

能量元及其在连续介质动力计算中的应用*

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摘 要

本文讨论有限差分法与有限元法相结合问题, 阐述一种能量元计算格式, 具有两法各自特点

有限差分法与有限单元法的相结合问题是近年来数值计算方法研究中值得探讨的一个课题. 有限元法由于其网格灵活性与程序统一性而越来越得到广泛应用与发展. 鉴于目前大部份机器的速度与容量有限, 讨论如何提高精度和减少计算时间是目前普遍感兴趣的一个问题. 本文用差分代替能量表达式中微分及用结点坐标参数表示运算矩阵建立起一种能量元计算格式使其具有有限元的灵活性与统一性特点, 而又保持了差分法计算精度较大与速度较高的优点.

一、连续介质动力问题的能量元

以结构平面应变问题为例, 其动力问题的变分方程为

$$\delta V - \int_{L_f} (F_x \delta u + F_y \delta v) dL - \iint_S (H_x \delta u + H_y \delta v) dS + \iint_S \rho \left(\frac{\partial^2 u}{\partial t^2} \delta u + \frac{\partial^2 v}{\partial t^2} \delta v \right) dS = 0 \quad (1.1)$$

其中 F_x , F_y 为边载荷, H_x , H_y 为体积力. 而变形能

$$V = \frac{G}{2} \iint_S \left\{ 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] + \frac{\mu}{1-2\mu} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right\} dS \quad (1.2)$$

将连续体划分为若干个三角形或矩形单元, 在单元 S_k 的变形能 V_k 中微商用差商来近

* 张耀勤、郭琳、席静珠参加有关计算工作.

似. 对于任意三角形域 (图 1) 取

$$\left. \begin{aligned} \frac{\partial u}{\partial x} &= \frac{(y_j - y_m)(u_i - u_j) + (y_j - y_i)(u_j - u_m)}{(y_i - y_m)(x_i - x_j) + (y_j - y_i)(x_j - x_m)} \\ &= M_{iK}u_i + M_{jK}u_j + M_{mK}u_m \\ \frac{\partial v}{\partial x} &= M_{iK}v_i + M_{jK}v_j + M_{mK}v_m \end{aligned} \right\} \text{若 } x_i \geq x_j \geq x_m \quad (1.3, a)$$

$$\left. \begin{aligned} \frac{\partial u}{\partial y} &= N_{iK}u_i + N_{jK}u_j + N_{mK}u_m \\ \frac{\partial v}{\partial y} &= N_{iK}v_i + N_{jK}v_j + N_{mK}v_m \end{aligned} \right\} \text{若 } y_i \geq y_j \geq y_m \quad (1.3, b)$$

其中,

$$M_{iK} = b_i M_0, \quad N_{iK} = a_i M_0 \quad (i, j, m) \quad (1.4, a)$$

$$M_0 = \frac{1}{amb_i - aib_m}, \quad N_0 = \frac{1}{aib_m - amb_i} \quad (1.4, b)$$

$$a_i = x_j - x_m, \quad b_i = y_j - y_m \quad (i, j, m) \quad (1.4, c)$$

对于矩形域 (图 2) 取

$$\left. \begin{aligned} \frac{\partial u}{\partial x} &= M_K[(u_j - u_i) + (u_m - u_n)] \\ \frac{\partial v}{\partial x} &= M_K[(v_j - v_i) + (v_m - v_n)] \\ \frac{\partial u}{\partial y} &= N_K[(u_j - u_m) + (u_i - u_n)] \\ \frac{\partial v}{\partial y} &= N_K[(v_j - v_m) + (v_i - v_n)] \end{aligned} \right\} \quad (1.5)$$

其中,

$$\left. \begin{aligned} M_K &= \frac{1}{2(x_j - x_i)} = \frac{1}{2(x_m - x_n)} \\ N_K &= \frac{1}{2(y_j - y_m)} = \frac{1}{2(y_i - y_n)} \end{aligned} \right\} \quad (1.6)$$

将式 (1.3) — (1.6) 代入式 (1.2) 求

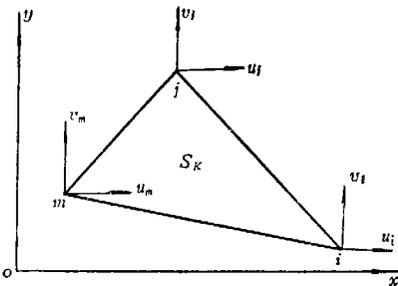


图 1

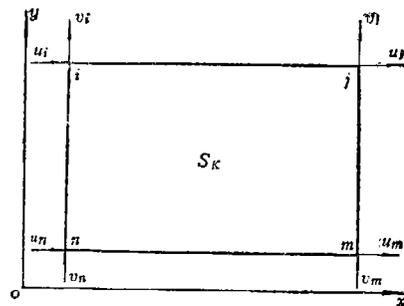


图 2

得单元变形能 V_K , 则其对各结点位移偏导数

$$\{V'\} = [B] \{\delta\} \tag{1.7, a}$$

$$\{V'\} = \begin{cases} \left[\frac{\partial V_K}{\partial u_i} \frac{\partial V_K}{\partial v_i} \frac{\partial V_K}{\partial u_j} \frac{\partial V_K}{\partial v_j} \frac{\partial V_K}{\partial u_m} \frac{\partial V_K}{\partial v_m} \right]^T, & \text{对于三角形元} \\ \left[\frac{\partial V_K}{\partial u_i} \frac{\partial V_K}{\partial v_i} \frac{\partial V_K}{\partial u_j} \frac{\partial V_K}{\partial v_j} \frac{\partial V_K}{\partial u_m} \frac{\partial V_K}{\partial v_m} \frac{\partial V_K}{\partial u_n} \frac{\partial V_K}{\partial v_n} \right]^T, & \text{对于矩形元} \end{cases} \tag{1.7, b}$$

$$\{\delta\} = \begin{cases} [u_i \ v_i \ u_j \ v_j \ u_m \ v_m]^T, & \text{对于三角形元} \\ [u_i \ v_i \ u_j \ v_j \ u_m \ v_m \ u_n \ v_n]^T, & \text{对于矩形元} \end{cases} \tag{1.7, c}$$

$[B]$ 为偏能矩阵, 对于三角形元为含 $G_K, \mu_K, M_{iK}, N_{iK}$ 等及面积 Δ_K 的六阶方阵

$$[B] = G_K \Delta_K \begin{pmatrix} a_{ii} & & & & & \\ c_{ii} & b_{ii} & & & & \\ & & \text{对} & & & \\ a_{ij} & c_{ji} & a_{jj} & & & \\ c_{ij} & b_{ji} & c_{jj} & b_{jj} & & \\ & & & & \text{称} & \\ a_{im} & c_{mi} & a_{mj} & c_{mj} & a_{mm} & \\ c_{im} & b_{im} & c_{jm} & b_{jm} & c_{mm} & b_{mm} \end{pmatrix} \tag{1.7, d}$$

其中,

$$\left. \begin{aligned} a_{ij} &= \frac{2(1-\mu_K)}{1-2\mu_K} M_{iK} M_{jK} + N_{iK} N_{jK} \\ b_{ij} &= \frac{2(1-\mu_K)}{1-2\mu_K} N_{iK} N_{jK} + M_{iK} M_{jK} \\ c_{ij} &= \frac{2\mu_K}{1-2\mu_K} M_{iK} N_{jK} + M_{jK} N_{iK} \end{aligned} \right\} \tag{1.7, e}$$

对于矩形元为含 G_K, μ_K, M_K, N_K 及面积 Δ_K 的八阶方阵

$$[B] = G_K \Delta_K \begin{pmatrix} a & & & & & & & \\ -b & c & & & & & & \\ -g & -d & a & & & & & \\ d & e & b & c & & & & \\ -a & b & g & -d & a & & & \\ b & -c & d & -e & -b & c & & \\ g & d & -a & b & -g & -d & a & \\ -d & -e & -b & -c & d & e & b & c \end{pmatrix} \tag{1.7, f}$$

其中,

$$\left. \begin{aligned} a &= \frac{2(1-\mu_K)}{1-2\mu_K} M_K^2 + N_K^2, \quad b = \frac{1}{1-2\mu_K} M_K N_K, \quad c = \frac{2(1-\mu_K)}{1-2\mu_K} N_K^2 + M_K^2 \\ d &= \frac{1-4\mu_K}{1-2\mu_K} M_K N_K, \quad e = \frac{2(1-\mu_K)}{1-2\mu_K} N_K^2 - M_K^2, \quad g = \frac{2(1-\mu_K)}{1-2\mu_K} M_K^2 - N_K^2 \end{aligned} \right\} \tag{1.7, g}$$

对于有 M 个单元 N 个结点的结构有

$$V = \sum_{K=1}^M V_K = V[u_1, \dots, u_N, v_1, \dots, v_N]$$

$$\delta V = \sum_{i=1}^N \left[\frac{\partial V}{\partial u_i} \delta u_i + \frac{\partial V}{\partial v_i} \delta v_i \right] \quad (1.8)$$

边界上 (图 3),

$$\int_L (F_x \delta u + F_y \delta v) dL = \sum_{i=1}^N \alpha(i) [F_{x_i} \delta u_i + F_{y_i} \delta v_i] \quad (1.9, a)$$

$$\alpha(i) = \begin{cases} 0, & i \text{ 点为面内点} \\ \frac{\lambda'_{i,i} + \lambda''_{i,i}}{2}, & i \text{ 点为边上点} \end{cases} \quad (1.9, b)$$

面内 (图 1, 2),

$$\iint_S \left\{ \left[H_x - \rho \frac{\partial^2 u}{\partial t^2} \right] \delta u + \left[H_y - \rho \frac{\partial^2 v}{\partial t^2} \right] \delta v \right\} dS = \sum_{i=1}^N \left\{ \left[\gamma_x(i) - \beta(i) \frac{\partial^2 u_i}{\partial t^2} \right] \delta u_i + \left[\gamma_y(i) - \beta(i) \frac{\partial^2 v_i}{\partial t^2} \right] \delta v_i \right\} \quad (1.10, a)$$

其中,

$$\beta(i) = \sum_K^i \frac{\rho_K \Delta_K}{c_K}, \quad \gamma_x(i) = \sum_K^i \frac{H_{x1K} \Delta_K}{c_K}, \quad \gamma_y(i) = \sum_K^i \frac{H_{y1K} \Delta_K}{c_K} \quad (1.10, b)$$

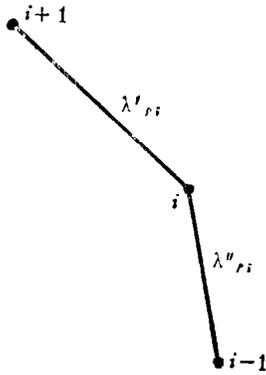


图 3

\sum_K^i 表示对 i 结点邻近各单元取和.

$$c_K = \begin{cases} 3, & \text{若第 } K \text{ 个单元为三角形.} \\ 4, & \text{若第 } K \text{ 个单元为矩形.} \end{cases} \quad (1.10, c)$$

而

$$\frac{\partial^2 u_i}{\partial t^2} = \frac{u_i^{(r+1)} - 2u_i^{(r)} + u_i^{(r-1)}}{\tau^2}, \quad \frac{\partial^2 v_i}{\partial t^2} = \frac{v_i^{(r+1)} - 2v_i^{(r)} + v_i^{(r-1)}}{\tau^2} \quad (1.11)$$

将式(1.7)–(1.11)代入式(1.1)考虑到 δu_i , δv_i 的任意性, 则得下列用矩阵表示的基本计算方程组

$$\{\delta\}^{(r+1)} = [D] \{\delta\}^{(r)} - \{\delta\}^{(r-1)} + \{R\}^{(r)} \quad (1.12)$$

式中位移列阵

$$\{\delta\} = [u_1 \ v_1 \ u_2 \ v_2 \ \dots \ u_N \ v_N]^T \quad (1.13, a)$$

运算矩阵 $[D]$ 为 $2N$ 阶方阵, 其第 p 行 q 列元素

$$D_{pq} = 2\delta_{pq} - \frac{\tau}{\beta(I)} \sum_n B_{pq}^n \quad (1.13, b)$$

其中,

$$\delta_{pq} = \begin{cases} 1, & p=q \\ 0, & p \neq q \end{cases}$$

I 为 $\{\delta\}$ 中第 p 个位移所在结点号,

J 为 $\{\delta\}$ 中第 q 个位移所在结点号,

$$\sum_n \begin{cases} \text{对 } I \text{ 点周围单元求和,} & \text{若 } I=J, \\ \text{对 } IJ \text{ 边两侧单元求和,} & \text{若 } I \neq J, \text{ 而 } I, J \text{ 为一边两端,} \\ \text{零,} & \text{若 } I \neq J, \text{ 而 } I, J \text{ 不是一边两端,} \end{cases}$$

B_{pq}^n 为第 n 个单元 $[B]$ 矩阵中第 p, q 位移对应的行, 列元素.

荷载列阵 $\{R\}$ 的元素

$$\left. \begin{aligned} R_{2I-1} &= \frac{\tau^2 \alpha(I)}{\beta(I)} F_{xI} + \frac{\tau^2}{\beta(I)} \gamma_x(I) \\ R_{2I} &= \frac{\tau^2 \alpha(I)}{\beta(I)} F_{yI} + \frac{\tau^2}{\beta(I)} \gamma_y(I) \end{aligned} \right\} \quad (1.13, c)$$

有了 $[D]$ 及 $\{R\}$ 即可按式 (1.12) 由第 $r, r-1$ 时刻的位移求第 $r+1$ 时刻位移.

对于空间问题, 其动力问题的变分方程为

$$\begin{aligned} \delta V - \iint_{S_F} (F_x \delta u + F_y \delta v + F_z \delta w) dS - \iiint_{\Omega} (H_x \delta u + H_y \delta v + H_z \delta w) d\Omega \\ + \iiint_{\Omega} \rho \left(\frac{\partial^2 u}{\partial t^2} \delta u + \frac{\partial^2 v}{\partial t^2} \delta v + \frac{\partial^2 w}{\partial t^2} \delta w \right) d\Omega = 0 \end{aligned} \quad (1.14)$$

应变能

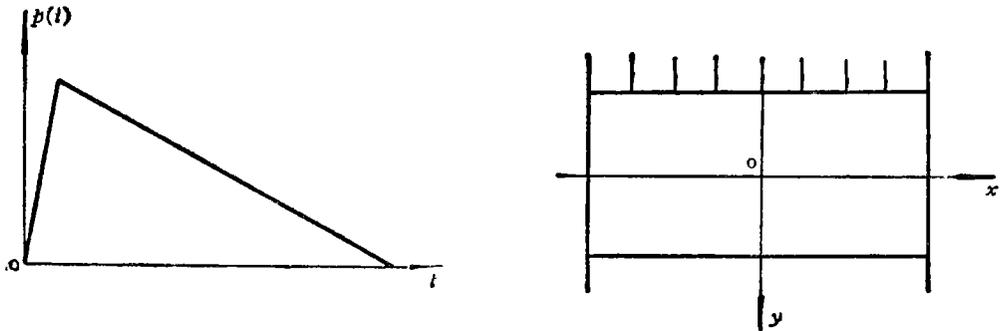


图 4

$$\begin{aligned}
 V = \frac{G}{2} \iiint_{\Omega} \left\{ 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right] + \frac{2\mu}{1-2\mu} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)^2 \right. \\
 \left. + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)^2 \right\} d\Omega \quad (1.15)
 \end{aligned}$$

有同样的计算格式(1.12)及类似(1.13)式的 $\{\delta\}$, $[D]$, $\{R\}$.

二、算 例

上表面受冲击荷载的二端支承长方形块体的平面应变问题(图4),按划分为 12×4 网格矩形能量元计算.其某些数字结果与同网格三角形有限元、四倍网格矩形有限元以及相应厚板理论解析介^[1]比较于下表:

方 法		$\sigma_{x \max}$ (kg/cm ²)	$M_{z \max}$ (kg-cm/cm)	$\tau_{xy \max}$ (kg/cm ²)	$Q_{z \max}$ (kg/cm)	
数值计算	有限元	三角形元(同网格)	3.39	206	0.899	14.1
		矩形元(四倍网格)	4.19	268	1.18	17.2
	能 量 元	4.03	257	1.22	17.0	
解 析 介		—	251	—	17.8	

三、讨 论

1. 能量元计算格式的矩阵元素(1.13,b)、(1.13,c)由任意形状、大小、方位单元的结点坐标参数 $x_i, y_i, x_j, y_j, \dots$ 及其材料常数 G_K, μ_K, ρ_K ,几何参数 Δ_K, λ_{pi} 等组成,并对任意结构的动力问题归结为离散化方程(1.12),因此能量元具有有限元法的网格、材料、边界灵活性与程序统一性的特点.

2. 变分方程(1.1), (1.14)与一般弹性动力学微分方程在力学上是等价的.在连续介质力学的多数问题和大部份区域(如结构的平缓应力区,自由振动及法向挠度计算等)其差分法^[2]比有限元法精度高、速度快^[3].计算实践表明,能量元也比有限元收敛快,计算时间要少.

3. 上述能量元叙述主要针对连续介质动力问题.显然也可应用于其它一些学科的数值计算,只要采用类似方法基于数学上相应的变分方程建立计算格式即可.

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Energy Element and Its Application in the Dynamic Calculation of Continuous Medium

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Abstract

The present paper discusses the problem related to the combination of the finite difference method and the finite element method, as well as describes an energy element calculating mode including the special features of the above two methods.