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# 对“巴伦布朗特挟沙水流时均运动方程”的讨论*

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## 摘要

本文通过推导证明，认为巴伦布朗特的挟沙水流运动方程，采用了液体不均质性只影响重力项的假定后，使时间平均运动方程难于成立．本文同时给出了挟沙水流时间平均运动方程的准确形式．

Г. И. 巴伦布朗特采用浑水质心速度代表单位体积的不均匀液体重心的运动速度，表示为

$$V_i=\frac{d_1(1-\rho)v_i+d_2\rho w_i}{D} \tag{1}$$

同时认为在细颗粒情况下，水流加速度相对小于重力加速度，颗粒水平分速与水流速度一致，垂直分速差一颗粒沉速．该条件一般形式为

$$w_i=v_i-a\delta_{i3},\quad \delta_{ij}\begin{cases}0, & 如\ i\neq j\\ 1, & 如\ i=j\end{cases} \tag{2}$$

提出挟沙水流运动方程组如下：

$$\frac{\partial}{\partial t}DV_i+\frac{\partial DV_iV_\alpha}{\partial X_\alpha}=-Dg_i-\frac{\partial P}{\partial X_i}-d_1d_2\frac{\partial}{\partial X_3}\left[\frac{a^2\rho(1-\rho)}{D}\right]\delta_{i\alpha}+\frac{\partial\tau_{\alpha i}}{\partial X_\alpha} \tag{3}$$

$$\frac{\partial}{\partial t}D+\frac{\partial}{\partial X_\alpha}DV_\alpha=0 \tag{4}$$

$$\frac{\partial}{\partial X_\alpha}\left[V_\alpha+a_\alpha(d_2-d_1)\frac{\rho(1-\rho)}{D}\right]=0 \tag{5}$$

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* 蔡树棠推荐．

在对方程(3)，(4)，(5)取平均时，巴氏假定液体不均质性只影响重力项．这样得出时均运动方程组如下：

$$\frac{\partial \overline{V}_i}{\partial t}+\overline{V}_\alpha\frac{\partial \overline{V}_i}{\partial X_\alpha}=\frac{-\partial \overline{V'_i V'_\alpha}}{\partial X_\alpha}-(1+\sigma\bar{\rho})g_i$$

$$+\frac{1}{d_1}\frac{\partial \bar{\tau}_{\alpha i}}{\partial X_\alpha}-\frac{1}{d_1}\frac{\partial \overline{P}}{\partial X_i} \tag{6}$$

$$\frac{\partial \bar{\rho}}{\partial t}+\overline{V}_\alpha\frac{\partial \bar{\rho}}{\partial X_\alpha}+\frac{\partial \overline{\rho' V'_\alpha}}{\partial X_\alpha}=a\frac{\partial \bar{\rho}}{\partial X_3} \tag{7}$$

$$\frac{\partial \overline{V}_\alpha}{\partial X_\alpha}=-a\sigma\frac{\partial \bar{\rho}}{\partial X_3} \tag{8}$$

$$\frac{\partial V_\alpha}{\partial X_\alpha}=0 \qquad \text{(对与质量无关的问题)} \tag{9}$$

$$\frac{\partial B}{\partial t}+\overline{V}_\alpha\frac{\partial B}{\partial X_\alpha}=-\overline{D'V'_3}g_i-\frac{\partial \overline{P'V_\alpha}}{\partial X_\alpha}+\frac{\partial q_\alpha}{\partial X_\alpha}$$

$$-Q-\frac{\partial}{\partial X_\alpha}\left[\frac{1}{2}d_1\sum_{i=1}^{3}\overline{V'^2_i V'_\alpha}\right]-\frac{1}{2}d_1\overline{V'_\alpha V'_\beta}e_{\alpha\beta} \tag{10}$$

方程(1)至(5)是巴氏为挟沙水流提供了一个挟沙水流运动的一般表达式．如果我们采用方程(3)至(5)，将各变数直接引入脉动值与平均值，则方程组变为

$$\frac{\partial}{\partial t}(\overline{D}+D')(\overline{V}_i+V'_i)+\frac{\partial}{\partial X_\alpha}(\overline{D}+D')(\overline{V}_i+V'_i)(\overline{V}_\alpha+V'_\alpha)$$

$$=\frac{-\partial(\overline{P}+P')}{\partial X_i}+\frac{\partial(\bar{\tau}_{\alpha i}+\tau_{\alpha i})}{\partial X_\alpha}-(\overline{D}+D')g_i$$

$$-d_1d_2\frac{\partial}{\partial X_\alpha}\frac{a_\alpha a_i(\bar{\rho}+\rho')(1-\rho-\rho')}{D} \tag{11}$$

$$\frac{\partial}{\partial t}(\overline{D}+D')+\frac{\partial}{\partial X_\alpha}(\overline{D}+D')(\overline{V}_\alpha+V'_\alpha)=0 \tag{12}$$

$$\frac{\partial}{\partial X_\alpha}\left[(\overline{V}_\alpha+V'_\alpha)+(d_2-d_1)\frac{(\bar{\rho}+\rho')(1-\bar{\rho}-\rho')}{D}\right] \tag{13}$$

对各式取时间平均得时均运动情况下不均质流体运动方程组．本文所得方程组与蔡树棠教授曾经在地球物理学报[2]上发表的结果一致．其方程组如下：

$$\frac{\partial}{\partial t}\overline{V}_i+\overline{V}_\alpha\frac{\partial \overline{V}_i}{\partial X_\alpha}=-\frac{1}{\overline{D}}\frac{\partial}{\partial t}\overline{V'_i D'}+\frac{\overline{V}_i}{\overline{D}}\frac{\partial}{\partial X_\alpha}\overline{V'_\alpha D'}$$

$$-\frac{1}{\overline{D}}\frac{\partial}{\partial X_i}\overline{P}+\frac{1}{\overline{D}}\frac{\partial}{\partial X_\alpha}\bar{\tau}_{i\alpha}-\frac{d_1d_2}{\overline{D}}\frac{\partial}{\partial X_\alpha}\left(\frac{\overline{a_\alpha a_i\rho(1-\rho)}}{D}\right)$$

$$-g_i+\frac{1}{\overline{D}}\frac{\partial}{\partial X_\alpha}P_{i\alpha} \tag{14}$$

$$\frac{\partial}{\partial X_a}\left[\overline{V}_a+(d_2-d_1)\frac{\overline{\rho(1-\rho)a_a}}{D}\right]=0 \tag{15}$$

$$\frac{\partial}{\partial t}\overline{D}+\frac{\partial}{\partial X_a}(\overline{D}\overline{V}_a)+\frac{\partial}{\partial X_a}(\overline{D'V'_a})=0 \tag{16}$$

$$\frac{\partial}{\partial t}B+\frac{\partial}{\partial X_a}(B\overline{V}_a)=-\frac{\partial P_{ia}}{\partial X_a}\overline{V}_i+\overline{N'_iV'_i}-\overline{D'V'_i}g_i$$

$$+\overline{V}_i\frac{\partial}{\partial t}\overline{V'_iD'}-\frac{\overline{V}_i^2}{2}\frac{\partial}{\partial X_a}\overline{D'V'_a}-\frac{\partial}{\partial X_a}M_a \tag{17}$$

式中

$$P_{ia}=-(\overline{D}\,\overline{V'_iV'_a}+\overline{V}_i\,\overline{D'V'_a}+\overline{V}_a\,\overline{D'V'_i}+\overline{D'V'_iV'_a})$$

$$B=\overline{D\frac{V_i^2}{2}}-\overline{D}\frac{\overline{V}_i^2}{2}$$

$$N_i=\frac{\partial}{\partial X_a}\left[\overline{P}-P+\tau_{ia}-\overline{\tau}_{ia}+d_1d_2\frac{\overline{a_aa_i\rho(1-\rho)}}{D}-d_1d_2\frac{a_aa_i\rho(1-\rho)}{D}\right]$$

$$M_a=\left[\overline{D}\,\overline{V}_i\overline{V'_iV'_a}+\overline{D}\,\overline{\frac{V_i'^2}{2}V'_a}+\frac{\overline{V}_i^2}{2}\overline{V'_aD'}\right.$$

$$\left.+\overline{V}_i\,\overline{V'_iV'_aD'}+\overline{\frac{V_i'^2}{2}V'_aD'}\right]$$

通过推导认为巴氏在对方程取时间平均时采用液体不均质性只影响重力项假定后，使方程（6）难于成立．根据具体研究对象作出假定简化方程是一般常采用的，但应在普遍条件下推出时均方程再作出假定，省掉方程中同级小项．下面为了证明（6）式不能成立，我们写出清水紊流情况的纳维也——斯托克方程如下：

$$\frac{\partial V_i}{\partial t}+V_a\frac{\partial V_i}{\partial X_a}=-g_i-\frac{\partial P}{d_1\partial X_i}+\frac{\partial\tau_{ia}}{d_1\partial X_a} \tag{18}$$

$$\frac{\partial V_a}{\partial X_a}=0 \tag{19}$$

对（18），（19）式取时均后，得

$$d_1\frac{\partial\overline{V}_i}{\partial t}+d_1V_a\frac{\partial\overline{V}_i}{\partial X_a}=\frac{\partial}{\partial X_a}(-\overline{dV_i'V_a'})-d_1\frac{\partial(\overline{P}+g_i)}{\partial X_i}+\frac{\partial\overline{\tau}_{ia}}{\partial X_a} \tag{20}$$

$$\frac{\partial\overline{V}_a}{\partial X_a}=0 \tag{21}$$

比较（20）式和（6）式，可以看出（6）式中的运动方程中$\sigma\rho g$ 项是不能存在的．即方程（6）要成立，必须$\sigma\rho g_i$ 项等于零．而重力加速度是一常值，这是由比萨斜塔试验得出的，如方程（6）中存在$\sigma\rho g_i$ 项只能表示挟沙水流重力加速度会受密度影响．这显然在物理意义上不对．

现在讨论通常应用的二元、恒定、均匀情况，远离边壁、浑水浓度不大和忽略流速分布等项影响之后，由（6）式得到

$$\frac{\partial}{\partial y}(-\overline{V'_x V'_y})=(1+\sigma\rho)g\sin\alpha \tag{22}$$

实际上该方程是一不等式，（22）式不能成立．因为在该条件下运动方程应为

$$\frac{\partial}{\partial y}(-\overline{V'_x V'_y})=g\sin\alpha \tag{23}$$

如根据本文所得方程（14），得到在上述条件下，浑水运动方程应为

$$\frac{\partial}{\partial y}(\overline{1+\sigma\rho)V'_x V'_y}=(1+\sigma\rho)g\sin\alpha \tag{24}$$

也即是表明清浑水的紊动剪切力是不相同的．

如果不考虑不均质密度项对其它各项的影响，尤其是对紊动粘剪力项的影响，则在脉动能量方程中就不能出现悬浮功这一项．只可能得出清水情况下脉动能方程如下：

$$\frac{\partial}{\partial t}\left(\frac{1}{2}d_1\overline{V'_i V'_i}\right)+\frac{\partial}{\partial X_a}\left(\frac{1}{2}d_1\overline{V'_i V'_i V_a}\right)+\frac{\partial}{\partial X_a}\left(\frac{1}{2}d_1\overline{V'_i V'_i V'_a}\right)$$

$$=-(d_1\overline{V'_i V'_a})\frac{\partial\bar{V}_i}{\partial X_a}+\frac{\partial}{\partial X_a}(\overline{-P'V'_i})+\overline{V'_i\frac{\partial\tau_{ai}'}{\partial X_a}} \tag{25}$$

为便于与原文作比较，本文尽量采用巴伦布朗特所用符号，现将各符号意义说明如下：

$d_2, d_1$——颗粒和液体密度

$V_i$——液体沿 $X_i$ 轴的速度分量

$\rho$——悬沙相对体积

$P$——不均质液体某点总的水动压力

$D$——浑水密度

$a$——泥沙静水沉速

$B$——紊动能

$\tau$——剪切应力

$Q$——不均质液体单位体积内脉动能的损耗

$q_\alpha$——紊动造成的分子转移，沿$X_a$轴产生脉动能的流动密度向量分量

工作中曾与蔡树棠教授进行过讨论，谨此致谢．

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# Discussion on Barenblatt's Equations of Mean Movement of the Inhomogeneous Fluid

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**Abstract**

In this paper, it is proved that Barenblatt's equations of mean movement of the inhomogeneous fluid is not correct, because his equations are based on the assumption that the characteristics of the inhomogeneity of fluid only affect on gravitational terms. The correct form of the equation of mean movement of the inhomogeneous fluid is derived in this paper.