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关于含小参数的拟线性椭圆型方程的狄立克雷问题

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摘　　要

本文研究最高阶导数项含小参数的拟线性椭圆型方程的狄立克雷问题，在退化方程的特征是曲线和区域是凸域的一般情形下，给出构造一致有效渐近解的方法，并证明当小参数是充分小时，狄立克雷问题的解是存在和唯一．

一、前　　言

关于最高阶导数项含小参数的椭圆型方程的狄立克雷问题的研究，七十年代以来，已开始进入到非线性方程的领域．Berger和Fraenkel[1]，以及Fife[2]等先后研究了退化方程不具特征的拟线性椭圆型方程的狄立克雷问题．但在实际问题中也常遇见退化方程具有特征的情形（例如考察流体在磁场中的管道中的流动[3]），将涉及解在边界与特征相切的点的近旁性质的研究，长期以来（即使对于线性方程）一直是比较困难的问题．1971年，Grasman[4]首先应用Eckhaus给出的匹配方法，研究了二阶线性椭圆型方程在圆域上的狄立克雷问题；1973—1978年以来，Van Harten[5]又研究了拟线性椭圆型方程的狄立克雷问题．但只考虑退化方程的特征是直线和区域是圆域，以及特征是曲线而区域是环域的狄立克雷问题．本文应用在边界层构造校正项的方法研究一般凸域上的狄立克雷问题，拓广和改进了Holland[6]以及Van Harten[5]的工作．本文和其它工作一样，提供了研究含有小参数的非线性椭圆型方程边值问题解的存在性和唯一性的一个途径．

1979年Howes[7]也研究了这一种类型和其它类型的非线性椭圆型方程的狄立克雷问题，但只涉及某些特殊情形，没有讨论解的渐近展开式．

二、形式渐近解

为了简单起见，考察两个自变量的情形（完全类似地可以推广到多个自变量的情形）设方程是

$$\varepsilon\left(\sum_{i,j=1}^{2} a_{i,j}(x_1,x_2,u)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{2} a_i(x_1,x_2,u)\frac{\partial u}{\partial x}+a(x_1,x_2,u)\right)$$

$$-p(x_1,x_2,u)\sum_{i=1}^{2} A_i(x_1,x_2)\frac{\partial u}{\partial x}-A(x_1,x_2,u)=0,\qquad (x_1,x_2)\in\Omega$$

其中$a_{i,j}=a_{j,i}$, $0<\varepsilon\leqslant\varepsilon_0$,（$0<\varepsilon_0\ll 1$），$\Omega$ 表示 x_1, x_2 平面上的有界凸域，和在 Ω 上成立：

$$\sum_{i,j=1}^{2} a_{i,j}(x_1,x_2,w_0)\xi_i\xi_j\geqslant\delta_1\sum_{i=1}^{2}\xi_i^2,\quad p(x_1,x_2,w_0)\geqslant\delta_2,\ \sum_{i=1}^{2}A_i^2(x_1,x_2)\geqslant\delta_3$$

δ_i,（$i=1,2,3$），是正的常数，w_0表示退化边值问题(2.1.8)—(2.1.9)的解.

我们知道在上面的条件下，可作自变量的非奇异变换将方程简化成

$$N_\varepsilon[u]\equiv\varepsilon\left(\sum_{i,j=1}^{2} a_{i,j}(x_1,x_2,u)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{2} a_i(x_1,x_2,u)\frac{\partial u}{\partial x_i}+a(x_1,x_2,u)\right)$$

$$-\frac{\partial u}{\partial x_2}-A(x_1,x_2,u)=0\qquad (x_1,x_2)\in\Omega \tag{2.0.1}$$

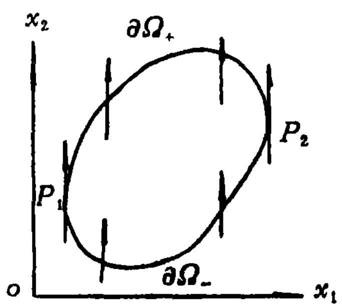

图1 边界和$\Omega_-\partial\Omega_+$

（仍用原来的字母表示坐标变换后相应的量）. 设边值条件是

$$u(x_1,x_2)|_{\partial\Omega}=\varphi(x_1,x_2)|_{\partial\Omega} \tag{2.0.2}$$

其中$\partial\Omega$表示Ω的边界. 今后假设$\partial\Omega$是充分光滑，边值函数φ沿$\partial\Omega$是充分光滑，和系数$a_{i,j}$,…等是 $\Omega\times(-\infty<u<\infty)$ 中的充分次可微函数.

二·一 解的外部展开式

假设 $\partial\Omega$ 与退化方程的特征 $x_1=c_1$和$x_2=c_2$，（$c_1<c_2$）相切于P_1和P_2点. 以$\partial\Omega_-$表示$\partial\Omega$的被特征$x_1=c$,（$c_1<c<c_2$），按x_2的正向穿入区域的部分，设其方程是$x_2=x_{2(-)}(x_1)$；以$\partial\Omega_+$表示$\partial\Omega$的被特征穿出区域的部分，设其方程是$x_2=x_{2(-)}(x_1)$.

假设狄立克雷问题(2.0.1)—(2.0.2) 的解的m阶近似式是

$$W_m(x_1,x_2;\varepsilon)=\sum_{i=0}^{m}\varepsilon^i w_i(x_1,x_2) \tag{2.1.1}$$

代入方程(2.0.1)，并将各系数关于ε展开

$$a_{i,j}(x_1,x_2,W_m)=\sum_{k=0}^{m}\varepsilon^k a_{i,j}^{(k)}+\varepsilon^{m+1}a_{i,j}^{(m+1)}(\theta\varepsilon),(0<\theta<1) \tag{2.1.2}$$

其中 $a_{i,j}^{(0)}=a_{i,j}(x_1,x_2,w_0)$, $a_{i,j}^{(1)}=\dfrac{\partial a_{i,j}(x_1,x_2,w_0)}{\partial u}w_1$和

$$a_{i,j}^{(k)}=\frac{\partial a_{i,j}(x_1,x_2,w_0)}{\partial u}w_k+a_{i,j}^{(k,r)}(x_1,x_2,w_0,\cdots,w_{k-1})$$

其中

$$a_{i,j}^{(k,r)}=\sum_{\substack{l_1+2l_2+\cdots+pl_p=k\\(p<k)}}\frac{1}{l_1!\cdots l_p!}\frac{\partial^{l_1+\cdots+l_p}a_{i,j}(x_1,x_2,w_0)}{\partial u^{l_1+\cdots+l_p}}w_1^{l_1}\cdots w_p^{l_p}$$

等等，再令ε的前$m+1$项较低幂的系数为零，得到关于w_i的递推方程：

$$\frac{\partial w_0}{\partial x_2}+A(x_1,x_2,w_0)=0 \tag{2.1.3}$$

$$\frac{\partial w_1}{\partial x_2}+\frac{\partial A(x_1,x_2,w_0)}{\partial u}w_1=\sum_{i,j=1}^{2}a_{i,j}^{(0)}\frac{\partial^2 w_0}{\partial x_i\partial x_j}+\sum_{i=1}^{2}a_i^{(0)}\frac{\partial w_0}{\partial x}+a^{(0)}\equiv F_1 \tag{2.1.4}$$

$$\frac{\partial w_k}{\partial x_2}+\frac{\partial A(x_1,x_2,w_0)}{\partial u}w_k=\Big[\sum_{l=0}^{k-1}\Big(\sum_{i,j=1}^{2}a_{i,j}^{(l)}\frac{\partial^2 w_{k-1-l}}{\partial x_i\partial x_j}+\sum_{i=1}^{2}a_i^{(l)}\frac{\partial w_{k-1-l}}{\partial x_i}\Big)$$

$$+a^{(k-1)}\Big]+A^{(k,r)}\equiv F_k\qquad(k=2,3,\cdots,m) \tag{2.1.5}$$

再在$\partial\Omega_-$上给出边值条件

$$w_0\Big|_{\partial\Omega_-}=\varphi(x_1,x_{2(-)}(x_1)) \tag{2.1.6}$$

$$w_k\Big|_{\partial\Omega_-}=0\qquad(k=1,2,\cdots,m) \tag{2.1.7}$$

作假设

(H_1)：退化边值问题

$$\frac{\partial w_0}{\partial x_2}+A(x_1,x_2,w_0)=0$$

$$w_0\Big|_{\partial\Omega_-}=\varphi(x_1,x_{2(-)}(x_1))$$

在区域$\bar{\Omega}\backslash[\partial\Omega\cap(x_1=c_1,\ x_1=c_2)]$存在充分次可微的解$w_0$，并且$w_0\in C(\bar{\Omega})$，其中$\bar{\Omega}=\Omega+\partial\Omega$.

将w_0代入关于w_1的定解问题(2.1.4)、(2.1.7)(取$k=1$)解得

$$w_1=\int_{x_{2(-)}(x_1)}^{x_2}F_1\exp\Big(\int_{x_2}^{s}\frac{\partial A(x_1,t,w_0(x_1,t))}{\partial u}dt\Big)ds$$

因为$x_2=x_{2(-)}(x_1)$的导数在切点P_1、P_2具有奇性所以$\frac{\partial w_0}{\partial x_1}$等在$P_1$、$P_2$一般也具有奇性. 除去切点后，可以根据(2.1.5)、(2.1.7)逐步地求出w_k，$(k=1,2,\cdots,m)$，并且当$c_1+\eta_0\leqslant x_1\leqslant c_2-\eta_0$时$N_\varepsilon[W_m]=O(\varepsilon^{m+1})$，$\eta_0$是小于$\frac{c_2-c_1}{2}$的正数.

为了使W_m在边界$\partial\Omega_+$和切点P_1、P_2也满足边值条件(2.0.2)，下面再分别在它们的邻域构造边界层项.

二·二 常型边界层 (Ordinary Boundary Layer)

先在$\partial\Omega_+$的邻域构造边界层项. 在$\partial\Omega$的邻域建立局部座标系：过$\partial\Omega$的每一点P作长为

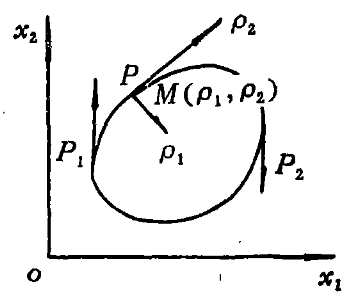

图 2 局部坐标系

$\rho_1^{(0)}$的内法线，并取$\rho_1^{(0)}$充分小使各内法线互不相交；对于$\partial\Omega$邻域中的每一点M，取该点沿内法线到$\partial\Omega$的距离作为M点的ρ_1坐标（$0\leqslant\rho_1\leqslant\rho_1^{(0)}$），取此内法线的原点(顺时针方向)到切点$P_1$的弧长作为$M$点的$\rho_2$坐标. 记$P_2$点的$\rho_2$坐标为$\rho_2^{(0)}$，（或$-\rho_2^{(-2)}$）. 在$(\rho_1,\rho_2)$坐标系方程（2.0.1）成为

$$\tilde{N}_\varepsilon[u]\equiv\varepsilon\Big(\sum_{i,j=1}^{2}\tilde{a}_{i,j}(\rho_1,\rho_2,u)\frac{\partial^2 u}{\partial\rho_i\partial\rho_j}+\sum_{i=1}^{2}\tilde{a}_i(\rho_1,\rho_2,u)\frac{\partial u}{\partial\rho_i}+\tilde{a}(\rho_1,\rho_2,u)\Big)$$

$$-\Big(\alpha(\rho_1,\rho_2)\frac{\partial u}{\partial\rho_1}+\beta(\rho_1,\rho_2)\frac{\partial u}{\partial\rho_2}\Big)-\tilde{A}(\rho_1,\rho_2,u)=0 \tag{2.2.1}$$

$$u\Big|_{\rho_1=0}=\tilde{\varphi}(\rho_2)$$

易知在$\partial\Omega_+$上$\alpha(0,\rho_2)=\cos(\widehat{\rho_1,x_2})<0$，（$0<\rho_2<\rho_2^{(0)}$）.

再在$\partial\Omega_+$的邻域$D_{\rho_1(0)}=\Big\{(\rho_1,\rho_2)\Big|0\leqslant\rho_1\leqslant\rho_1^{(0)},\ 0<\rho_2<\rho_2^{(0)}\Big\}$作伸展变换

$$\tau=\frac{\rho_1}{\varepsilon}$$

方程(2.2.1)在$D_{\rho_1^0}$具有形式：

$$\begin{aligned}\tilde{N}_\varepsilon[u]\equiv\varepsilon^{-1}\Big\{&\Big[\tilde{a}_{1,1}(\varepsilon\tau,\rho_2,u)\frac{\partial^2 u}{\partial\tau^2}-\alpha(\varepsilon\tau,\rho_2)\frac{\partial u}{\partial\tau}\Big]+\varepsilon\Big[2\tilde{a}_{1,2}(\varepsilon\tau,\rho_2,u)\frac{\partial^2 u}{\partial\tau\partial\rho_2}\\&+\tilde{a}_1(\varepsilon\tau,\rho_2,u)\frac{\partial u}{\partial\tau}-\beta(\varepsilon\tau,\rho_2)\frac{\partial u}{\partial\rho_2}-\tilde{A}(\varepsilon\tau,\rho_2,u)\Big]\\&+\varepsilon^2\Big[\tilde{a}_{2,2}(\varepsilon\tau,\rho_2,u)\frac{\partial^2 u}{\partial\rho_2^2}+\tilde{a}_2(\varepsilon\tau,\rho_2,u)\frac{\partial u}{\partial\rho_2}+\tilde{a}(\varepsilon\tau,\rho_2,u)\Big]\Big\}=0\end{aligned} \tag{2.2.3}$$

假设边界层项的展开式是

$$V_m(\tau,\rho_2;\varepsilon)=\sum_{i=0}^{m+1}\varepsilon^i v_i(\tau,\rho_2) \tag{2.2.4}$$

作函数

$$U_m=\tilde{W}_m(\rho_1,\rho_2;\varepsilon)+V_m(\tau,\rho_2;\varepsilon) \tag{2.2.5}$$

其中$\tilde{W}_m$表示W_m在(ρ_1,ρ_2)坐标系中的表示式. 将(2.2.5)代入方程(2.2.3)，并将各系数按ε展开

$$\tilde{a}_{i,j}(\varepsilon\tau,\rho_2,U_m)=\sum_{k=0}^{m+1}\varepsilon^k\tilde{a}_{i,j}^{(\tau,k)}+\varepsilon^{m+2}\tilde{a}_{i,j}^{(\tau,m+2)}(\theta\varepsilon),\ (0<\theta<1),$$

其中$\tilde{a}_{i,j}^{(\tau,0)}=\tilde{a}_{i,j}(0,\rho_2,\tilde{w}_0(0,\rho_2)+v_0(\tau,\rho_2))$，和$\tilde{a}_{i,j}^{(\tau,k)}=\dfrac{\partial\tilde{a}_{i,j}(0,\rho_2,\tilde{w}_0+v_0)}{\partial u}$

$$\cdot\left[v_k(\tau,\rho_2)+\widetilde{w}_k(0,\rho_2)\right]+\widetilde{a}_{i,j}^{(\tau,k,r)}(\tau,\rho_2,\widetilde{w}_0,v_0,\cdots,\widetilde{w}_{k-1},v_{k-1})$$ 其中

$$\widetilde{a}_{i,j}^{(\tau,k,r)}=\sum_{\substack{m_1+l_1+\cdots+pl_p=k\\(p<k)}}\frac{1}{m_1!l_1!\cdots l_p!}\frac{\partial^{m_1+l_1+\cdots+l_p}\widetilde{a}_{i,j}(0,\rho_2,\widetilde{w}_0+v_0)}{\partial\rho_1^{m_1}\partial u^{l_1+\cdots+l_p}}$$

$$\cdot\tau^{m_1}\left(\frac{\partial U_m}{\partial\varepsilon}\right)^{l_1}\cdots\left(\frac{\partial^p U_m}{\partial\varepsilon^p}\right)^{l_p}\Bigg|_{\varepsilon=0}+\frac{\partial\widetilde{a}_{i,j}(0,\rho_2,\widetilde{w}_0+v_0)}{\partial u}\sum_{l=1}^{k}\frac{1}{l!}\frac{\partial^l\widetilde{w}_{k-l}}{\partial\rho_1^l}\tau^l$$

$$\frac{\partial^i U_m}{\partial\varepsilon^i}\bigg|_{\varepsilon=0}=i!\left[\sum_{l=1}^{i}\frac{1}{l!}\frac{\partial^l\widetilde{w}_{i-l}}{\partial\rho_1^l}\tau^l+(v_i(\tau,\rho_2)+\widetilde{w}_i(0,\rho_2))\right],\quad(i=1,2,\cdots,p)$$

$$\widetilde{N}_\varepsilon[U_m]\equiv\widetilde{N}_\varepsilon[\widetilde{W}_m]+\widetilde{N}_\varepsilon[\widetilde{W}_m+V_m]-\widetilde{N}_\varepsilon[\widetilde{W}_m]\tag{2.2.6}$$

顾及$\widetilde{N}_\varepsilon[\widetilde{W}_m]=O(\varepsilon^{m+1})$，在(2.2.6)式令$\varepsilon^i$，$(i=-1,0,\cdots,m)$的系数为零，则得到关于$v_i$，$(i=0,1,\cdots,m+1)$的递推方程：

$$\widetilde{a}_{1,1}(0,\rho_2,\widetilde{w}_0+v_0)\frac{\partial^2 v_0}{\partial\tau^2}-\alpha(0,\rho_2)\frac{\partial v_0}{\partial\tau}=0\tag{2.2.7}$$

$$\widetilde{a}_{1,1}(0,\rho_2,\widetilde{w}_0+v_0)\frac{\partial^2 v_1}{\partial\tau^2}-\alpha(0,\rho_2)\frac{\partial v_1}{\partial\tau}+\frac{\partial\widetilde{a}_{1,1}(0,\rho_2,\widetilde{w}_0+v_0)}{\partial u}\frac{\partial^2 v_0}{\partial\tau^2}v_1$$

$$=-\frac{\partial\widetilde{a}_{1,1}(0,\rho_2,\widetilde{w}_0+v_0)}{\partial u}\frac{\partial^2 v_0}{\partial\tau^2}\widetilde{w}_1-\left(\frac{\partial\widetilde{a}_{1,1}(0,\rho_2,\widetilde{w}_0+v_0)}{\partial u}\frac{\partial\widetilde{w}_0}{\partial\rho_1}\tau\right.$$

$$\left.+\frac{\partial\widetilde{a}_{1,1}(0,\rho_2,\widetilde{w}_0+v_0)}{\partial\rho_1}\tau\right)\frac{\partial^2 v_0}{\partial\tau^2}+\frac{\partial\alpha(0,\rho_2)}{\partial\rho_1}\tau\frac{\partial v_0}{\partial\tau}-2\widetilde{a}_{1,2}(0,\rho_2,\widetilde{w}_0+v_0)$$

$$-\widetilde{a}_1(0,\rho_2,\widetilde{w}_0+v_0)\frac{\partial v_0}{\partial\tau}+\beta(0,\rho_2)\frac{\partial v_0}{\partial\tau}+\widetilde{A}(0,\rho_2,\widetilde{w}_0+v_0)-\widetilde{A}(0,\rho_2,\widetilde{w}_0)$$

$$\equiv G_1(\rho_2,\widetilde{w}_0,v_0)\tag{2.2.8}$$

$$\widetilde{a}_{1,1}(0,\rho_2,\widetilde{w}_0+v_0)\frac{\partial^2 v_k}{\partial\tau^2}-\alpha(0,\rho_2)\frac{\partial v_k}{\partial\tau}+\frac{\partial\widetilde{a}_{1,1}(0,\rho_2,\widetilde{w}_0+v_0)}{\partial u}\frac{\partial^2 v_0}{\partial\tau^2}v_k$$

$$=-\frac{\partial\widetilde{a}_{1,1}}{\partial u}\frac{\partial^2 v_0}{\partial\tau^2}\widetilde{w}_k-\widetilde{a}_{1,1}^{(\tau,k,r)}\frac{\partial^2 v_0}{\partial\tau^2}+\alpha^{(k)}\frac{\partial v_0}{\partial\tau}-\sum_{l=1}^{k-1}\left(\widetilde{a}_{1,1}^{(\tau,l)}\frac{\partial^2 v_{k-l}}{\partial\tau^2}\right.$$

$$\left.+\alpha^{(l)}\frac{\partial v_{k-l}}{\partial\tau}\right)-\sum_{l=0}^{k-1}\left(2\widetilde{a}_{1,2}^{(\tau,l)}\frac{\partial^2 v_{k-1-l}}{\partial\tau\partial\rho_2}+\widetilde{a}_1^{(\tau,l)}\frac{\partial v_{k-1-l}}{\partial\tau}-\beta^{(l)}\frac{\partial v_{k-1-l}}{\partial\rho_2}\right)$$

$$-\sum_{l=0}^{k-2}\left(\widetilde{a}_{2,2}^{(\tau,l)}\frac{\partial^2 v_{k-2-l}}{\partial\rho_2^2}+\widetilde{a}_2^{(\tau,l)}\frac{\partial v_{k-2-l}}{\partial\rho_2}\right)-\widetilde{a}^{(\tau,k-2,\alpha)}-\sum_{l=0}^{k-2}\left(\sum_{i,j=1}^{2}\widetilde{a}_{i,j}^{(\tau,l,\alpha)}\right.$$

$$\left.\cdot\frac{\partial^2 w_{k-2-l}}{\partial\rho_i\partial\rho_j}+\sum_{i=1}^{2}\widetilde{a}_i^{(\tau,l,\alpha)}\frac{\partial\widetilde{w}_{k-2-l}}{\partial\rho_i}\right)+\widetilde{A}^{(\tau,k-1,\alpha)}\equiv G_k,\quad(k=2,3,\cdots,m+1)\tag{2.2.9}$$

其中$\alpha^{(k)}$、$\beta^{(k)}$表示$\alpha(\varepsilon\tau,\rho_2)$、$\beta(\varepsilon\tau,\rho_2)$按$\varepsilon$展开的$\varepsilon^k$幂的系数；

$a^{(k)}=\frac{1}{k!}\frac{\partial^k a(0,\rho_2)}{\partial\rho_1{}^k}\tau^k,\cdots,\tilde{a}_{i,j}^{(r,l,\alpha)}=\tilde{a}_{i,j}^{(r,l)}(0,\rho_2,\tilde{w}_0+v_0)-\tilde{a}_{i,j}^{(r,l)}(0,\rho_2,\tilde{w}_0)$ 等等.

再给出$v_k,(k=0,1,\cdots,m+1)$的边界条件以抵消$\tilde{w}_k$在$\partial\Omega_+$上的值

$$v_0\Big|_{\tau=0}=\tilde{\varphi}(\rho_2)-\tilde{w}_0(0,\rho_2),\quad \lim_{\tau\to\infty}v_0=0,\quad (0<\rho_2<\rho_2^{(2)}) \tag{2.2.10}$$

$$v_k\Big|_{\tau=0}=-\tilde{w}_k(0,\rho_2),\quad \lim_{\tau\to\infty}v_k=0,\quad (0<\rho_2<\rho_2^{(0)}),\quad (k=1,2,\cdots,m+1) \tag{2.2.11}$$

在上式中取$\tilde{w}_{m+1}\equiv 0$.

作假设

(H_2): $\tilde{a}_{1,1}(0,\rho_2,w_0(0,\rho_2)+Z)\geqslant\delta_4>0$, $\int_0^z\frac{dt}{\tilde{a}_{1,1}(0,\rho_2,w_0+t)}>0$, 其中$0<|Z|\leqslant|\tilde{\varphi}(\rho_2)-\tilde{w}_0(0,\rho_2)|$, $\mathrm{sign}(z)=\mathrm{sign}(\tilde{\varphi}(\rho_2)-\tilde{w}_0(0,\rho_2))$.

应用解常微分方程的方法求得(2.2.7)、(2.2.10)的解为

$$\tau=\int_{v_0}^{\tilde{\varphi}(\rho_2)-\tilde{w}_0(0,\rho_2)}\left(\int_0^z\frac{-a(0,\rho_2)}{\tilde{a}_{1,1}(0,\rho_2,\tilde{w}_0+t)}dt\right)^{-1}dz,\quad (0<\rho_2<\rho_2^{(0)})$$

可以证明

$$v_0=C(\rho_2)\exp\left(\frac{a(0,\rho_2)}{\tilde{a}_{1,1}(0,\rho_2,\tilde{w}_0)}\tau\right)(1+o(1)),\quad 当\tau\to\infty$$

因$a(0,\rho_2)=\cos\,(\widehat{\rho_1,x_2})<0$, 所以当$\tau\to\infty$时$v_0\to 0$.

将v_0代入方程(2.2.8), 因$\frac{\partial v_0}{\partial\tau}$是(2.2.8)所对应的齐次方程的特解, 如果令$v_1=\frac{\partial v_0}{\partial\tau}\tilde{v}_1$, 则得到关于$\tilde{v}_1$的可降阶的方程:

$$\int_0^{v_0}\frac{a(0,\rho_2)}{\tilde{a}_{1,1}(0,\rho_2,\tilde{w}_0+t)}dt\left(\tilde{a}_{1,1}(0,\rho_2,\tilde{w}_0+v_0)\frac{\partial^2\tilde{v}_1}{\partial\tau^2}+a(0,\rho_2)\frac{\partial\tilde{v}_1}{\partial\tau}\right)=G_1(\rho_2,\tilde{w}_0,v_0)$$

解得

$$v_1=\left(\int_0^{v_0(\tau,\rho_2)}\frac{a(0,\rho_2)}{\tilde{a}_{1,1}(0,\rho_2,\tilde{w}_0+t)}dt\right)\left\{\int_0^\tau\int_0^z\Big[\tilde{a}_{1,1}(0,\rho_2,\tilde{w}_0+v_0(t,\rho_2))\right.$$

$$\cdot\int_0^{v_0(t,\rho_2)}\frac{a(0,\rho_2)}{\tilde{a}_{1,1}(0,\rho_2,\tilde{w}_0+s)}ds\Big]^{-1}G_1(\rho_2,\tilde{w}_0,v_0(t,\rho_2))\exp$$

$$\cdot\left(\int_t^z\frac{-a(0,\rho_2)}{\tilde{a}_{1,1}(0,\rho_2,\tilde{w}_0+v_0(s,\rho_2))}ds\right)dt\,dz-\tilde{w}_1(0,\rho_2)$$

$$\left.\cdot\left(\int_0^{\tilde{\varphi}(\rho_2)-\tilde{w}_0(0,\rho_2)}\frac{a(0,\rho_2)}{\tilde{a}_{1,1}(0,\rho_2,\tilde{w}_0+t)}dt\right)\right\}$$

因当$\tau\to\infty$时$G(\rho,\tilde{w}_0,v_0(\tau,\rho_2))\to 0$, 所以当$\tau\to\infty$时$v_1\to 0$; 但在切点$P_1$、$P_2$,因$a(0,\rho_2)=0$所以$v_1$将具有奇性. 将求得的$v_0$、$v_1$再代入方程(2.2.9)(取$k=2$), 又可以根据边值条件(2.2.11)求出$v_2$等等.

再引进无限次可微的切断函数;

$$X(t)=\begin{cases}1, & 当t\leqslant\frac{1}{2}\\ 0, & 当t\geqslant 1\end{cases} \tag{2.2.12}$$

作函数 $V_m^{(0)}=X\left(\frac{\rho_1}{\rho_1^{(0)}}\right)V_m$ 当 $(x_1,x_2)\in D_{\rho_1^{(0)}}$； $V_m^{(0)}=0$ 当 $(x_1,x_2)\in\bar{\Omega}\setminus D_{\rho_1^{(0)}}$，有定理

定理 1： 在正则四边形 $\Pi_{\eta_0}=\{(x_1,x_2)|c_1+\eta_0\leqslant x_1\leqslant c_2-\eta_0,\ x_{2(-)}(x_1)\leqslant x_2\leqslant x_{2(+)}(x_1)\}$ 中，

$$U_m=W_m(x_1,x_2;\varepsilon)+V_m^{(0)}(x_1,x_2;\varepsilon)$$

是狄立克雷问题(2.0.1)－(2.0.2)的形式渐近解，即成立

$$N_\varepsilon[U_m]=O(\varepsilon^{m+1}),\quad (x_1,x_2)\in\Pi_{\eta_0}$$

$$U_m\Big|_{\partial\Omega\cap\Pi_{\eta_0}}=\varphi\Big|_{\partial\Omega\cap\Pi_{\eta_0}}$$

证：在区域 $\Pi_{\eta_0}\cap D_{\frac{1}{2}\rho_1^{(0)}}$ 中，因 $U_m=\widetilde{W}_m+V_m$，所以

$$N_\varepsilon[U_m]=\tilde{N}_\varepsilon[\widetilde{W}_m]+\tilde{N}_\varepsilon[\widetilde{W}_m+V_m]-\tilde{N}_\varepsilon[\widetilde{W}_m]=O(\varepsilon^{m+1})+\varepsilon^{m+1}\Bigg[\Bigg(\sum_{l=1}^{m+2}\tilde{a}_{1,1}^{(\tau,l)}$$

$$\cdot\frac{\partial^2U_{m+2-l}}{\partial\tau^2}+\cdots+\sum_{l=0}^{m}\sum_{i,j=1}^{2}\tilde{a}_{i,j}^{(\tau,l)}\frac{\partial^2\tilde{v}_{m-l}}{\partial\rho_i\partial\rho_j}\Bigg)+\cdots+\varepsilon^{m+3}\Bigg(\tilde{a}_{2,2}^{(\tau,m+2)}\frac{\partial^2v_{m+1}}{\partial\rho_1^2}$$

$$+\tilde{a}_2^{(\tau,m+2)}\frac{\partial v_{m+1}}{\partial\rho_2}\Bigg)\Bigg]=O(\varepsilon^{m+1})$$

在区域 $\Pi_{\eta_0}\cap(D_{\rho_1^{(0)}}\setminus D_{\frac{1}{2}\rho_1^{(0)}})$ 中，$U_m=\widetilde{W}_m+XV_m$，所以

$$N_\varepsilon[U_m]=O(\varepsilon^{m+1})+\tilde{N}_\varepsilon[\widetilde{W}_m+XV_m]-\tilde{N}_\varepsilon[\widetilde{W}_m]=O(\varepsilon^{m+1})+\varepsilon^{-1}$$

$$\Bigg\{\Bigg[\tilde{a}_{1,1}(\varepsilon\tau,\rho_2,\widetilde{W}_m+XV_m)\frac{\partial^2(XV_m)}{\partial\tau^2}+(\tilde{a}_{1,1}(\varepsilon\tau,\rho_2,\widetilde{W}_m+XV_m)-\tilde{a}_{1,1}$$

$$(\varepsilon\tau,\rho_2,\widetilde{W}_m))\frac{\partial^2\widetilde{W}_m}{\partial\tau^2}-\alpha(\varepsilon\tau,\rho_2)\frac{\partial(XV_m)}{\partial\tau}\Bigg]+\varepsilon[\cdots]+\varepsilon^2[\cdots]\Bigg\}$$

因在该区域中 v_i，$(i=0,1,\cdots,m+1)$ 及其各阶导数是渐近地为零，又 $\tilde{a}_{1,1}(\varepsilon\tau,\rho_2,\widetilde{W}_m+XV_m)-\tilde{a}_{1,1}(\varepsilon\tau,\rho_2,\widetilde{W}_m)=\frac{\partial\tilde{a}_{1,1}(\widetilde{W}_m+\theta XV_m)}{\partial u}XV_m$，所以 $N_\varepsilon[U_m]=O(\varepsilon^{m+1})$. 在区域 $\Pi_{\eta_0}\cap(\bar{\Omega}\setminus D_{\rho_1^{(0)}})$ 中，因 $U_m=W_m$，所以 $N_\varepsilon[U_m]=O(\varepsilon^{m+1})$. 又在边界 $\Pi_{\eta_0}\cap\partial\Omega$ 上 U_m 显然取值 φ，定理证毕.

二·三 预备知识—$\widetilde{W}_m$ 和 V_m 在切点邻域的展开式

为了在切点的邻域构造边界层项，先考察 $\widetilde{W}_m$ 和 V_m 在切点邻域的展开式. 先考察 P_1 点的邻域 $D_1=\left\{(\rho_1,\rho_2)\,\Big|\,0\leqslant\rho_1\leqslant\rho_1^{(0)},\ |\rho_2|\leqslant\delta_0\right\}$，$\delta_0$ 是小于 $\frac{1}{2}\rho_2^{(2)}$、$\frac{1}{2}\rho_2^{(-2)}$ 的某正数. 作伸展变换

$$\xi=\frac{\rho_1}{\varepsilon_1^2},\qquad \eta=\frac{\rho_2}{\varepsilon_1},\qquad \left(\varepsilon_1=\varepsilon^{\frac{1}{3}}\right),$$

在(ξ,η)坐标系，确定w_0的方程(2.1.3)具有形式：

$$\alpha(\varepsilon_1^2\xi,\varepsilon_1\eta)\frac{\partial\hat{w}_0}{\partial\xi}+\varepsilon_1\beta(\varepsilon_1^2\xi,\varepsilon_1\eta)\frac{\partial\hat{w}_0}{\partial\eta}+\varepsilon_1^2\tilde{A}(\varepsilon_1^2\xi,\varepsilon_1\eta,\hat{w}_0)=0 \tag{2.3.1}$$

其中$\hat{w}_0$表示w_0在(ξ,η)坐标系的表示式．假设$\hat{w}_0$的展开式为

$$\hat{w}_0=\sum_{k=0}^{3m+1}\varepsilon_1^k\hat{w}_{0,k}+\varepsilon_1^{3m+2}r^{(0)} \tag{2.3.2}$$

代入方程(2.3.1)，并将各系数按ε_1展开

$$\alpha(\varepsilon_1^2\xi,\varepsilon_1\eta)=\sum_{i=0}^{3m+1}\varepsilon_1^i\hat{\alpha}^{(i)}+\varepsilon_1^{3m+2}\hat{\alpha}^{(3m+2)}(\theta\varepsilon_1),\quad(0<\theta<1) \tag{2.3.3}$$

$$\tilde{A}(\varepsilon_1^2\xi,\varepsilon_1\eta,\hat{w}_0)=\sum_{i=0}^{3m+1}\varepsilon_1^i\hat{A}^{(i)}+\varepsilon_1^{3m+2}\hat{A}^{(3m+2)}(\theta\varepsilon_1) \tag{2.3.4}$$

其中 $\hat{\alpha}^{(0)}=\alpha(0,0)=\cos\frac{\pi}{2}=0$，　$\hat{\alpha}^{(1)}=\frac{\partial\alpha(0,0)}{\partial\rho_2}\eta$，　$\hat{\alpha}^{(i)}=\sum\limits_{2l_1+l_2=i}\frac{1}{(2l_1)!\,l_2!}$ $\cdot\frac{\partial^{l_1+l_2}\alpha(0,0)}{\partial\rho_1^{l_1}\partial\rho_2^{l_2}}(2\xi)^{l_1}\eta^{l_2}$；$\hat{A}^{(0)}=\tilde{A}(0,0,\hat{w}_{0,0})$，$\hat{A}^{(1)}=\frac{\partial\tilde{A}(0,0,\hat{w}_{0,0})}{\partial u}\hat{w}_{0,1}+\frac{\partial\tilde{A}(0,0,\hat{w}_{0,0})}{\partial\rho_2}\eta$，

$\hat{A}^{(i)}=\frac{\partial\tilde{A}(0,0,\hat{w}_{0,0})}{\partial u}\hat{w}_{0,i}+\hat{A}^{(i,r)}(\xi,\eta,\hat{w}_{0,0},\cdots,\hat{w}_{0,i-1})$，而 $\hat{A}^{(i,r)}=\sum\limits_{\substack{2m_1+m_2+l_1+\cdots+pl_p=i\\(p<i)}}$

$\frac{1}{(2m_1)!\,m_2!\,l_1!\cdots l_p!}\ \frac{\partial^{m_1+\cdots+l_p}\tilde{A}(0,0,\hat{w}_{0,0})}{\partial\rho_1^{m_1}\partial\rho_2^{m_2}\partial u^{l_1+\cdots+l_p}}(2\xi)^{m_1}\eta^{m_2}\hat{w}_{0,1}^{l_1}\cdots\hat{w}_{0,p}^{l_p}$，　$(i\ 2,3,\cdots,m+1)$ 等等．再令 ε_1^k，$(k=1,2,\cdots,3m+2)$，的系数为零，得到关于 $\hat{w}_{0,k}$，$(k=0,1,\cdots,3m+1)$ 的递推方程（为了讨论确定起见，假设$\frac{\partial\alpha(0,0)}{\partial\rho_2}\neq0$，即边界和特征在 P_1、P_2是零阶相切，关于高阶相切的情形，可以类似地讨论．由于在 P_1 点的邻域当 ρ_1 由负值增大到正值时，$\cos(\widehat{\rho_1,x_1})$ 由正值减小到负值，所以$\frac{\partial\alpha(0,0)}{\partial\rho_2}=-b<0$)：

$$-b\eta\frac{\partial\hat{w}_{0,0}}{\partial\xi}+\beta_0\frac{\partial\hat{w}_{0,0}}{\partial\eta}=0,\quad(\beta_0\equiv\beta(0,0)) \tag{2.3.5}$$

$$-b\eta\frac{\partial\hat{w}_{0,k}}{\partial\xi}+\beta_0\frac{\partial\hat{w}_{0,k}}{\partial\eta}=-\sum_{i=1}^{k}\left(\hat{\alpha}^{(i+1)}\frac{\partial\hat{w}_{0,k-i}}{\partial\xi}+\beta^{(i)}\frac{\partial\hat{w}_{0,k-i}}{\partial\eta}\right)-\hat{A}^{(k-1)}$$
$$\equiv F_{0,k}(\xi,\eta,\hat{w}_{0,0},\cdots,\hat{w}_{0,k-1}),\quad(k=1,2,\cdots,3m+1) \tag{2.3.6}$$

又从(2.1.6)得到$\hat{w}_{0,k}$的边值条件：

$$\hat{w}_{0,k}\Big|_{\xi=0}=\frac{\eta^k}{k!}\tilde{\varphi}^{(k)}(0),\quad\left(-\frac{\delta_0}{\varepsilon_1}\leqslant\eta<0\right),\quad(k=0,1,\cdots,3m+1) \tag{2.3.7}$$

作坐标变换

$$\sigma=\xi+\frac{b}{2\beta_0}\eta^2,\qquad t=\frac{\eta}{\beta_0} \tag{2.3.8}$$

递推方程和边值条件化为：$\dfrac{\partial\hat{\omega}_{0,0}}{\partial t}=0$，$\hat{\omega}_{0,0}\Big|_{t=-\sqrt{\frac{2\sigma}{b\beta_0}}}=\tilde{\varphi}(0)$；和

$$\frac{\partial\hat{\omega}_{0,k}}{\partial t}=\tilde{F}_{0,k}(\sigma,t,\hat{\omega}_{0,0},\cdots,\hat{\omega}_{0,k-1})$$

$$\hat{\omega}_{0,k}\Big|_{t=-\sqrt{\frac{2\sigma}{b\beta_0}}}=\left(-\sqrt{\frac{2\beta_0\sigma}{b}}\right)^k\frac{\tilde{\varphi}^{(k)}(0)}{k!}\ (k=0,1,\cdots,3m+1)$$

可以逐步地解得

$$\hat{\omega}_{0,0}=\tilde{\varphi}(0),\ \hat{\omega}_{0,1}=-\tilde{A}(0,0,\tilde{\varphi}(0))\left(t+\sqrt{\frac{2\sigma}{b\beta_0}}\right)-\sqrt{\frac{2\beta_0\sigma}{b}}\ \tilde{\varphi}^1(0)$$

等等．为了今后讨论方便起见，将 $\sigma^{\frac{1}{2}}$和 t的q次齐次多项式所组成的集合记作 R_q，它的元素是 $R_q^{(i)}=\sum\limits_{l+m=q}a_{l,m}^{(i)}\ \sigma^{\frac{l}{2}}t^m$．易知具有性质：

(i) $R_{q_1}^{(i)}\ R_{q_2}^{(j)}\in R_{q_1+q_2}$，$\dfrac{\partial R_q^{(i)}}{\partial\sigma}\in R_{q-2}$，$\dfrac{\partial R_q^{(i)}}{\partial t}\in R_{q-1}$

(ii) $R_q^{(i)}=O\left(\sigma^{\frac{q}{2}}\right)=O\left(\varepsilon_1^{-q}\Phi^{\frac{q}{2}}\right)$，$\left(\Phi\equiv\rho_1+\dfrac{b}{2\beta_0}\ \rho_2^2\right)$

可以应用数学归纳法证知 $\hat{\omega}_{0,k}\in R_k$；事实上，若 $\hat{\omega}_{0,j}\in R_j$，$(j=0,1,\cdots,k-1)$，则 $\hat{\alpha}^{(i+1)}\in R_{i+1}$，$\beta^{(i)}\in R_i$，$\dfrac{\partial\hat{\omega}_{0,k-i}}{\partial\sigma}\in R_{k-i-2}$等等，所以$\hat{\omega}_{0,k}\in R_k$．

下面再估计余项$r^{(0)}$．将(2.3.2)式代入关于$\hat{\omega}_0$的方程(2.3.1)和边值条件得

$$\alpha(\rho_1,\rho_2)\frac{\partial r^{(0)}}{\partial\rho_1}+\beta(\rho_1,\rho_2)\frac{\partial r^{(0)}}{\partial\rho_2}=H^{(0)}(\rho_1,\rho_2;\varepsilon_1) \tag{2.3.9}$$

$$r^{(0)}\Big|_{\rho_1=0}=\varepsilon_1^{-(3m+2)}\ \frac{\rho_2^{3m+2}}{(3m+2)!}\tilde{\varphi}^{(3m+2)}(\theta\rho_2),\quad(0<\theta<1) \tag{2.3.10}$$

其中$H^{(0)}(\rho_1,\rho_2;\varepsilon_1)$在$(\xi,\eta)$坐标系的表示式是

$$H^{(0)}=\varepsilon_1^{-1}\left\{\left[\sum_{i=1}^{3m+1}\left(\hat{\alpha}^{(i+1)}\ \frac{\partial\hat{\omega}_{0,3m+2-i}}{\partial\xi}+\beta^{(i)}\frac{\partial\hat{\omega}_{0,3m+2-i}}{\partial\eta}\right)+\hat{A}^{(3m+1)}\right]\right.$$

$$\left.+\varepsilon_1\left[\sum_{i=1}^{3m}\left(\hat{\alpha}^{(i+2)}\frac{\partial\hat{\omega}_{0,3m+2-i}}{\partial\xi}+\beta^{(i+1)}\ \frac{\partial\omega_{0,3m+2-i}}{\partial\eta}+\hat{A}^{(3m+2)}\right]+\cdots\right\}$$

易知$H^{(0)}=O\left(\varepsilon_1^{-(3m+2)}\ \Phi^{\frac{3m+1}{2}}\right)$，所以

$$r^{(0)}=\int_{x_{2(-)}(x_1)}^{x_2}H^{(0)}(\rho_1(x_1,x_2),\rho_2(x_1,x_2),w_0;\varepsilon_1)\,dx_2$$

$$+\varepsilon_1^{-(3m+2)}\frac{[\rho_2(x_1,x_{2(-)}(x_1))]^{3m+2}}{(3m+2)!}\tilde{\varphi}^{(3m+2)}(\theta\rho_2)=O\left(\varepsilon_1^{-(3m+2)}\Phi^{\frac{3m+2}{2}}\right)\qquad(2.3.11)$$

上面利用了关系式：$|x_2-x_{2(-)}(x_1)|\leqslant 2|\rho_2|=O(\Phi^{\frac{1}{2}})$．再将上式对$x_1$和$x_2$求偏导，考虑到

$$\left|\frac{dx_{2(-)}(x_1)}{dx_1}\right|=\left|-\frac{\partial\rho_1(x_1,x_{2(-)}(x_1))}{\partial x_1}\Big/\frac{\partial\rho_1(x_1,x_{2(-)}(x_1))}{\partial x_2}\right|\leqslant\frac{1}{|a(0,\varepsilon_1\eta)|}=O\left(\Phi^{\frac{-1}{2}}\right),$$

可以证明

$$\frac{\partial^l r^{(0)}}{\partial x_1^{l_1}\partial x_2^{l_2}}=O\left(\varepsilon_1^{-(3m+2)}\Phi^{\frac{3m+2-2l_1-l_2}{2}}\right)\qquad(2.3.12)$$

一般地，应用数学归纳法可以证明$\hat{\omega}_i$具有展开式

$$\hat{\omega}_i=\varepsilon_1^{-3i}\left(\sum_{j=0}^{3m+1}\varepsilon_1^j\hat{\omega}_{i,j}+\varepsilon_1^{3m+2}r^{(i)}\right),\quad(i=0,1,\cdots,m)\qquad(2.3.13)$$

其中$\hat{\omega}_{i,0}=0$，$\hat{\omega}_{i,j}\in R_{j-3i}$，和成立估计式

$$\frac{\partial^l r^{(i)}}{\partial x_1^{l_1}\partial x_2^{l_2}}=O\left(\varepsilon_1^{-(3m+2-3i)}\Phi^{\frac{3m+2-3i-2l_1-l_2}{2}}\right)\qquad(2.3.14)$$

事实上，若假设上面的断言对于$i=0,1,\cdots,k-1$皆成立，考察关于$\hat{\omega}_k$的定解问题

$$a(\varepsilon_1^2\xi,\varepsilon_1\eta)\frac{\partial\hat{\omega}_k}{\partial\xi}+\varepsilon_1\beta(\varepsilon_1^2\xi,\varepsilon_1\eta)\frac{\partial\hat{\omega}_k}{\partial\eta}+\varepsilon_1^2\frac{\partial\bar{A}(\varepsilon_1^2\xi,\varepsilon_1\eta,\hat{\omega}_0)}{\partial u}\hat{\omega}_k$$

$$=\varepsilon_1^{-2}\left\{\sum_{l=0}^{k-1}\left[\tilde{a}_{1,1}^{(l)}\frac{\partial^2\hat{\omega}_{k-1-l}}{\partial\xi^2}+2\varepsilon_1\tilde{a}_{1,2}^{(l)}\frac{\partial^2\hat{\omega}_{k-1-l}}{\partial\xi\partial\eta}+\varepsilon_1^2\left(\tilde{a}_{2,2}^{(l)}\frac{\partial^2\hat{\omega}_{k-1-l}}{\partial\eta^2}+\tilde{a}_1^{(l)}\frac{\partial\hat{\omega}_{k-1-l}}{\partial\xi}\right)\right.\right.$$

$$\left.\left.+\varepsilon_1^3\tilde{a}_2^{(l)}\frac{\partial\hat{\omega}_{k-1-l}}{\partial\eta}\right]+\varepsilon_1^4\tilde{a}^{(k-1)}\right\}+\varepsilon_1^2\bar{A}^{(k,r)}\qquad(2.3.15)$$

$$\hat{\omega}_k\Big|_{\xi=0}=0\qquad(2.3.16)$$

假设$\hat{\omega}_k$的展开式是

$$\hat{\omega}_k=\varepsilon_1^{-3k}\left(\sum_{j=0}^{3m+1}\varepsilon_1^j\hat{\omega}_{k,j}+\varepsilon_1^{3m+2}r^{(k)}\right)\qquad(2.3.17)$$

代入(2.3.15)－(2.3.16)，并将各系数按ε_1展开（参见(2.3.4)式）

$$\tilde{a}_{1,1}^{(l)}=\varepsilon_1^{-3l}\sum_{i=0}^{3m+1}\varepsilon_1^i\hat{\tilde{a}}_{1,1}^{(l)(i)}+\varepsilon_1^{3m+2}\hat{\tilde{a}}_{1,1}^{(l)(3m+2)}(\theta\varepsilon_1),\quad(0<\theta<1)\qquad(2.3.18)$$

其中$\hat{\tilde{a}}_{1,1}^{(0)(0)}=\tilde{a}_{1,1}(0,0,\tilde{\varphi}(0)),\cdots,\hat{\tilde{a}}_{1,1}^{(0)(i)}=\dfrac{\partial\tilde{a}_{1,1}(0,0,\tilde{\varphi}(0))}{\partial u}\hat{\omega}_{0,i}+\sum\limits_{\substack{2m_1+m_2+l_1+\cdots+pl_p=i\\(p<i)}}$

$$\frac{1}{(2m_1)!\cdots l_p!}\frac{\partial^{m_1+\cdots+l_p}\tilde{a}_{1,1}(0,0,\tilde{\varphi}(0))}{\partial\rho_1^{m_1}\partial\rho_2^{m_2}\partial u^{l_1+\cdots+l_p}}(2\xi)^{m_1}\eta^{m_2}\hat{\omega}_{0,1}^{l_1}\cdots\hat{\omega}_{0,p}^{l_p},$$ 以及

$$\hat{\tilde{a}}_{1,1}^{(l)(i)}=\sum_{j=0}^{i}\hat{\tilde{a}}_{1,1,u}^{(j)}\hat{\omega}_{l,i-j}+\sum_{l_1+\cdots+pl_p=l}\frac{1}{l_1!\cdots l_p!}\sum_{j=0}\hat{\tilde{a}}_{1,1,u(l_1\cdots l_p)}^{(j)}B_{-j},$$

其中 $\hat{\alpha}^{(l)}_{1,1,u}$，$\hat{\alpha}^{(l)}_{1,1,u(l_1\cdots l_p)}$ 分别表示 $\dfrac{\partial\tilde{\alpha}_{1,1}(\varepsilon_1^2\xi,\varepsilon_1\eta,\hat{\omega}_0)}{\partial u}$、$\dfrac{\partial^{l_1+\cdots+l_p}\tilde{\alpha}_{1,1}(\varepsilon_1^2\xi,\varepsilon_1\eta,\hat{\omega}_0)}{\partial u^{l_1+\cdots+l_p}}$ 关于 ε_1 的展开式中 ε_1^j 项的系数，B_j 表示乘积 $\prod\limits_{\alpha=1}^{p}(\hat{\omega}_{\alpha,0}+\varepsilon_1\hat{\omega}_{\alpha,1}+\cdots+\varepsilon_1^{3m+1}\hat{\omega}_{\alpha,3m+1}+\varepsilon_1^{3m+2}r^{(\alpha)})^{l_\alpha}$ 中 ε_1^j 项的系数：

$$B_j=\sum_{t_1+\cdots+t_p=j}\prod_{\alpha=1}^{p}\left(\sum_{\substack{i_1+2i_2+\cdots+t_\alpha i_{t_\alpha}=t_\alpha\\(i_0+\cdots+i_{t_\alpha}=l_\alpha)}}\frac{l_\alpha!}{i_0!\cdots i_{t_\alpha}!}\hat{\omega}_{\alpha,0}^{i_0}\cdots\hat{\omega}_{\alpha,t_\alpha}^{i_{t_\alpha}}\right)\tag{2.3.19}$$

等等. 再令前 $3m+2$ 项 ε_1 的各次幂的系数为零，得到关于 $\hat{\omega}_{k,j}$ 的递推方程和定解条件：

$$-b\eta\frac{\partial\hat{\omega}_{k,0}}{\partial\xi}+\beta_0\frac{\partial\hat{\omega}_{k,0}}{\partial\eta}=\tilde{\alpha}_{1,1}(0,0,\tilde{\varphi}(0))\frac{\partial^2\hat{\omega}_{k-1,0}}{\partial\xi^2}\tag{2.3.20}$$

$$-b\eta\frac{\partial\hat{\omega}_{k,1}}{\partial\xi}+\beta_0\frac{\partial\hat{\omega}_{k,1}}{\partial\eta}=\tilde{\alpha}_{1,1}(0,0,\tilde{\varphi}(0))\frac{\partial^2\hat{\omega}_{k-1,1}}{\partial\xi^2}\tag{2.3.21}$$

$$\begin{aligned}-b\eta\frac{\partial\hat{\omega}_{k,j}}{\partial\xi}+\beta_0\frac{\partial\hat{\omega}_{k,j}}{\partial\eta}=&-\sum_{i=1}^{j}\left(\hat{\alpha}^{(i+1)}\frac{\partial\hat{\omega}_{k,j-i}}{\partial\xi}+\hat{\beta}^{(i)}\frac{\partial\hat{\omega}_{k,j-i}}{\partial\eta}\right)+\sum_{i=0}^{j-1}\\ &\hat{A}_u^{(i)}\hat{\omega}_{k,j-1-i}+\sum_{q=0}^{k-1}\Bigg[\sum_{i=0}^{j}\hat{\alpha}^{(q)(i)}_{1,1}\frac{\partial^2\hat{\omega}_{k-1-q,j-i}}{\partial\xi^2}+2\sum_{i=0}^{j-1}\hat{\alpha}^{(q)(i)}_{1,2}\frac{\partial^2\hat{\omega}_{k-1-q,j-1-i}}{\partial\xi\partial\eta}\\ &+\sum_{i=0}^{j-2}\left(\hat{\alpha}^{(q)(i)}_{2,2}\frac{\partial^2\hat{\omega}_{k-1-q,j-2-i}}{\partial\eta^2}+\hat{\alpha}^{(q)(i)}_{1}\frac{\partial\hat{\omega}_{k-1-q,j-2-i}}{\partial\xi}\right)+\sum_{i=0}^{j-3}\hat{\alpha}^{(q)(i)}_{2}\frac{\partial\hat{\omega}_{k-1-q,j-3-i}}{\partial\eta}\Bigg]\\ &+\hat{\alpha}^{(k-1)(j-4)}+\sum_{\substack{l_1+\cdots+pl_p=k\\(p<k)}}\frac{1}{l_1!\cdots l_p!}\sum_{q=0}^{j-1}\hat{A}^{(q)}_{u(l_1\cdots l_p)}B_{j-1-q}\equiv F_{k,j}\end{aligned}\tag{2.3.22}$$

$$\hat{\omega}_{k,j}\Big|_{\xi=0}=0,\qquad(j=2,3,\cdots,3m+1)\tag{2.3.23}$$

从(2.3.20)、(2.3.23)知 $\hat{\omega}_{k,0}=0$. 再引入 (σ,t) 坐标系，因 $F_{k,1}\in R_{-3}$，所以 $\hat{\omega}_{k,1}\in R_{1-3k}$. 如此，可以逐步地推知 $\hat{\omega}_{k,j}\in R_{j-3k}$. 又 $r^{(k)}$ 确定于下面的定解问题：

$$\alpha(\rho_1,\rho_2)\frac{\partial r^{(k)}}{\partial\rho_1}+\beta(\rho_1,\rho_2)\frac{\partial r^{(k)}}{\partial\rho_2}+\tilde{A}_u(\rho_1,\rho_2,\tilde{\omega}_0)r^{(k)}=H^{(k)}(\rho_1,\rho_2;\varepsilon_1)$$

$$r^{(k)}\Big|_{\rho_1=0}=0$$

其中 $H^{(k)}$ 在 (ξ,η) 坐标系的表示式是

$$\begin{aligned}H^{(k)}=\varepsilon_1^{-1}\Bigg\{&-\left[\sum_{i=1}^{3m+1}\hat{\alpha}^{(i+1)}\frac{\partial\hat{\omega}_{k,3m+2-i}}{\partial\xi}+\varepsilon\sum_{i=1}^{3m}\hat{\alpha}^{(i+2)}\frac{\partial\hat{\omega}_{k,3m+2-i}}{\partial\xi}+\cdots\right]\\ &+\sum_{q=0}^{k-1}\Bigg[\Bigg(\sum_{i=1}^{3m+2}\hat{\alpha}^{(q)(i)}_{1,1}\frac{\partial^2\hat{\omega}_{k-1-q,3m+2-i}}{\partial\xi^2}+\varepsilon\sum_{i=1}^{3m+1}\hat{\alpha}^{(q)(i+1)}_{1,1}\frac{\partial^2\hat{\omega}_{k-1-q,3m+2-i}}{\partial\xi^2}+\cdots\Bigg)\end{aligned}$$

$$+2\left(\sum_{i=0}^{3m+1}\hat{\tilde{a}}_{1,2}^{(q)(i)}\frac{\partial^2\hat{w}_{k-1-2,3m+1-i}}{\partial\xi\partial\eta}+\cdots\right)+\cdots\Big]+\left(\hat{\tilde{a}}^{(k-1)(3m-2)}\quad+\cdots\right)\Big\}$$

因$H^{(k)}=O\left(\varepsilon_1^{-(3m-3k+2)}\Phi^{\frac{3m+1-3k}{2}}\right)$，所以

$$r^{(k)}=\int_{x_{2(-)}(x_1)}^{x_2}H^{(k)}(\rho_1(x_1,t),\rho_2(x_1,t);\varepsilon_1)\exp\left(-\int_t^{x_2}A_u(x_1,s,w_0(x_1,s))ds\right)dt$$

类似$r^{(0)}$地可以证明

$$\frac{\partial^l r^{(k)}}{\partial x_1^{l_1}\partial x_2^{l_2}}=O\left(\varepsilon_1^{-(3m+2-3k)}\Phi^{\frac{3m+2-3k-2l_1-l_2}{2}}\right)\tag{2.3.24}$$

所以(2.3.13)、(2.3.14)式对于$i=k$也成立.

下面再求$v_i(\tau,\rho_2)$，$(i=0,1,\cdots,m+1)$，在D_1的展开式．在(ξ,η)坐标系下$\hat{v}^{(0)}$确定于下面的定解问题：

$$\varepsilon_1^2\tilde{a}_{1,1}(0,\varepsilon_1\eta,\tilde{w}_0(0,\varepsilon_1\eta)+v_0\left(\frac{\xi}{\varepsilon_1},\ \varepsilon_1\eta\right)\Big)\frac{\partial^2\hat{v}_0}{\partial\xi^2}-\varepsilon_1\alpha(0,\varepsilon_1\eta)\frac{\partial\hat{v}_0}{\partial\xi}=0\tag{2.3.25}$$

$$\hat{v}_0|_{\xi=0}=\tilde{\varphi}(\varepsilon_1\eta)-\tilde{w}_0(0,\varepsilon_1\eta)\tag{2.3.26}$$

假设$\hat{v}_0$的展开式是

$$\hat{v}_0=\sum_{k=0}^{3m+1}\varepsilon_1^k\hat{v}_{0,k}+\varepsilon_1^{3m+2}\psi^{(0)}\tag{2.3.27}$$

代入(2.3.25)—(2.3.26)，并将各系数按ε_1展开

$$\tilde{a}_{1,1}\left(0,\varepsilon_1\eta,\hat{w}_0(0,\varepsilon_1\eta)+\hat{v}_0\left(\frac{\xi}{\varepsilon_1},\varepsilon_1\eta\right)\right)=\sum_{i=0}^{3m+1}\varepsilon_1^i\hat{\tilde{a}}_{1,1}^{(\tau,i)}+\varepsilon_1^{3m+2}\hat{\tilde{a}}^{(\tau,3m+2)}(\theta\varepsilon_1)$$

其中$\hat{\tilde{a}}_{1,1}^{(\tau,0)}=\tilde{a}_{1,1}(0,0,\hat{w}_{0,0}+\hat{v}_{0,0})$，

$$\hat{\tilde{a}}_{1,1}^{(\tau,1)}=\frac{\partial\tilde{a}_{1,1}(0,0,\hat{w}_{0,0}+\hat{v}_{0,0})}{\partial u}(\hat{w}_{0,1}+\hat{v}_{0,1})+\frac{\partial\tilde{a}_{1,1}(0,0,\hat{w}_{0,0}+\hat{v}_{0,0})}{\partial\rho_2}\eta,$$

$$\hat{\tilde{a}}_{1,1}^{(\tau,i)}=\frac{\partial\tilde{a}_{1,1}(0,0,\hat{w}_{0,0}+\hat{v}_{0,0})}{\partial u}(\hat{w}_{0,i}+\hat{v}_{0,i})+\hat{\tilde{a}}_{1,1}^{(\tau,i,r)}(\eta,\hat{w}_{0,0}+\hat{v}_{0,0},\cdots,\hat{w}_{0,i-1}+\hat{v}_{0,i-1}),$$

而$\hat{\tilde{a}}^{(\tau,i,r)}=\sum\limits_{\substack{m_2+l_1+\cdots+pl_p=i\\(p<i)}}\dfrac{1}{m_2!l_1!\cdots l_p!}\eta^{m_2}(\hat{w}_{0,1}+\hat{v}_{0,1})^{l_1}\cdots(\hat{w}_{0,p}+\hat{v}_{0,p})^{l_p}$

等等；令ε_1的最低次幂的系数为零，得到确定$\hat{v}_{0,0}$的方程和边值条件：

$$\tilde{a}_{1,1}(0,0,\tilde{\varphi}(0)+\hat{v}_{0,0})\frac{\partial^2\hat{v}_{0,0}}{\partial\xi^2}+b\eta\frac{\partial\hat{v}_{0,0}}{\partial\xi}=0,\ \hat{v}_{0,0}|_{\xi=0}=0\tag{2.3.28}$$

所以$\hat{v}_{0,0}=0$．再令ε_1的较高次幂的系数为零，得到确定$\hat{v}_{0,k}$，$(k=1,2,\cdots,3m+1)$，的递推方程和边值条件：

$$\tilde{a}_{1,1}(0,0,\tilde{\varphi}(0))\frac{\partial^2\hat{v}_{0,k}}{\partial\xi^2}+b\eta\frac{\partial\hat{v}_{0,k}}{\partial\xi}=-\sum_{i=1}^{k-1}\left(\hat{\tilde{a}}_{1,1}^{(\tau,i)}\frac{\partial^2\hat{v}_{0,k-i}}{\partial\xi^2}-\hat{a}_0^{(i+1)}\frac{\partial\hat{v}_{0,k-i}}{\partial\xi}\right)\tag{2.3.29}$$

$$\hat{v}_{0,k}|_{\xi=0}=\frac{\eta^k}{k!}\tilde{\varphi}^{(k)}(0)-\hat{\omega}_{0,k}(0,\eta),\quad (k=1,2,\cdots,3m+1) \tag{2.3.30}$$

其中 $\hat{\alpha}_0^{(i+1)}=\hat{\alpha}^{(i+1)}(0,\eta)$. 在 (2.3.29)—(2.3.30) 中，令 $k=1$，解得

$$\hat{v}_{0,1}=e^{\frac{-b}{a_0}\xi\eta}\left(2\tilde{A}(0,0,\tilde{\varphi}(0))\frac{\eta}{\beta_0}+2\tilde{\varphi}'(0)\eta\right)$$

为了今后讨论方便起见，引进函数集合 $S_{q,n}$，其元素是

$$S_{q,n}^{(i)}=e^{\frac{-nb}{a_0}\xi\eta}\sum_{l-m=q}\beta_{l,m}^{(i)}\eta^l\xi^m$$

n 是正整数，易知具有性质：

(i) $\quad S_{q_1,n_1}^{(i)}S_{q_2,n_2}^{(j)}\in S_{q_1+q_2,\,n_1+n_2},\quad \frac{\partial S_{q,n}^{(i)}}{\partial\xi}\in S_{q+1,n},\quad \frac{\partial S_{q,n}^{(i)}}{\partial\eta}\in S_{q-1,n}$

(ii) $\quad S_{q,n}^{(i)}=O\left(\eta^q e^{\frac{-nb}{2a_0}\xi\eta}\right)=O\left(\varepsilon_1^{-q}\rho_2^q e^{\frac{-nb}{2a_0}\tau\rho_2}\right)$

因 $\hat{v}_{0,1}\in S_{1,1}$，可以应用数学归纳法证知 $\hat{v}_{0,k}\in S_{k,1}$.

再估计 $\psi^{(0)}$. 将 (2.3.27) 式代入 (2.3.25)—(2.3.26) 得到确定 $\psi^{(0)}$ 的定解问题：

$$\tilde{a}_{1,1}(0,\rho_2,\tilde{\omega}_0(0,\rho_2)+v_0(\tau,\rho_2))\frac{\partial^2\psi^{(0)}}{\partial\tau^2}-\alpha(0,\rho_2)\frac{\partial\psi^{(0)}}{\partial\tau}=K^{(0)} \tag{2.3.31}$$

$$\psi^{(0)}|_{\xi=0}=\varepsilon^{-(3m+2)}\frac{\rho_2^{(3m+2)}}{(3m+2)!}\tilde{\varphi}^{(3m+2)}(\theta\rho_2)-r^{(0)}\equiv k^{(0)} \tag{2.3.32}$$

其中$K^{(0)}$ 在 (ξ,η) 坐标系的表示式为

$$K^{(0)}\equiv\varepsilon_1^2\left[\left(\sum_{i=1}^{3m+1}\hat{\tilde{a}}_{1,1}^{(\tau,i)}\frac{\partial^2 v_{0,3m+2-i}}{\partial\xi^2}+\varepsilon_1\sum_{i=1}^{3m}\hat{\tilde{a}}_{1,1}^{(\tau,i+1)}\frac{\partial^2 V_{0,3m+2-i}}{\partial\xi^2}+\cdots\right)\right.$$

$$\left.-\left(\sum_{i=1}^{3m+1}\hat{\alpha}_0^{(i+1)}\frac{\partial\hat{v}_{0,3m+2-i}}{\partial\xi}+\cdots\right)\right]=O\left(\varepsilon_1^2\eta^{3m+4}e^{\frac{-b}{2a_0}\xi\eta}\right)$$

$$=O\left(\varepsilon_1^{-(3m+2)}\rho_2^{3m+4}e^{\frac{-b}{2a_0}\tau\rho_2}\right)$$

从(2.3.31)－(2.3.32)解得

$$\psi^{(0)}=\left(\int_0^\infty e^{-\int_0^r Q(t,\rho_2)dt}dr\right)^{-1}\left[\int_\tau^\infty e^{-\int_0^r Q(t,\rho_2)dt}dr\right.$$

$$\cdot\left(k^{(0)}+\int_0^\tau\int_0^r\frac{K^{(0)}}{\tilde{a}_{1,1}}e^{-\int_t^r Q(s,\rho_2)ds}dt\,dr\right)$$

$$\left.-\left(\int_0^\tau e^{-\int_0^r Q(t,\rho_2)dt}dr\right)\int_\tau^\infty\int_0^r\frac{K^{(0)}}{\tilde{a}_{1,1}}e^{-\int_t^r Q(s,\rho_2)ds}dt\,dr\right] \tag{2.3.33}$$

其中 $Q(\tau,\rho_2)=\dfrac{-\alpha(0,\rho_2)}{\tilde{a}_{1,1}(0,\rho_2,\tilde{\omega}_0+v_0)}>0$ 因 $\alpha(0,\rho_2)=-b\rho_2+\dfrac{1}{2}\dfrac{\partial^2\alpha(0,\theta\rho_2)}{\partial\rho_2^2}\rho_2^2$，所以当

ρ_2 充分小时（例如 $0<\rho_2\leqslant\frac{b}{m}$，其中 $m=\underset{0\leqslant\rho_2\leqslant\delta_0}{\mathrm{Sup}}\left|\frac{\partial^2 a(0,\rho_2)}{\partial\rho_2^2}\right|$）成立 $-a(0,\rho_2)\geqslant\frac{b}{2}\rho_2$，因此

$$Q(\tau,\rho_2)\geqslant\frac{b}{2M}\rho_2$$

其中 $M=\underset{0\leqslant\rho_2\leqslant\delta_0}{\mathrm{Sup}}|\tilde{a}_{1,1}(0,\rho_2,\tilde{w}_0+v_0)|$. 从(2.3.33)式知

$$\psi^{(0)}=O(\varepsilon_1^{-(3m+2)}\ \rho_2^{3m+2}\ e^{-\nu\rho_2\tau}),\left(\ \nu=\frac{b}{2M}\right)\tag{2.3.34}$$

将(2.3.33)式对 τ 和 ρ_2 求偏导数，可以类似地证知

$$\frac{\partial^{l_1+l_2}\psi^{(0)}}{\partial\tau^{l_1}\partial\rho_2^{l_2}}=O\ (\varepsilon_1^{-(3m+2)}\ \rho_2^{3m+2-l_2}\ e^{-\nu\rho_2\tau})\tag{2.3.35}$$

一般地，应用类似的证明(2.3.13)、(2.3.14)式的方法可以证明 $\hat{v}_i$，$(i=0,1,\cdots,m+1)$，具有展开式

$$\hat{v}_i=\varepsilon_1^{-3i}\left(\sum_{j=0}^{3m+1}\varepsilon_1^j\,\hat{v}_{i,j}+\varepsilon_1^{3m+2}\,\psi^{(i)}\right)$$

其中 $\hat{v}_{i,0}=0$，$\hat{v}_{i,j}\in\sum_{\mu=1}^{j-1}S_{j-3i,\mu}$，和成立估计式

$$\frac{\partial^{l_1+l_2}\psi^{(i)}}{\partial\tau^{l_1}\partial\rho_2^{l_2}}=O\ (\varepsilon_1^{-(3m+2-3i)}\ \rho_2^{3m+2-3i-l_2}\ e^{-\nu\rho_2\tau})$$

二·四　抛物型边界层

我们将只在 P_1 点的邻域构造边界层项，类似地可以得到 P_2 点邻域的边界层项．在 P_1 点的邻域 D_1 引进 (ξ,η) 坐标系，方程(2.0.1)和边值条件(2.0.2)化为

$$\begin{aligned}&\varepsilon_1^3\left(\tilde{a}_{1,1}(\varepsilon_1^2\xi,\varepsilon_1\eta,u)\frac{\partial^2 u}{\partial\xi^2}-a(\varepsilon_1^2\xi,\ \varepsilon_1\eta)\varepsilon_1^{-1}\frac{\partial u}{\partial\xi}-\beta\ (\varepsilon_1^2\xi,\varepsilon_1\eta)\frac{\partial u}{\partial\eta}\right)\\&\quad+\varepsilon\left(2\tilde{a}_{1,2}\frac{\partial^2 u}{\partial\xi\partial\eta}-\tilde{A}\ (\varepsilon_1^2\xi,\varepsilon_1\eta,u)\right)+\varepsilon_1^2\left(\tilde{a}_{2,2}\frac{\partial^2 u}{\partial\eta^2}+\tilde{a}_1\frac{\partial u}{\partial\xi}\right)\\&\quad+\varepsilon_1^3\tilde{a}_2\frac{\partial u}{\partial\eta}+\varepsilon_1^4\tilde{a}=0\end{aligned}\tag{2.4.1}$$

$$u|_{\xi=0}=\tilde{\varphi}(\varepsilon_1\eta),\quad|\eta|\leqslant\frac{\delta_0}{\varepsilon_1}\tag{2.4.2}$$

假设狄立克雷问题的解在 P_1 点的邻域具有展开式

$$Y_{3m}(\xi,\eta;\varepsilon_1)=\sum_{i=0}^{3m+1}\varepsilon_1^i y_i\tag{2.4.3}$$

代入(2.4.1)—(2.4.2)，并将各系数按 ε_1 展开（参见(2.3.3)、(2.3.4)），令 ε_1 的最低次幂

的系数为零，得到确定 y_0 的定解问题：

$$E[y_0]\equiv a_0\frac{\partial^2 y_0}{\partial\xi^2}+b\,\eta\,\frac{\partial y_0}{\partial\xi}-\beta_0\frac{\partial y_0}{\partial\eta}=0$$

$$y_0|_{\xi=0}=\tilde{\varphi}(0)$$

所以 $y_0=\tilde{\varphi}(0)=\hat{\omega}_{0,0}$. 再令 ε_1 的较高次幂的系数为零，得到确定 y_i，$(i=1,2,\cdots,3m+1)$的定解问题：

$$E[y_i]=\sum_{l=1}^{i-1}\left(\tilde{a}_{1,1}^{(l)}\frac{\partial^2 y_{i-l}}{\partial\xi^2}-\hat{a}^{(l+1)}\frac{\partial y_{i-l}}{\partial\xi}-\beta^{(l)}\frac{\partial y_{i-l}}{\partial\eta}\right)$$

$$-\left(2\sum_{l=0}^{i-2}\tilde{a}_{1,2}^{(l)}\frac{\partial^2 y_{i-1-l}}{\partial\xi\partial\eta}-\bar{A}^{(i-1)}\right)-\sum_{l=0}^{i-3}\left(\tilde{a}_{2,2}^{(l)}\frac{\partial^2 y_{i-2-l}}{\partial\eta^2}\right.$$

$$\left.+\tilde{a}_1^{(l)}\frac{\partial y_{i-2-l}}{\partial\xi}\right)-\sum_{l=0}^{i-4}\tilde{a}_2^{(l)}\frac{\partial y_{i-3-l}}{\partial\eta}-\tilde{a}^{(i-4)} \qquad (2.4.4)$$

$$y_i|_{\xi=0}=\frac{\eta^i}{i!}\tilde{\varphi}^{(i)}(0),\quad |\eta|\leqslant\frac{\delta_0}{\varepsilon_1} \qquad (2.4.5)$$

无法求此定解问题的解，我们采用文献[8]的方法考察它们的解在 P_1 点邻域的性质.

以D表示区域：$D=\{(\xi,\eta)|\xi\geqslant 0,\eta\leqslant -1|\}$，以$K_m(D)$表示在$D$中无限次可微和满足估计式

$$\frac{\partial^l v}{\partial\xi^{l_1}\partial\eta^{l_2}}=0\left[\left(\frac{2\beta_0}{b}\xi+\eta^2\right)^{\frac{-m+l_2+2l_1}{2}}\right] \qquad (2.4.6)$$

的函数 v 所组成的集合，有下面的引理

引理1 （[8]的引理6)：设 $F(\xi,\eta)\in K_r(D)$，r是某正整数，则存在 $v(\xi,\eta)\in K_{r-1}(D)$ 使

$$E[v]=F,\qquad v|_{\xi=0}=0 \qquad (2.4.7)$$

定理 2：定解问题 (2.4.4)—(2.4.5)，$(i=1,2,\cdots,3m+1)$，当 $\xi>0$ 时存在无限次可微的解 y_i，并且当 $\eta\leqslant -1$ 时成立

$$y_i=\sum_{j=0}^{N}y_{i,j}(\sigma,t)+\beta_N^{(i)} \qquad (2.4.8)$$

其中 $y_{i,j}\in R_{i-3j}$，$\beta_N^{(i)}\in K_{3N+2-6(i-1)}$. N是正整数.

证：取 $i=1$，y_1 确定于定解问题（在 (σ,t) 坐标系)：

$$a_0\frac{\partial^2 y_1}{\partial\sigma^2}-\frac{\partial y_1}{\partial t}=\bar{A}(0,0,\tilde{\varphi}(0)),\quad y_1\Big|_{t=-\sqrt{\frac{2\sigma}{b\beta_0}}}=-\sqrt{\frac{2\beta_0\sigma}{b}}\tilde{\varphi}'(0) \qquad (2.4.9)$$

假设当 $\eta\leqslant -1$ 时，y_1 可分解成

$$y_1=\sum_{j=0}^{N}y_{1,j}(\sigma,t)+\beta_N^{(1)} \qquad (2.4.10)$$

代入 (2.4.9) 式，并取 $y_{1,j}$，$(j=0,1,\cdots,N)$，是下面的定解问题的解

$$\frac{\partial y_{1,0}}{\partial t}=-\tilde{A}(0,0,\tilde{\varphi}(0)),\quad y_{1,0}\Big|_{t=-\sqrt{\frac{2\sigma}{b\beta_0}}}=-\sqrt{\frac{2\beta_0\sigma}{b}}\tilde{\varphi}'(0) \tag{2.4.11}$$

$$\frac{\partial y_{1,j}}{\partial t}=a_0\frac{\partial^2 y_{1,j-1}}{\partial\sigma^2},\quad y_{1,j}\Big|_{t=-\sqrt{\frac{2\sigma}{b\beta_0}}}=0,\quad (j=1,2,\cdots,N) \tag{2.4.12}$$

则得到关于 $\beta_N^{(1)}$ 的定解问题：

$$E[\beta_N^{(1)}]=-a_0\frac{\partial^2 y_{1,N}}{\partial\sigma^2},\quad \beta_N^{(1)}|_{\xi=0}=0$$

从 (2.4.11)—(2.4.12) 可以逐步地推知 $y_{1,j}\in R_{1-3j}$. 又因 $a_0\frac{\partial^2 y_{1,N}}{\partial a^2}\in K_{3N+3}(D)$，所以根据引理 1 知 $\beta_N^{(1)}\in K_{3N+2}(D)$. 当 $\eta>-1$ 时，则 y_1 确定于抛物型方程的边值问题：

$$a_0\frac{\partial^2 y_1}{\partial\xi^2}+b\eta\frac{\partial y_1}{\partial\xi}-\beta_0\frac{\partial y_1}{\partial\eta}=\tilde{A}(0,0,\tilde{\varphi}(0))$$

$$y_1|_{\xi=0}=\eta\tilde{\varphi}'(0),\quad y_1|_{\eta=-1}=(y_{1,0}+\cdots+y_{1,N}+\beta_N^{(1)})|_{\eta=-1}$$

所以 $i=1$ 时定理成立.

假设定理当 $i=1,2,\cdots,k-1$ 时成立，假设 y_k 当 $\eta\leqslant-1$ 时可以分解成：

$$y_k=\sum_{j=0}^{N}y_{k,j}(\sigma,t)+\beta_N^{(k)}$$

代入确定 y_k 的定解问题 (2.4.4)—(2.4.5) (取 $i=k$)，并用 (σ,t) 坐标表示出，得

$$\begin{aligned}
&a_0\frac{\partial^2}{\partial\sigma^2}(y_{k,0}+y_{k,1}+\cdots+y_{k,N}+\beta_N^{(k)})-\frac{\partial}{\partial t}(y_{k,0}+y_{k,1}+\cdots+y_{k,N}+\beta_N^{(k)})\\
&=-\sum_{l=1}^{k-1}\Big[\hat{a}_{1,1}^{(l)}\frac{\partial^2}{\partial\sigma^2}\Big(\sum_{j=0}^{N}y_{k-l,j}+\beta_N^{(k-l)}\Big)-\hat{a}^{(l+1)}\frac{\partial}{\partial\sigma}\Big(\sum_{j=0}^{N}y_{k-l,j}+\beta_N^{(k-l)}\Big)\\
&\quad-\beta^{(l)}\Big(\frac{b\eta}{\beta_0}\frac{\partial}{\partial\sigma}+\frac{1}{\beta_0}\frac{\partial}{\partial t}\Big)\Big(\sum_{j=0}^{N}y_{k-l,j}+\beta_N^{(k-l)}\Big)\Big]\\
&\quad-2\sum_{l=0}^{k-2}\hat{a}_{1,2}^{(l)}\Big(\frac{b\eta}{\beta_0}\frac{\partial^2}{\partial\sigma^2}+\frac{1}{\beta_0}\frac{\partial^2}{\partial\sigma\partial t}\Big)\Big(\sum_{j=0}^{N}y_{k-1-l,j}+\beta_N^{(k-1-l)}\Big)\\
&\quad-\sum_{l=0}^{k-3}\Big[\hat{a}_{2,2}^{(l)}\Big(\frac{b\eta}{\beta_0}\frac{\partial}{\partial\sigma}+\frac{1}{\beta_0}\frac{\partial}{\partial t}\Big)^{(2)}\Big(\sum_{j=0}^{N}y_{k-2-l,j}+\beta_N^{(k-2-l)}\Big)\\
&\quad+\hat{a}_1^{(l)}\frac{\partial}{\partial\sigma}\Big(\sum_{j=0}^{N}y_{k-2-l,j}+\beta_N^{(k-2-l)}\Big)\Big]
\end{aligned}$$

$$-\sum_{l=0}^{k-3}\hat{a}_2^{(l)}\left(\frac{b\eta}{\beta_0}\frac{\partial}{\partial\sigma}+\frac{1}{\beta}\frac{\partial}{\partial t}\right)\left(\sum_{j=0}^{N}y_{k-3-l,j}+\beta_N^{(k-3-l)}\right)+\hat{A}^{(k-1)}$$

$$-\hat{a}^{(k-4)}(y_{k,0}+y_{k,1}+\cdots+y_{k,N}+\beta_N^{(k)})\Big|_{t=-\sqrt{\frac{2\sigma}{b\beta_0}}}=\frac{1}{k!}\left(-\sqrt{\frac{2\beta_0\sigma}{b}}\right)^k\tilde{\varphi}^{(k)}(0)$$

将 $a_{1,1}^{(l)}$ 分解成 $a_{1,1}^{(l)}=\sum_{m=0}^{N}\hat{a}^{(l)(m)}+\hat{a}_{1,1}^{(l)(N+1)}$，其中

$$\hat{a}_{1,1}^{(l)(m)}=\frac{\partial\tilde{a}_{1,1}(0,0,\tilde{\varphi}(0))}{\partial u}y_{l,m}$$

$$+\sum_{\substack{2m_1+m_2+l_1+\cdots+pl_p=l\\(p<l)}}\frac{1}{(2m_1)!m_2!l_1!\cdots l_p!}\frac{\partial^{m_1+m_2+l_1+\cdots+l_p}\tilde{a}_{1,1}(0,0,\tilde{\varphi}(0))}{\partial\rho_1^{m_1}\partial\rho_2^{m_2}\partial u^{l_1+\cdots+l_p}}$$

$$\cdot(2\xi)^{m_1}(\eta)^{m_2}\sum_{n_1+\cdots+n_p=m}\prod_{\alpha=1}^{p}\left(\sum_{\substack{i_0+i_1+\cdots+i_{n_\alpha}=l_\alpha\\(i_1+2i_2+\cdots+n_\alpha i_{n_\alpha}=n_\alpha)}}\frac{l_\alpha!}{i_0!i_1!\cdots i_{n_\alpha}}y_{\alpha,0}^{i_0}\cdots y_{\alpha,n_\alpha}^{i_{n_\alpha}}\right)$$

$$\hat{a}_{1,1}^{(l)(N+1)}=\frac{\partial\tilde{a}_{1,1}(0,0,\tilde{\varphi}(0))}{\partial u}\beta_N^{(l)}+R_{1,1}^{(l)}$$

$R_{1,1}^{(l)}$ 表示属于$K_{3N+2-6(l-1)}$ 的项．取 $y_{k,j}$，$(j=0,1,\cdots,N)$ 满足定解问题

$$-\frac{\partial y_{k,0}}{\partial t}=\sum_{l=1}^{k-1}\left[\hat{a}^{(l+1)}\frac{\partial y_{k-l}}{\partial\sigma}+\beta^{(l)}\left(\frac{b\eta}{\beta_0}\frac{\partial}{\partial\sigma}+\frac{1}{\beta_0}\frac{\partial}{\partial t}\right)y_{k-l,0}\right]+\hat{A}^{(k-1)(0)}$$

$$y_{k,0}\Big|_{t=-\sqrt{\frac{2\sigma}{b\beta_0}}}=\frac{1}{k!}\left(-\sqrt{\frac{2\beta_0\sigma}{b}}\right)^k\tilde{\varphi}^{(k)}(0)$$

$$-\frac{\partial y_{k,j}}{\partial t}=-a_0\frac{\partial^2 y_{k,j-1}}{\partial\sigma^2}-\sum_{l=1}^{k-1}\left[\sum_{m=0}^{j-1}\hat{a}_{1,1}^{(l)(m)}\frac{\partial^2 y_{k-l,j-1-m}}{\partial\sigma^2}-\hat{a}^{(l+1)}\frac{\partial y_{k-l,j}}{\partial\sigma}\right.$$

$$\left.-\beta^{(l)}\left(\frac{b\eta}{\beta_0}\frac{\partial}{\partial\sigma}+\frac{1}{\beta_0}\frac{\partial}{\partial t}\right)y_{k-l,j}\right]$$

$$-2\sum_{l=0}^{k-2}\sum_{m=0}^{j-1}\hat{a}_{1,2}^{(l)(m)}\left(\frac{b\eta}{\beta_0}\frac{\partial^2}{\partial\sigma^2}+\frac{1}{\beta_0}\frac{\partial^2}{\partial\sigma\partial t}\right)y_{k-1-l,j-1-m}$$

$$-\sum_{l=0}^{k-3}\left[\sum_{m=0}^{j-1}\hat{a}_{2,2}^{(l)(m)}\left(\frac{b\eta}{\beta_0}\frac{\partial}{\partial\sigma}+\frac{1}{\beta_0}\frac{\partial}{\partial t}\right)^{(2)}y_{k-2-l,j-1-m}\right.$$

$$\left.+\sum_{m=0}^{j-1}\hat{a}_1^{(l)(m)}\frac{\partial y_{k-2-l,j-1-m}}{\partial\sigma}\right]$$

$$-\sum_{l=0}^{k-4}\sum_{m=0}^{j-1}\hat{a}_2^{(l)(m)}\left(\frac{b\eta}{\beta_0}\frac{\partial y_{k-3-l,j-1-m}}{\partial\sigma}+\frac{1}{\beta_0}\frac{\partial y_{k-3-l,j-1-m}}{\partial t}\right)+\hat{\hat{A}}^{(k-1)(j)}-\hat{a}^{(k-4)(j-1)}$$

$$y_{k,j}\Big|_{t=-\sqrt{\frac{2\sigma}{b\beta_0}}}=0,(j=1,2,\cdots,N)$$

则得到关于 $\beta_N^{(k)}$ 的定解问题：

$$E[\beta_N^{(k)}]=-a_0\frac{\partial^2 y_{k,N}}{\partial\sigma^2}-\sum_{l=1}^{k-1}\Bigg[\left(\sum_{m=0}^{N}\hat{a}_{1,1}^{(l)(m)}\frac{\partial^2 y_{k-l,N-m}}{\partial\sigma^2}+\cdots+\hat{a}_{1,1}^{(l)(N+1)}\frac{\partial^2\beta_N^{(k-l)}}{\partial\sigma^2}\right)$$

$$-\hat{a}^{(l+1)}\frac{\partial\beta_N^{(k-l)}}{\partial\sigma}-\beta^{(l)}\frac{\partial\beta_N^{(k-l)}}{\partial\sigma}\Bigg]+\cdots+\hat{\hat{A}}^{(k-1)(N+1)}-(\hat{\hat{a}}^{(k-4)(N)}+\hat{\hat{a}}^{(k-4)(N+1)})$$

$$\beta_N^{(k)}|_{\xi=0}=0$$

因微分方程的右端是属于$K_{3N-6k+9}$，根据引理 1 知 $\beta_N^{(k)}\in K_{3N+2-6(k-1)}$．当$\eta>-1$时，定义$y_k$是抛物型方程 (2.4.4) 的满足边值条件

$$y_k|_{\xi=0}=\frac{\eta^k}{k!}\varphi^{(k)}(0),\quad y_k|_{\eta=-1}=(y_{k,0}+y_{k,1}+\cdots+y_{k,N}+\beta_N^{(k)})|_{\eta=-1}$$

的解，所以当 $i=k$ 时定理也成立，证毕．

注 1：因 $y_{i,j}$ 和 $\hat{\omega}_{j,i}$，($i=0,1,\cdots,3m+1$；$j=0,1,\cdots,m$) 满足同一微分方程和定解条件，所以 $y_{i,j}=\hat{\omega}_{j,i}$．

注 2：当 η 取有限值时，只要 ξ 的值充分大，分解式 (2.4.8) 仍然成立．

下面再考察 y_i 当 $\eta\to\infty$ 时的性质．

引理 2 ([8]的引理 7)：设无限次可微的函数 $v(\xi,\eta)$ 在区域$G_1=\{(\xi,\eta)|\xi>0,\ \eta>1\}$ 满足方程：

$$E[v]=\Psi(\xi,\eta)$$

其中$\Psi=O\left[\left(\frac{2\beta_0}{b}\xi+\eta^2\right)^{-\frac{m}{2}}+\eta^{-l}e^{-\gamma\xi\eta}\right]$，$0<\gamma<1$，以及$m$和 l 是满足不等式$m\leqslant l+3$ 的充分大的正数，如果

$$v|_{\xi=0}=O(\eta^{1-m}),\quad v|_{\eta=1}=O\left[\left(\frac{2\beta_0}{b}\xi+1\right)^{\frac{1-m}{2}}\right]$$

则在G_1 成立 $v=O\left[\left(\frac{2\beta_0}{b}\xi+\eta^2\right)^{\frac{1-m}{2}}\right]$．

定理 3：定解问题 (2.4.4)－(2.4.5)，($i=0,1,\cdots,3m+1$) 的解 y_i 在区域 G_1 可以分解成

$$y_i=\sum_{j=0}^{N}y_{i,j}(\sigma,t)+\sum_{j=0}^{N}\tilde{y}_{i,j}(\xi,\eta)+\beta_N^{(i)}$$

其中$y_{i,j}\in R_{i-3j}$，$\tilde{y}_{i,j}\in\sum_{\mu=1}^{j-1}S_{i-3j,\mu}$，和$y_{i,j}=\hat{\omega}_{j,i}$，$\tilde{y}_{i,j}=\hat{v}_{j,i}$，($i=0,1,\cdots,3m+1;j=0,1,$

$\cdots,\ m)$，$\tilde{y}_{i,m+1}=\hat{v}_{m+1,i}$，并且成立估计式

$$\frac{\partial^l \beta_N^{(i)}}{\partial\xi^{l_1}\partial\eta^{l_2}}=O\left[\left(\frac{2\beta_0}{b}\xi+\eta^2\right)^{\frac{-3(N+1)+1-6(i-1)+P_0}{2}}\right]$$

P_0 是只与 l_1、l_2 有关的常数.

证：假设y_1可以分解成 $y_1=\sum\limits_{j=0}^{N}y_{1,j}(\sigma,t)+\sum\limits_{j=0}^{N}\tilde{y}_{1,j}(\xi,\eta)+\beta_N^{(1)}$，代入(2.4.4)－(2.4.5)（取 $i=1$）得

$$\left(a_0\frac{\partial^2}{\partial\sigma^2}+\frac{\partial}{\partial t}\right)\sum_{j=0}^{N}y_{1,j}+\left(a_0\frac{\partial^2}{\partial\xi^2}+b\eta\frac{\partial}{\partial\xi}-\beta_0\frac{\partial}{\partial\eta}\right)\left(\sum_{j=0}^{N}\tilde{y}_{1,j}+\beta_N^{(1)}\right)$$

$$=\tilde{A}(0,0,\tilde{\varphi}(0))\Big)$$

$$\sum_{j=0}^{N}y_{1,j}\Big|_{t=\sqrt{\frac{2\sigma}{b\beta_0}}}+\sum_{j=0}^{N}\tilde{y}_{1,j}|_{\xi=0}+\beta_N^{(1)}|_{\xi=0}=\frac{\eta^i}{i!}\tilde{\varphi}^{(i)}(0)$$

取 $y_{1,j}$，$(j=0,1,\cdots,N)$ 是下面定解问题的解

$$-\frac{\partial y_{1,0}}{\partial t}=\tilde{A}(0,0,\tilde{\varphi}(0)),\quad y_{1,0}\Big|_{t=\sqrt{\frac{2\sigma}{b\beta_0}}}=\hat{w}_{0,1}(\sigma,t)\Big|_{t=\sqrt{\frac{2\sigma}{b\beta_0}}}$$

$$-\frac{\partial y_{1,j}}{\partial t}=-a_0\frac{\partial^2 y_{1,j-1}}{\partial\sigma^2},\quad y_{1,j}\Big|_{t=\sqrt{\frac{2\sigma}{b\beta_0}}}=w_{j,1}(\sigma,t)\Big|_{t=\sqrt{\frac{2\sigma}{b\beta_0}}},\quad (j=1,2,\cdots,N)$$

再取 $\tilde{y}_{1,j}$，$(j=0,1,\cdots,N)$，是下面定解问题的解

$$a_0\frac{\partial^2\tilde{y}_{1,0}}{\partial\xi^2}+b\eta\frac{\partial\tilde{y}_{1,0}}{\partial\xi}=0,\quad \tilde{y}_{1,0}|_{\xi=0}=\eta\,\tilde{\varphi}'(0)-\hat{w}_{0,1}(0,\eta)$$

$$a_0\frac{\partial^2\tilde{y}_{1,j}}{\partial\xi^2}+b\eta\frac{\partial\tilde{y}_{1,j}}{\partial\xi}=\beta_0\frac{\partial\tilde{y}_{1,j-1}}{\partial\eta},\quad \tilde{y}_{1,j}|_{\xi=0}=-\hat{w}_{j,1}(0,\eta),\quad (j=1,2,\cdots,N)$$

则得到关于 $\beta_N^{(1)}$ 的定解问题

$$a_0\frac{\partial^2\beta_N^{(1)}}{\partial\xi^2}+b\eta\frac{\partial\beta_N^{(1)}}{\partial\xi}=-a_0\frac{\partial^2 y_{1,N}}{\partial\sigma^2}+\beta_0\frac{\partial\tilde{y}_{1,N}}{\partial\eta}\equiv Q^{(1)}$$

$$\beta_N^{(1)}|_{\xi=0}=0,\quad \beta_N^{(1)}|_{\eta=1}=\left(y_1-\sum_{j=0}^{N}y_{1,j}-\sum_{j=0}^{N}\tilde{y}_{1,j}\right)_{\eta=1}\equiv q^{(1)}$$

因 $\frac{\partial^2 y_{1,N}}{\partial\sigma^2}\in R_{-3-3N}$，$\frac{\partial\tilde{y}_{1,N}}{\partial\eta}\in S_{-3N,1}$，所以$Q^{(1)}=O\left[\left(\frac{2\beta_0}{b}\xi+\eta^2\right)^{\frac{-3-3N}{2}}+\eta^{-3N}e^{-\gamma\xi\eta}\right]$；又因 $q^{(1)}=O[\beta_N^{(1)}]+O(e^{-\gamma\xi})=O\left[\left(\frac{2\beta_0}{b}\xi+1\right)^{\frac{-3N-2}{2}}\right]$，根据引理2知$\beta_N^{(1)}=O\Big[\Big(\frac{2\beta_0}{b}\xi+$

$+\eta^2\Big)^{\frac{-3(N+1)+1}{2}}\Big]$. 再根据抛物型方程解的先验估计知（参见[8]的引理8）：

$$\frac{\partial^l \beta_N^{(1)}}{\partial\xi^{l_1}\partial\eta^{l_2}}=O\left[\left(\frac{2\beta_0}{b}\xi+\eta^2\right)^{\frac{-3(N+1)+1+P_0}{2}}\right]$$

P_0 是只与 l_1、l_2 有关的常数，所以当 $i=1$ 时定理成立．应用定理 2 中类似的方法可以证明对于任意i定理都成立，并且当 $i=0,1,\cdots,3m+1, j=0,1,\cdots,m$ 时 $y_{i,j}=\hat{w}_{j,i}$，$\tilde{y}_{i,j}=\hat{v}_{j,i}$，$\tilde{y}_{i,m+1}==\hat{v}_{m+1,i}$．

二·五 形式渐近解

下面再根据外部展开式和边界层项构造形式渐近解．

将切点 $P_i(i=1,2)$，的邻域 D_i 分成三部分：

$\Pi_1^{(i)}=\{(\rho_1,\rho_2)\,|\,0\leqslant\rho_1\leqslant\varepsilon_1,\ |\rho_2-\rho_2^{(i)}|\leqslant\varepsilon_1^{\frac{1}{2}}\}$，

$\Pi_2^{(i)}=\{(\rho_1,\rho_2)\,|\,0\leqslant\rho_1\leqslant\frac{1}{2}\varepsilon_1,\ |\rho_2-\rho_2^{(i)}|\leqslant\frac{1}{2}\varepsilon_1^{\frac{1}{2}}\}$，

$\Pi_3^{(i)}=\Pi_1^{(i)}\diagdown\Pi_2^{(i)}$，其中令 $\rho_2^{(1)}=0$．应用截断函数X (t) (参见 (2.2.12) 式) 作和式：

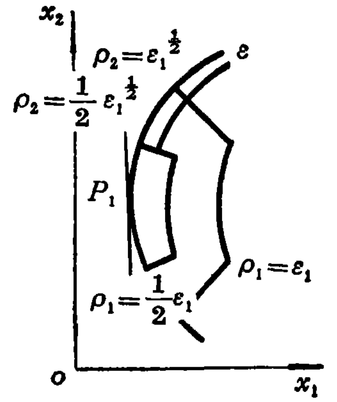

图 3 P_1点邻域的边界层

$$U_m=W_m(x_1,x_2;\varepsilon)[1-X_1^{(0)}(x_1,x_2)][1-X_2^{(0)}(x_1,x_2)]+V_m(x_1,x_2;\varepsilon)X(\varepsilon_1\xi)[1-X_1(x_1,x_2)]$$

$$[1-X_2(x_1,x_2)]+Y_{3m}^{(1)}(\xi,\eta;\varepsilon_1)X_1^{(0)}(x_1,x_2)+Y_{3m}^{(2)}(\xi,\eta';\varepsilon_1)X_2^{(0)}(x_1,x_2)\qquad(2.5.1)$$

其中

$$X_i^{(0)}=\begin{cases}X(\varepsilon_1\xi)X(\varepsilon_1^{\frac{1}{2}}\eta^{(i)})X(-\varepsilon_1^{\frac{1}{2}}\eta^{(i)}), & 当(x_1,x_2)\in D_i\\ 0, & 当(x_1,x_2)\in\bar{\Omega}\diagdown D_i\end{cases},\quad X_i=\begin{cases}X(\varepsilon_1^{\frac{1}{2}}\eta), & 当(x_1,x_2)\in D_i\\ 0, & 当(x_1,x_2)\in\bar{\Omega}\diagdown D_i\end{cases};$$

记 $\eta^{(1)}=\eta$，$\eta^{(2)}=\eta'$，η' 表示 P_2点的邻域中相应的η坐标：$\eta'=\dfrac{\rho_2^{(2)}-\rho_2}{\varepsilon_1}$．有定理

定理 4：由 (2.5.1) 式给出的U_m是狄立克雷问题 (2.0.1)—(2.0.2) 的形式渐近解，即成立

$$N_\varepsilon[U_m]=O\left(\varepsilon_1^{\frac{3m-2}{2}}\right),\quad U_m\Big|_{\partial\Omega}=\tilde{\varphi}(\rho_2)-\varepsilon_1^{3m+2}\varphi_m$$

其中

$$\varphi_m=\frac{\eta^{3m+2}}{(3m+2)!}\tilde{\varphi}^{(3m+2)}(\theta\varepsilon_1\eta)X_1+\frac{(\eta')^{3m+2}}{(3m+2)!}\tilde{\varphi}^{(3m+2)}(\theta\varepsilon_1\eta')X_2,\quad(0\leqslant\theta\leqslant1)\qquad(2.5.2)$$

证：在区域$\Pi_2^{(1)}$中，因 $0<\xi\leqslant\frac{1}{2}\varepsilon_1^{-1}$，$|\eta|\leqslant\frac{1}{2}\varepsilon_1^{\frac{-1}{2}}$，所以

$$N_\varepsilon[U_m]=N_\varepsilon[Y_{3m}^{(1)}]=\varepsilon_1^{3m+1}\left\{\left[\sum_{l=1}^{3m+1}\left(a_{1,1}^{(l)}\frac{\partial^2 y_{3m+2-l}}{\partial\xi^2}-\hat{\alpha}^{(l+1)}\frac{\partial y_{3m+2-l}}{\partial\xi}-\beta^{(l)}\frac{\partial y_{3m+2-l}}{\partial\eta}\right)\right.\right.$$

$$+\varepsilon_1\sum_{l=1}^{3m+1}\left(\hat{a}_{1,1}^{(l+1)}\frac{\partial^2 y_{3m+2-l}}{\partial\xi^2}-\hat{\alpha}^{(l+2)}\frac{\partial y_{3m+2-l}}{\partial\xi}-\beta^{(l+1)}\frac{\partial y_{3m+2-l}}{\partial\eta}\right)+\varepsilon_1^2(\cdots)+\cdots\Big]$$

$$+\left[\left(2\sum_{l=0}^{3m+1}\hat{a}_{1,2}^{(l)}\frac{\partial^2 y_{3m+1-l}}{\partial\xi\partial\eta}-\hat{A}^{(3m+1)}\right)+\varepsilon_1\left(2\sum_{l=0}^{3m+1}\hat{a}_{1,2}^{(l)}\frac{\partial^2 y_{3m+1-l}}{\partial\xi\partial\eta}-\hat{A}^{(3m+2)}\right)+\cdots\right]$$

$$+\cdots+\left[\hat{a}^{(3m-2)}+\varepsilon_1\hat{a}^{(3m-1)}+\cdots\right]\Big\}$$

当 $\eta\leqslant-1$ 时，从定理 2 和关系式 $\sigma=O(\varepsilon_1^{-1})$，$\sigma\geqslant\frac{b}{2\beta_0}>0$ 知：若取 N 充分地大（例如取 $N\geqslant5m+P_0$），则 $\hat{a}_{1,1}^{(l)}\frac{\partial^2 y_{3m+2-l}}{\partial\xi}=O\left(\varepsilon_1^{\frac{-(3m+1)}{2}}\right)$ 等等，所以 $N_\varepsilon[U_m]=O\left(\varepsilon_1^{\frac{3m-2}{2}}\right)$. 当 $\eta\geqslant1$ 时，从定理 3 知 $\hat{a}_{1,1}^{(l)}\frac{\partial^2 y_{3m+2-l}}{\partial\xi^2}=O\left(\varepsilon_1^{\frac{-3m-4}{2}}\right)$ 等等，所以

$$N_\varepsilon[U_m]=O\left(\varepsilon_1^{\frac{3m-2}{2}}\right)$$

当 $-1\leqslant\eta\leqslant1$ 时，利用抛物型方程的极值原理和估计解的导数的方法（例如 [9] 中 § 1— § 2 的方法）可以知道 $\hat{a}_{1,1}^{(l)}\frac{\partial^2 y_{3m+2-l}}{\partial\xi^2}=O\left(\varepsilon_1^{\frac{-3m-4}{2}}\right)$ 等等，仍成立 $N_\varepsilon[U_m]=O\left(\varepsilon_1^{\frac{3m-2}{2}}\right)$.

将区域 $D_1\setminus\Pi_1^{(1)}$ 分成五个子区域：

$$D_{1,1}^{(0)}=\{(\xi,\eta)\,|\,0<\xi\leqslant\tfrac{1}{2}\varepsilon_1^{-1},\ \ \varepsilon_1^{\frac{-1}{2}}<\eta\leqslant\delta_0\varepsilon_1^{-1}\}$$

$$D_{1,2}^{(0)}=\{(\xi,\eta)\,|\,0<\xi\leqslant\tfrac{1}{2}\varepsilon_1^{-1},\ \ -\delta_0\varepsilon_1^{-1}\leqslant\eta<-\varepsilon_1^{\frac{-1}{2}}\}$$

$$D_{1,3}^{(0)}=\{(\xi,\eta)\,|\,\tfrac{1}{2}\varepsilon_1^{-1}<\xi\leqslant\rho_1^{(0)}\varepsilon_1^{-2},\ \ \varepsilon_1^{\frac{-1}{2}}<\eta\leqslant\delta_0\varepsilon_1^{-1}\}$$

$$D_{1,4}^{(0)}=\{(\xi,\eta)\,|\,\tfrac{1}{2}\varepsilon_1^{-1}<\xi\leqslant\rho_1^{(0)}\varepsilon_1^{-2},\ \ -\delta_0\varepsilon_1^{-1}\leqslant\eta<-\varepsilon_1^{\frac{-1}{2}}\}$$

$$D_{1,5}^{(0)}=\{(\xi,\eta)\,|\,\varepsilon_1^{-1}<\xi\leqslant\rho_1^{(0)}\varepsilon_1^{-2},\ \ |\eta|\leqslant\varepsilon_1^{\frac{-1}{2}}\}$$

当 $(\xi,\eta)\in D_{1,1}^{(0)}$ 时成立

$$N_\varepsilon[U_m]=\tilde{N}_\varepsilon[\widehat{W}_m+V_m]=O(\varepsilon^{m+1})+\varepsilon^{m+1}\Bigg\{\Bigg(\sum_{l=1}^{m+2}\tilde{a}_{1,1}^{(r,l)}\frac{\partial^2\hat{v}_{m+2-l}}{\partial\xi^2}\varepsilon_1^2+\cdots$$

$$+\sum_{l=0}^{m}\left(\tilde{a}_{1,1}^{(r,l)}\frac{\partial^2\hat{w}_{m-l}}{\partial\xi^2}\varepsilon_1^{-4}+2\tilde{a}_{1,2}^{(r,l)}\frac{\partial^2\hat{w}_{m-l}}{\partial\xi\partial\eta}\varepsilon_1^{-3}+\tilde{a}_{2,2}^{(r,l)}\frac{\partial^2\hat{w}_{m-1}}{\partial\eta^2}\varepsilon_1^2\right)\Bigg]$$

$$+\cdots+\varepsilon^{m+3}\left(\tilde{a}_{2,2}^{(r,m+2)}\frac{\partial^2\hat{v}_{m+1}}{\partial\eta^2}\varepsilon_1^{-2}+\tilde{a}_2^{(r,m+2)}\frac{\partial\hat{v}_{m+1}}{\partial\eta}\varepsilon_1^{-2}\right)\Bigg\}$$

因在 $D_{1,1}^{(0)}$ 中 $\sigma=O(\varepsilon_1^{-2})$，$\sigma\geqslant\frac{b}{2\beta_0}\varepsilon_1^{-1}$，而有 $\tilde{a}_{1,1}^{(r,l)}\frac{\partial^2\hat{v}_{m+2-l}}{\partial\xi^2}=O\left(\varepsilon_1^{\frac{-3m+6l-12}{2}}\right)$ 等等，所以

$N_\varepsilon[U_m]=O\left(\varepsilon_1^{\frac{3m-2}{2}}\right)$. 当$(\xi,\eta)\in D_{1,2}^{(0)}$时同理有$N_\varepsilon[U_m]=O\left(\varepsilon_1^{\frac{3m-2}{3}}\right)$. 当$(\xi,\eta)\in D_{1,3}^{(0)}$时，因$N_\varepsilon[U_m]=\tilde{N}_\varepsilon[\tilde{W}_m+X(\varepsilon_1\xi)V_m]$，只需注意当$\varepsilon_1\to 0$时，$v_i$及其各阶导数在$D_{1,3}^{(0)}$内是渐近地为零，可以证明$N_\varepsilon[U_m]=O\left(\varepsilon_1^{\frac{3m-2}{2}}\right)$. 当$(\xi,\eta)\in D_{1,4}^{(0)}$时同理可证$N_\varepsilon[U_m]=O\left(\varepsilon_1^{\frac{3m-2}{2}}\right)$.

当$(\xi,\eta)\in D_{1,6}^{(0)}$时，显然有$N_\varepsilon[U_m]=N_\varepsilon[W_m]=O\left(\varepsilon_1^{\frac{3m-2}{2}}\right)$.

在区域$\Pi_1\backslash\Pi_2$，分成四个子区域讨论. 当$(\xi,\eta)\in D_{1,1}^{(in)}=\left\{(\xi,\eta)\,\middle|\,0<\xi\leqslant\frac{1}{2}\varepsilon_1^{-1},\ \frac{1}{2}\varepsilon_1^{\frac{-1}{2}}<\eta\leqslant\varepsilon_1^{\frac{-1}{2}}\right\}$时有

$$N_\varepsilon[U_m]=\tilde{N}_\varepsilon[\tilde{W}_m+V_m+X(\varepsilon_1^{\frac{1}{2}}\eta)(-\tilde{W}_m-V_m+Y_{3m}^{(1)})]=O\left(\varepsilon_1^{\frac{3m-2}{2}}\right)+\tilde{N}_\varepsilon[\tilde{W}_m+V_m$$

$$+X(\varepsilon_1^{\frac{1}{2}}\eta)(-\tilde{W}_m-V_m+Y_{3m}^{(1)})]-\tilde{N}_\varepsilon[\tilde{W}_m+V_m],$$ 只需注意在该区域内

$$Y_{3m}^{(1)}-\tilde{W}_m-V_m=\sum_{i=1}^{3m+1}\varepsilon_1^i\left[\sum_{l=m+1}^{N}(y_{i,l}+\tilde{y}_{i,l})+\beta_N^{(i)}\right]-\varepsilon_1^{3m+2}\left(\sum_{i=0}^{m}r^{(i)}+\sum_{i=0}^{m+1}\psi^{(i)}\right)$$

$$=O\left(\varepsilon_1^{\frac{3m-2}{2}}\right)$$

可以类似地证明 $N_\varepsilon[U_m]=O\left(\varepsilon_1^{\frac{3m-2}{2}}\right)$. 当$(\xi,\eta)\in D_{1,2}^{(in)}=\left\{(\xi,\eta)\,\middle|\,\frac{1}{2}\varepsilon_1^{-1}<\xi\leqslant\varepsilon_1^{-1},\ 1\leqslant\eta\leqslant\varepsilon_1^{\frac{-1}{2}}\right\}$时有$N_\varepsilon[U_m]=\tilde{N}_\varepsilon[\tilde{W}_m+V_m+X(\varepsilon_1\xi)X(\varepsilon_1^{\frac{1}{2}}\eta)(Y_{3m}^{(1)}-\tilde{W}_m-V_m)$
$-(1-X(\varepsilon_1\xi))V_m]$，只需注意在该区域内$v_i$，$(i=0,1,\cdots,m+1)$及其各阶导数当$\varepsilon_1\to 0$时都渐近地为零，可以类似地证明$N_\varepsilon[U_m]=O\left(\varepsilon_1^{\frac{3m-2}{2}}\right)$. 当$(\xi,\eta)\in D_{1,3}^{(in)}=\left\{(\xi,\eta)\,\middle|\,\frac{1}{2}\varepsilon_1^{-1}<\xi\leqslant\varepsilon_1^{-1},\ -\frac{1}{2}\varepsilon_1^{\frac{-1}{2}}\leqslant\eta<1\right\}$时有$N_\varepsilon[U_m]=\tilde{N}_\varepsilon[\tilde{W}_m+X(\varepsilon_1\xi)(Y_{3m}^{(1)}-\tilde{W}_m)]$，只需注意在该区域内

$$Y_{3m}^{(1)}-\tilde{W}_m=\sum_{i=1}^{3m+1}\varepsilon_1^i\left(\sum_{l=m+1}^{N}y_{i,l}+\beta_N^{(i)}\right)-\varepsilon_1^{3m+2}\sum_{i=0}^{m}r^{(i)}=O\left(\varepsilon_1^{\frac{3m+2}{2}}\right)$$

可以同样地证明$N_\varepsilon[U_m]=O\left(\varepsilon_1^{\frac{3m-2}{2}}\right)$. 当$(\xi,\eta)\in D_{1,4}^{(in)}=\left\{(\xi,\eta)\,\middle|\,0<\xi\leqslant\varepsilon_1^{-1},\ -\varepsilon_1^{\frac{-1}{2}}\leqslant\eta<-\frac{1}{2}\varepsilon_1^{\frac{-1}{2}}\right\}$时有$N_\varepsilon[U_m]=\tilde{N}_\varepsilon[\tilde{W}_m+X(\varepsilon_1\xi)X(-\varepsilon_1^{\frac{1}{2}}\eta)(Y_{3m}^{(1)}-\tilde{W}_m)]$，可以同样地

证明 $N_\varepsilon[U_m]=O\left(\varepsilon_1^{\frac{3m-2}{2}}\right)$.

在边界 $\partial\Omega$ 上显然满足(2.5.2)中的边值条件，再考虑到定理1，立刻可得到上述定理.

三、余 项 估 计

添加假设

$(H3)$：在区域 Ω 上成立 $\dfrac{\partial A(x_1,x_2,U_m)}{\partial u}\geqslant\delta>0$，其中 U_m 是由(2.5.1)式给出的形式渐近解，δ 是常数.

以 L_ε 表示对应于 N_ε 的线性化微分算子：

$$L_\varepsilon\equiv\left[\varepsilon\left(\sum_{i,j=1}^{2}a_{i,j}(x_1,x_2,U_m)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{2}a_i(x_1,x_2,U_m)\frac{\partial}{\partial x_i}+a(x_1,x_2,U_m)\right)-\frac{\partial}{\partial x_2}\right]+\varepsilon\left(\sum_{i,j=1}^{2}\frac{\partial a_{i,j}(x_1,x_2,U_m)}{\partial u}\frac{\partial^2 U_m}{\partial x_i\partial x_j}+\sum_{i=1}^{2}\frac{\partial a_i(x_1,x_2,U_m)}{\partial u}\frac{\partial U_m}{\partial x_i}+\frac{\partial a(x_1,x_2,U_m)}{\partial u}\right)-\frac{\partial A(x_1,x_2,U_m)}{\partial u} \tag{3.1}$$

以 Z_m 表示狄立克雷问题的真解 u_ε 对于 U_m 的余项：$Z_m=u_\varepsilon-U_m$ 显然有

$$L_\varepsilon[Z_m]\equiv N_\varepsilon[u_\varepsilon]-N_\varepsilon[U_m]+R(Z_m)=\varepsilon_1^{\frac{3m-2}{2}}\Phi_m(x_1,x_2;\varepsilon)+R(Z_m) \tag{3.2}$$

$$Z_m|_{\partial\Omega}=\Psi_m \tag{3.3}$$

其中 $\Phi_m=O(1)$，$\Psi_m=\varepsilon_1^{3m+2}\varphi_m$，$\varphi_m$ 由(2.5.2)式定义，和

$$R(Z_m)\equiv\varepsilon\left[\sum_{i,j=1}^{2}(a_{i,j}(x_1,x_2,U_m)-a_{i,j}(x_1,x_2,U_m+Z_m))\frac{\partial^2(U_m+Z_m)}{\partial x_i\partial x_j}+\sum_{i=1}^{2}(a_i(x_1,x_2,U_m)-a_i(x_1,x_2,U_m+Z_m))\frac{\partial(U_m+Z_m)}{\partial x_i}\right]$$
$$+\varepsilon(a(x_1,x_2,U_m)-a(x_1,x_2,U_m+Z_m))-(A(x_1,x_2,U_m)-A(x_1,x_2,U_m+Z_m))$$
$$+\varepsilon\left(\sum_{i,j=1}^{2}\frac{\partial a_{i,j}(x_1,x_2,U_m)}{\partial u}\frac{\partial^2 U_m}{\partial x_i\partial x_j}+\sum_{i=1}^{2}\frac{\partial a_i(x_1,x_2,U_m)}{\partial u}\frac{\partial U_m}{\partial x_i}+\frac{\partial a(x_1,x_2,U_m)}{\partial u}\right)Z_m$$
$$-\frac{\partial A(x_1,x_2,U_m)}{\partial u}Z_m+\varepsilon a(x_1,x_2,U_m)Z_m \tag{3.4}$$

令 $\tilde{Z}_m=Z_m-\Psi_m$，得到关于 $\tilde{Z}_m$ 的边值问题：

$$L_\varepsilon[\tilde{Z}_m]=\varepsilon_1^{\frac{3m-2}{2}}\Phi_m^*(x_1,x_2;\varepsilon_1)+R(\tilde{Z}_m+\Psi_m) \tag{3.5}$$

$$\widetilde{Z}_m\Big|_{\partial\Omega}=0 \tag{3.6}$$

其中$\Phi_m^*=O(1)$．考察线性椭圆型方程的狄立克雷问题：

$$L_\varepsilon[w]=f(x_1,x_2),\quad (x_1,x_2)\in\Omega;\quad w\Big|_{\partial\Omega}=0 \tag{3.7}$$

若以$C_{2,\alpha}(\bar{\Omega})$，$(0<\alpha<1)$表示在$\bar{\Omega}$上具有二阶连续偏导数和

$$|u|_{2,\alpha}=\sum_{h=0}^{2}\sum_{|i|=h}\underset{\bar{\Omega}}{S_{up}}|D^{(i)}u|+\sum_{|i|=2}|D^{(i)}u|_{(\alpha),\Omega}$$

是有限值的函数空间，其中

$$i=(i_1,i_2),\quad D^{(i)}u\equiv\frac{\partial^i u}{\partial x_1^{i_1}\partial x_2^{i_2}},\quad |i|=i_1+i_2,$$

$$|u|_{(\alpha),\Omega}=\underset{p_1,p_2\in\bar{\Omega}}{S_{up}}\frac{|u(p_1)-u(p_2)|}{|p_1-p_2|^\alpha},$$

则在假设(H_3)下成立[10]

引理 3：设边界$\partial\Omega$是属于$C_{2,\alpha}$类，$f(x_1,x_2)\in C_{0,\alpha}(\bar{\Omega})$，则线性狄立克雷问题(3.7)存在唯一的解$w\in C_{2,\alpha}(\bar{\Omega})$，和

$$|w|_{2,\alpha}\leqslant\varepsilon^{-1}K|f|_{0,\alpha} \tag{3.8}$$

其中K是与ε无关的常数．

再以$C_{2,\alpha}^{(0)}(\bar{\Omega})$表示$C_{2,\alpha}(\bar{\Omega})$中在$\partial\Omega$取零值的函数所组成的子空间，在$C_{2,\alpha}^{(0)}(\bar{\Omega})$中定义算子方程：

$$u=T_\varepsilon[u]$$

其中

$$T_\varepsilon[u]=L_\varepsilon^{-1}\left[\varepsilon_1^{\frac{3m-2}{2}}\Phi_m^*(x_1,x_2;\varepsilon_1)+R(u+\Psi_m)\right]$$

以S_m表示$C_{2,\alpha}^{(0)}(\bar{\Omega})$中的球：$S_m=\left\{u\in C_{2,\alpha}^{(0)}(\bar{\Omega})\,\Big|\,|u|_{2,\alpha}\leqslant\varepsilon_1^{\frac{3m-9}{2}}\right\}$，有

引理 4：设$u\in S_m$，则$T_\varepsilon[u]\in S_m$．

证：根据引理3有

$$|T_\varepsilon[u]|_{2,\alpha}\leqslant\varepsilon^{-1}K\left(\varepsilon_1^{\frac{3m-2}{2}}|\Phi_m^*|_{0,\alpha}+|R(u+\Psi_m)|_{0,\alpha}\right)$$

因 $|R(u+\Psi_m)|_{0,\alpha}\leqslant\varepsilon\sum_{i,j=1}^{2}\left(\left|\frac{\partial^2 a_{ij}(x_1,x_2,U_m+\theta_2(U_m+\theta_1(u+\Psi_m)))}{\partial u^2}\;\frac{\partial^2 U_m}{\partial x_i\partial x_j}\theta_1(u+\Psi_m)^2\right|_{0,\alpha}\right.$

$$\left.+\left|\frac{\partial a_{ij}(x_1,x_2,U_m+\theta_1(u+\Psi_m))}{\partial u}\;\frac{\partial^2(u+\Psi_m)}{\partial x_i\partial x_j}(u+\Psi_m)\right|_{0,\alpha}\right)$$

$$+\varepsilon\sum_{i=1}^{2}\left(\left|\frac{\partial^2 a_i(x_1,x_2,U_m+\theta_2(U_m+\theta_1(u+\Psi_m)))}{\partial u^2}\;\frac{\partial U_m}{\partial x_i}\theta_1(u+\Psi_m)^2\right|_{0,\alpha}\right.$$

$$+\left|\frac{\partial a_i(x_1,x_2,U_m+\theta_1(u+\Psi_m))}{\partial u}\quad\frac{\partial(u+\Psi_m)}{\partial x_i}(u+\Psi_m)\right|_{0,\alpha}\Bigg)$$

$$+\left|\frac{\partial^2 a(x_1,x_2,U_m+\theta_2(U_m+\theta_1(u+\Psi_m)))}{\partial u^2}\right|_{0,\alpha}$$

$$+\left|\frac{\partial^2 A(x_1,x_2,U_m+\theta_2(U_m+\theta_1(u+\Psi_m)))}{\partial u^2}\theta_1(u+\Psi_m)^2\right|_{0,\alpha}\leqslant K_1\varepsilon_1^{3m-9},\ (0\leqslant\theta_1,\theta_2\leqslant 1)$$

所以

$$|T_\varepsilon[u]|_{2,\alpha}\leqslant K_2\varepsilon_1^{\frac{1}{2}}\varepsilon_1^{\frac{3m-9}{2}}$$

当 ε_1 是充分小并使 $K_2\varepsilon_1^{\frac{1}{2}}\leqslant 1$ 时则有 $T_\varepsilon[u]\in S_m$，证毕.

引理 5：若 $u_1\in S_m$，$u_2\in S_m$，其中 $m\geqslant 6$，则 $|T_\varepsilon[u_1]-T_\varepsilon[u_2]|_{2,\alpha}\leqslant r|u_1-u_2|_{2,\alpha}$，$r$ 是小于 1 的正数.

证：根据引理 3 有

$$|T_\varepsilon[u_1]-T_\varepsilon[u_2]|_{2,\alpha}\leqslant\varepsilon^{-1}K|R(u_1+\Psi_m)-R(u_2+\Psi_m)|_{0,\alpha}$$

因 $R(u_1+\Psi_m)-R(u_2+\Psi_m)=\varepsilon\sum\limits_{i,j=1}^{2}\Big\{a_{i,j}(x_1,x_2,U_m)\dfrac{\partial^2(u_1-u_2)}{\partial x_i\partial x_j}$

$$-\frac{\partial^2 a_{i,j}(x_1,x_2,U_m+\theta_2(\cdots))}{\partial u^2}[\Psi_m+u_2+\theta_1(u_1-u_2)]\,(u_1-u_2)\frac{\partial^2 U_m}{\partial x_i\partial x_j}$$

$$-\frac{\partial a_{i,j}(x_1,x_2,U_m+\Psi_m+u_2+\theta_1(u_1-u_2))}{\partial u}(u_1-u_2)\frac{\partial^2\Psi_m}{\partial x_i\partial x_j}$$

$$-\left(a_{i,j}(x_1,x_2,U_m+\Psi_m+u_1)\frac{\partial^2 u_1}{\partial x_i\partial x_j}-a_{i,j}(x_1,x_2,U_m+\Psi_m+u_2)\frac{\partial^2 u_2}{\partial x_i\partial x_j}\right)\Big\}$$

$$+\varepsilon\sum_{i=1}^{2}\Big[a_i(x_1,x_2,U_m)\frac{\partial(u_1-u_2)}{\partial x_i}-\frac{\partial a_i(x_1,x_2,U_m+\Psi_m+u_2+\theta_1(u_1-u_2))}{\partial u}$$

$$\cdot(u_1-u_2)\left(\frac{\partial U_m}{\partial x_i}+\frac{\partial\Psi_m}{\partial x_i}\right)+\frac{\partial a_i(x_1,x_2,U_m)}{\partial u}(u_1-u_2)\frac{\partial U_m}{\partial x_i}$$

$$-a_i(x_1,x_2,U_m+\Psi_m+u_1)\frac{\partial u_1}{\partial x_i}+a_i(x_1,x_2,U_m+\Psi_m+u_2)\frac{\partial u_2}{\partial x_i}\Big]$$

$$-\varepsilon\frac{\partial^2 a(x_1,x_2,U_m+\theta_2(\cdots))}{\partial u^2}[\Psi_m+u_2+\theta_1(u_1-u_2)](u_1-u_2)$$

$$+\frac{\partial^2 A(x_1,x_2,U_m+\theta_2(\cdots))}{\partial u^2}[\Psi_m+u_2+\theta_1(u_1-u_2)](u_1-u_2)$$

$+\varepsilon\, a(x_1,x_2,U_m)(u_1-u_2)$ 所以

$$|T_\varepsilon[u_1]-T_\varepsilon[u_2]|\leqslant\varepsilon^{-1}K\,K_2\,\varepsilon_1^{\frac{3m-9}{2}}|u_1-u_2|_{2,\alpha}.$$

当 $m\geqslant 6$ 时，可取 ε_1 充分小使 $\varepsilon^{-1}K\,K_2\varepsilon_1^{\frac{3m-9}{2}}\leqslant r<1$，证毕.

从引理 4 和 5 知 T_ε 是 S_m 上的压缩算子，所以算子方程(3.9)在 S_m中存在唯一的解u_0，得到

定 理 5：在假设(H_1)—(H_3)下，当 ε 充分小时，狄立克雷问题(2.0.1)—(2.0.2) 存在解 $u_\varepsilon \in C_{2,\alpha}(\bar{\Omega})$，并且在球$S_m$，$(m\geqslant 6)$，中是唯一的，它的一致有效的渐近展开式是

$$u_\varepsilon = U_m + Z_m \tag{3.10}$$

其中 U_m 由(2.5.1)式 给 出，$|Z_m|_{2,\alpha} = O\left(\varepsilon_1^{\frac{3m-9}{2}}\right)$，$(m\geqslant 6)$

应用[1]中类似的方法可以免除条件“$m\geqslant 6$”，即有

定 理 6：在定理 5 的条件下，解 $u_\varepsilon \in C_{2,\alpha}(\bar{\Omega})$在 球 $S_m^{(0)} = \left\{ u\in C_0 \,\middle|\, |u|_0 \leqslant \varepsilon_1^{\frac{3m+1}{2}} \right\}$ 中是唯一的，并且在展开式 (3.10) 中成 立$|Z_m|_0 = O\left(\varepsilon_1^{\frac{3m+1}{2}}\right)$，$(m\geqslant 0)$，其 中 $|\cdots|_0$ 表 示 连续函数空间的范数．

证：首先知道边值问题(3.5)—(3.6)的解 $\tilde{Z}_m$ 在 $S_m^{(0)}$ 中是唯一 的，否则设另有一解 $Z_m^{(1)}$，则有

$$L_\varepsilon[\tilde{Z}_m - \tilde{Z}_m^{(1)}] = R(\tilde{Z}_m + \Psi_m) - R(\tilde{Z}_m + \Psi_m)$$

根据引理 3 知$\tilde{Z}_m - \tilde{Z}_m^{(1)}|_{0,\alpha} \leqslant K_2 \varepsilon_1^{\frac{1}{2}} |\tilde{Z}_m - \tilde{Z}_m^{(1)}|_{2,\alpha}$，当 ε_1 充分小 并 使 $K_2 \varepsilon_1^{\frac{1}{2}} < 1$时，则导出矛盾．从 $\tilde{Z}_m$ 的唯一性可推出 Z_m 的唯一性．

作函数$Z = U_{m+6} - U_m + Z_{m+6}$，因$|Z_{m+6}|_0 = O\left(\varepsilon_1^{\frac{3m+9}{2}}\right)$又 $|U_{m+6} - U_m|_0 = O\left(\varepsilon_1^{\frac{3m+1}{2}}\right)$，所以$|Z|_0 = O\left(\varepsilon_1^{\frac{3m+1}{2}}\right)$．再可以验证$Z$和 Z_m 满足同一边值问题(3.2)—(3.3)，所以$Z = Z_m$，因 此 $|Z_m|_0 = O\left(\varepsilon_1^{\frac{3m+1}{2}}\right)$，定理证毕．

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On the Dirichlet Problem for a Quasilinear Elliptic Equation with a Small Parameter

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Abstract

The method of "boundary layer corrections" is developed to study the Dirichlet problem for a quasilinear elliptic equation in a bounded domain, when the degenerate equation has characteristics tangent to the boundary. The existence and uniqueness of solution have been proved. The uniformly valid asymptotic expansion of solution has been constructed.