omega}\backslash D_B \end{cases}$$

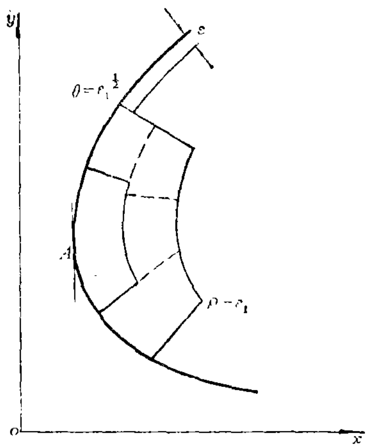


图3 切点A的邻域的边界层.

$X(t)$ 由 (2.2.10) 式定义；$\eta'=\dfrac{\theta_B-\theta}{\varepsilon_1}$；$W_m$、$V_m$和$Y_{3m}^{(A)}$ 分别由(2.1.1)、(2.2.3) 和 (2.4.2)式定义；$Y_{3m}^{(B)}$是解在切点$B$邻域的展开式．有定理

**定理4** $U_m$是狄立克雷问题(2.0.1)-(2.0.2)的形式渐近解，即成立

$$N_\varepsilon[U_m]=O(\varepsilon_1^{\frac{3m-2}{2}}),\quad U_m\Big|_{\partial\Omega_\varepsilon}=\varphi(\theta)-\varepsilon_1^{3m+2}\varphi_m \tag{2.5.2}$$

其中$\varphi_m=\dfrac{1}{(3m+2)!}\eta^{3m+2}\varphi^{(3m+2)}(\mu\varepsilon_1\eta)\Lambda_A+\dfrac{1}{(3m+2)!}\eta'^{3m+2}\varphi^{(3m+2)}(\mu\varepsilon_1\eta')\Lambda_B$

证：在区域$D_{A,2}$，因$0<\xi\leqslant\frac{1}{2}\varepsilon_1^{-1}$，$|\eta|\leqslant\frac{1}{2}\varepsilon_1^{\frac{-1}{2}}$，有

$$N_\varepsilon[U_m]=\hat{N}_\varepsilon[Y_{3m}^{(A)}]=\varepsilon_1^{3m+1}[(E_1[y_{3m+1}]+\cdots+\bar{E}_{3m+2}[y_0])+\cdots$$
$$+\varepsilon_1^{3m+1}\bar{E}_{3m+2}[y_{3m+1}]]-\varepsilon_1^{3m+1}\frac{1}{(3m+1)!}\frac{\partial^{3m+1}\widehat{F}(\varepsilon_1^2\xi,\varepsilon_1\eta,Y_{3m}^{(A)})}{\partial\varepsilon_1^{3m+1}}\Big|_{\varepsilon_1=\mu\varepsilon_1}$$

当$\eta\leqslant-1$时，利用展开式(2.4.9) 和关系式$\sigma=\xi+\dfrac{b}{2\beta_0}\eta^2=O(\varepsilon_1^{-1})$，并取$N=5m+p_0$可以证知$N_\varepsilon[U_m]=O(\varepsilon_1^{\frac{3m-2}{2}})$；当 $\eta\geqslant1$时，利用展开式(2.4.14)可以证知 $N_\varepsilon[U_m]=O(\varepsilon_1^{\frac{3m-2}{2}})$；当$-1\leqslant\eta\leqslant1$时，利用抛物型方程解的极值原理和导数估计方法(例如[6]中 § 1— § 2的方法)可以证知$N_\varepsilon[U_m]=O(\varepsilon_1^{\frac{3m-2}{2}})$．

在区域 $D_A\backslash D_{A,1}$，可以分别在五个子区域：$D_{A,1}^{(1)}=\left\{(\xi,\eta)\,|\,0<\xi\leqslant\frac{1}{2}\varepsilon_1^{-1},\ \varepsilon_1^{\frac{-1}{2}}<\eta\leqslant\delta_0\varepsilon_1^{-1}\right\}$，$D_{A,1}^{(2)}=\left\{(\xi,\eta)\,|\,0<\xi\leqslant\frac{1}{2}\varepsilon_1^{-1},\ -\delta_0\varepsilon_1^{-1}\leqslant\eta<-\varepsilon_1^{\frac{-1}{2}}\right\}$，$D_{A,1}^{(3)}=\left\{(\xi,\eta)\,|\,\frac{1}{2}\varepsilon_1^{-1}<\xi\leqslant\rho_0\varepsilon_1^{-2},\ \varepsilon_1^{\frac{-1}{2}}<\eta\leqslant\delta_0\varepsilon_1^{-1}\right\}$，$D_{A,1}^{(4)}=\left\{(\xi,\eta)\,|\,\frac{1}{2}\varepsilon_1^{-1}<\xi\leqslant\rho_0\varepsilon_1^{-2},\ -\delta_0\varepsilon_1^{-1}\leqslant\eta\leqslant-\varepsilon_1^{\frac{-1}{2}}\right\}$，$D_{A,1}^{(5)}=\left\{(\xi,\ \eta)\,|\,\varepsilon_1^{-1}<\xi\leqslant\rho_0\varepsilon_1^{-2},\ |\eta|\leqslant\varepsilon_1^{\frac{-1}{2}}\right\}$类似地证明 $N_\varepsilon[U_m]=O(\varepsilon_1^{\frac{3m-2}{2}})$，

在区域$D_{A,1}\setminus D_{A,2}$，可以分别在四个子区域：$D_{A,2}^{(1)}=\left\{(\xi,\eta)\mid 0<\xi\leqslant\frac{1}{2}\varepsilon_1^{-1},\ \frac{1}{2}\varepsilon_1^{\frac{-1}{2}}\leqslant\eta<\varepsilon_1^{\frac{-1}{2}}\right\}$、$D_{A,2}^{(2)}=\left\{(\xi,\eta)\mid\frac{1}{2}\varepsilon_1^{-1}<\xi\leqslant\varepsilon_1^{-1},\ 1\leqslant\eta\leqslant\varepsilon_1^{\frac{-1}{2}}\right\}$，$D_{A,2}^{(3)}=\left\{(\xi,\eta)\mid\frac{1}{2}\varepsilon_1^{-1}<\xi\leqslant\varepsilon_1^{-1},\ \frac{-1}{2}\varepsilon_1^{\frac{-1}{2}}\leqslant\eta<1\right\}$、$D_{A,2}^{(4)}=\left\{(\xi,\eta)\mid 0<\xi\leqslant\varepsilon_1^{-1},\ -\varepsilon_1^{\frac{-1}{2}}\leqslant\eta<\frac{-1}{2}\varepsilon_1^{\frac{-1}{2}}\right\}$类似地证明$N_\varepsilon[U_m]=O(\varepsilon_1^{\frac{3m-2}{2}})$.

在$B$点的邻域可以类似地证明$N_\varepsilon[U_m]=O(\varepsilon_1^{\frac{3m-2}{2}})$.

在边界$\partial\Omega$上显然满足(2.5.2)中的边界条件，再考虑到定理1，立刻得到所述定理.

## 三、余　项　估　计

再补充假设：

($H5$). 在区域$\Omega$上成立$F_u(x,y,U_m)\geqslant n_0>0$，其中$U_m$是由(2.5.1)式给出的形式渐近解，$n_0$是常数.

以$L_\varepsilon$表示对应于$N_\varepsilon$的线性化微分算子：

$$L_\varepsilon\equiv\varepsilon L_2-L_1-F_u(x,y,U_m)\tag{3.1}$$

以$Z_m$表示$U_m$的余项：$Z_m\equiv u_\varepsilon-U_m$，有

$$L_\varepsilon[Z_m]=\varepsilon_1^{\frac{3m-2}{2}}\Phi_m(x,y,\varepsilon_1)+F(x,y,u_\varepsilon)-F(x,y,U_m)-F_u(x,y,U_m)Z_m\tag{3.2}$$

$$Z_m|_{\partial\Omega}=\Psi_m\tag{3.3}$$

其中$\Phi_m=O(1)$，$\Psi_m=\varepsilon_1^{3m+2}\varphi_m$.

令$\tilde{Z}_m=Z_m-\Psi_m$，得到关于$\tilde{Z}_m$的边值问题：

$$L_\varepsilon[\tilde{Z}_m]=\varepsilon_1^{\frac{3m-2}{2}}\Phi_m^*(x,y,\varepsilon_1)+F(x,y,U_m+\tilde{Z}_m+\Psi_m)-F(x,y,U_m)$$

$$-F_u(x,y,U_m)(\tilde{Z}_m+\Psi_m)\tag{3.4}$$

$$\tilde{Z}_m|_{\partial\Omega}=0\tag{3.5}$$

其中$\Phi_m^*=O(1)$.

以$C_{2,\alpha}(\bar{\Omega})$，$(0<\alpha<1)$表示在$\bar{\Omega}$是二次连续可微，并以

$$|u|_{2,\alpha}=\sum_{k=0}^{2}\sum_{|i|=k}\operatorname*{Sup}_{\Omega}|D^{(i)}u|+\sum_{|i|=2}|D^{(i)}u|_{(\alpha),\Omega}$$

为范数的函数空间，其中$i=(i_1,\ i_2)$，$D^{(i)}u$表示$u$对$x$、$i_1$阶对$y$、$i_2$阶的偏导数，$|i|=i_1+i_2$，$|u|_{(\alpha),\Omega}=\operatorname*{Sup}_{P_1,P_2\in\bar{\Omega}}\frac{|u(P_1)-u(P_2)|}{|P_1-P_2|^\alpha}$. 在假设($H4$)下成立[7]

**引理3.** 考察狄立克雷问题：

$$L_\varepsilon[w]=f(x,y),\qquad w|_{\partial\Omega}=0\tag{3.6}$$

若$\partial\Omega$是属于$C_{2,\alpha}$，$f(x,y)\in C_{0,\alpha}(\bar{\Omega})$，则存在唯一的解$w\in C_{2,\alpha}(\bar{\Omega})$，和成立估计式

$$|w|_{2,\alpha}\leqslant\varepsilon^{-1}K|f|_{0,\alpha} \tag{3.7}$$

其中$K$是与$\varepsilon$无关的常数.

以$C_{2,\alpha}^{(0)}(\bar{\Omega})$表示$C_{2,\alpha}(\bar{\Omega})$中在$\partial\Omega$取零值的函数所构成的子空间. 在$C_{2,\alpha}^{(0)}(\bar{\Omega})$中定义算子方程:

$$u=T_{\varepsilon}[u] \tag{3.8}$$

其中

$$T_{\varepsilon}[u]\equiv L_{\varepsilon}^{-1}[\varepsilon_1^{\frac{3m-2}{2}}\Phi_m^*+F(x,y,U_m+u+\Psi_m)-F(x,y,U_m)$$
$$-F_u(x,y,U_m)(u+\Psi_m)]$$

以$S_m$表示$C_{2,\alpha}^{(0)}(\bar{\Omega})$中的球$S_m=\{u\in C_{2,\alpha}^{(0)}(\bar{\Omega})\Big|\,|u|_{2,\alpha}\leqslant\varepsilon_1^{\frac{3m-9}{2}}\}$，有

**引理4.** 若$u\in S_m$，则$T_{\varepsilon}[u]\in S_m$

证:因$F(x,y,U_m+u+\Psi_m)-F(x,y,U_m)-F_u(x,y,U_m)(u+\Psi_m)=F_{uu}(x,y,U_m+\mu_2\mu_1(u+\Psi_m))\mu_1(u+\Psi_m)^2$，$(0\leqslant\mu_1,\ \mu_2\leqslant1)$，从引理3有

$$|T_{\varepsilon}[u]|_{2,\alpha}\leqslant K_2\varepsilon_1^{\frac{1}{2}}\varepsilon_1^{\frac{3m-9}{2}}$$

$K_2$是与$\varepsilon_1$无关的常数，所以当$\varepsilon_1$充分小而使$K_2\varepsilon_1^{\frac{1}{2}}\leqslant1$时，则$T_{\varepsilon}[u]\in S_m$，证毕.

**引理5.** 若$u_1\in S_m,u_2\in S_m$，其中$m\geqslant6$，则成立$|T_{\varepsilon}[u_1]-T_{\varepsilon}[u_2]|_{2,\alpha}\leqslant r_0|u_1-u_2|_{2,\alpha}$，$r_0$是小于1的正数.

证：因$F(x,y,U_m+u_1+\Psi_m)-F_u(x,y,U_m)(u_1+\Psi_m)-F(x,y,U_m+u_2+\Psi_m)+F_u(x,y,U_m)(u_2+\Psi_m)=F_{uu}(x,y,U_m+\mu_2\tilde{u})\tilde{u}(u_1-u_2)$，其中$\tilde{u}=\Psi_m+u_2+\mu_1(u_1-u_2)$，$(0\leqslant\mu_1,\ \mu_2\leqslant1)$，从前面的引理可以证知

$$|T_{\varepsilon}[u_1]-T_{\varepsilon}[u_2]|_{2,\alpha}\leqslant\varepsilon^{-1}K_1K_2\varepsilon_1^{\frac{3m-9}{2}}$$

当$m\geqslant6$和$\varepsilon_1$充分小时，总可使$\varepsilon^{-1}K_1K_2\varepsilon_1^{\frac{3m-9}{2}}\leqslant r_0<1$，证毕.

从引理4和5知$T_{\varepsilon}$是$S_m$中的压缩算子，所以算子方程(3.8)在$S_m$中存在唯一的解$u_0$，得到

**定理5.** 在假设$(H1)$—$(H5)$下，当$\varepsilon$充分小时，狄立克雷问题(2.0.1)—(2.0.2)存在解$u_{\varepsilon}\in C_{2,\alpha}(\bar{\Omega})$，并且在球$S_m$，$(m\geqslant6)$中是唯一的，它的一致有效渐近展开式是

$$u_{\varepsilon}=U_m+Z_m \tag{3.10}$$

其中$U_m$由(2.5.1)式给出，$|Z_m|_{2,\alpha}=O(\varepsilon_1^{\frac{3m-9}{2}})$，$(m\geqslant6)$.

应用[1]中类似的方法可以消除条件“$m\geqslant6$”，即有

**定理6.** 在定理5的条件下，解$u_{\varepsilon}\in C_{2,\alpha}(\bar{\Omega})$在球$S_m^{(0)}=\left\{u\in C_0\Big|\,|u|_0\leqslant\varepsilon_1^{\frac{3m+1}{2}}\right\}$中是唯一的，和具有一致有效渐近展开式(3.10)，和$|Z_m|_0=O(\varepsilon_1^{\frac{3m+1}{2}})$，$(m\geqslant0)$. 其中$|\cdots|_0$表示连

续函数空间$C_0$的范数.

证：首先知道边值问题(3.5)—(3.6)的解$\tilde{Z}_m$在$S_m^{(0)}$中是唯一的；否则，设另有一解$\tilde{Z}_m^{(1)}$则

$$L_\varepsilon[\tilde{Z}_m-\tilde{Z}_m^{(1)}]=F_{uu}(x,y,U_m+\mu_2\bar{Z})\bar{Z}(\tilde{Z}_m-\tilde{Z}_m^{(1)})$$

$$(\tilde{Z}_m-\tilde{Z}_m^{(1)})\Big|_{\partial\Omega}=0$$

其中$\bar{Z}=\Psi_m+\tilde{Z}_m^{(1)}+\mu_1(\tilde{Z}_m-\tilde{Z}_m^{(1)})$，因$F_u\geqslant n_0>0$，所以

$$\left|\tilde{Z}_m-\tilde{Z}_m^{(1)}\right|_0\leqslant K_3\varepsilon_1^{\frac{1}{2}}\left|\tilde{Z}_m-\tilde{Z}_m^{(1)}\right|_0$$

当$\varepsilon_1$充分小时应成立$\tilde{Z}_m\equiv\tilde{Z}_m^{(1)}$，得出矛盾.

从$\tilde{Z}_m$的唯一性，推出$Z_m$的唯一性. 作函数$Z=U_{m+6}-U_m+Z_{m+6}$，因$|Z_{m+6}|_0=O\left(\varepsilon_1^{\frac{3m+9}{2}}\right)$和$|U_{m+6}-U_m|_0=O\left(\varepsilon_1^{\frac{3m+1}{2}}\right)$所以$|Z|_0=O\left(\varepsilon_1^{\frac{3m+1}{2}}\right)$. 由于$Z$和$Z_m$确定于同一边值问题(3.2)—(3.3)，所以$Z=Z_m$，因此$|Z_m|_0=O\left(\varepsilon_1^{\frac{3m+1}{2}}\right)$，证毕.

## 参 考 文 献


1. Berger, M. S. and Fraenkel, L. E., On the asymptotic solution of a nonlinear Dirichlet problem, *J. Math. Mech.*, **19**, 7(1970), 553—585.
2. Fife, P. C., Semilinear elliptic boundary value problems with small parameters, *Arch. Rat. Mech. and Anal.*, **52**, 2(1973), 205—232.
3. Holland, C. J., Singular perturbations in elliptic boundary value problems, *J. Diff. Equ.* **20**, 1(1976), 248—265.
4. Вишик, М. И. и Люстерник, Л. А., Регулярное вырождение и пограничный слой пля линейных дифференциальных уравнений с малым параметром, *УМН*., **12**, 5(1957), 3-122.
5. Леликова, Е. Ф., Об асимптотике решения эллиптического уравнения второго порядка с малым параметром при старшик производных, *Дифф. Урав.*, **12**, 10 (1976), 1852—1865.
6. Ильин, А. М. Калашников, А. С. и Олейник, О. А., Линейные уравнения второго порядка параболического типа, *УМН*., **17**, 5(1962), 3-146.
7. Ладыженская, О. А. и Уральцева, Н. Н., *Линейные и квазилинейные уравнения эллиптческого типа*, Изд. Наука, Москва, (1964).

# On the Uniformly Valid Asymptotic Solution of a Non-linear Dirichlet Problem

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**Abstract**

In this paper, Dirichlet problem of second order quasilinear elliptic equation with a small parameter at highest derivatives is studied. In case the degenerate equation has no singular point, and parameter $\varepsilon$ is sufficiently small, the existence and uniqueness of solution are proved. The uniformly valid asymptotic approximation of solution is derived in the entire region.