非均匀切向荷载下弹性半空间二阶效应的实例计算*

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（樊大钧推荐，1995年9月11日收到，1996年9月20日收到修改稿）

摘 要

继文[1]，本文详细讨论了一个实例，对二阶弹性材料在 z 方向的位移和应力作了数值计算。我们发现，二阶效应使 z 方向位移减小而使应力 t_{rz} 的值在荷载作用圆内减小而在圆外却增加。

关键词 弹性半空间 切向荷载 二阶弹性效应 积分变换

一、实 例

文[1]提供的非均匀切向荷载下弹性半空间二阶效应的一般解答，对于 $\delta>-1$ 的所有值都是适应的，但求解过程十分复杂。作为文[1]的一个实例，本文给出 $\delta=1/2$ 时的线性和二阶解答。

令 $\delta=1/2$，半空间表面 $z=0$ 处的线性解答为[1]

当 $r\leqslant a$ 时

$$v_r=-\frac{T}{4\mu}\left[\frac{(2-\eta)\sqrt{2\pi a}}{4}-\frac{(4-3\eta)\sqrt{\tau}\,r^2}{8\sqrt{2a^3}}\right]\cos\theta$$

$$v_\theta=\frac{T}{4\mu}\left[\frac{(2-\eta)\sqrt{2\pi a}}{4}-\frac{(4-\eta)\sqrt{\pi}\,r^2}{8\sqrt{2a^3}}\right]\sin\theta$$

$$v_z=-\frac{(1-2\eta)T}{4\mu}\ \frac{\sqrt{2a^2}}{3\sqrt{\pi}\,r}\left[1-\left(1-\frac{r^2}{a^2}\right)^{3/2}\right]\cos\theta$$

$$\tau_{rr}=T\frac{(4+\eta)\sqrt{\pi}\,r}{8\sqrt{2a^3}}\cos\theta,\quad \tau_{\theta\theta}=-T\frac{13\eta\sqrt{\pi}\,r}{8\sqrt{2a^3}}\cos\theta,\quad \tau_{zz}=0$$

$$\tau_{rz}=\frac{T}{\sqrt{2\pi a}}\left(1-\frac{r^2}{a^2}\right)^{1/2}\cos\theta,\quad \tau_{r\theta}=-T\frac{(2-\eta)\sqrt{\pi}\,r}{8\sqrt{2a^3}}\sin\theta$$

$$\tau_{\theta z}=-\frac{T}{\sqrt{2\pi a}}\left(1-\frac{r^2}{a^2}\right)^{1/2}\sin\theta$$

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* 湖南省自然科学基金资助课题

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当$r>a$时

$$v_r=-\frac{T\sqrt{a}}{16\mu\sqrt{2\pi}}\left[\frac{(4-\eta)r}{a}\sqrt{1-\frac{a^2}{r^2}}+\left(8-4\eta-(4-3\eta)\frac{r^2}{a^2}\right)\arcsin\frac{a}{r}-\frac{2\eta r}{a}\left(1-\frac{a^2}{r^2}\right)^{3/2}\right]\cos\theta$$

$$v_\theta=\frac{T\sqrt{a}}{16\mu\sqrt{2\pi}}\left[\frac{(4-3\eta)r}{a}\left(1-\frac{a^2}{r^2}\right)^{1/2}+(8-4\eta)-(4-\eta)\frac{r^2}{a^2}\arcsin\frac{a}{r}+\frac{2\eta r}{r}\left(1-\frac{a^2}{r^2}\right)^{3/2}\right]\sin\theta$$

$$v_z=-\frac{(1-2\eta)T}{4\mu}\,\frac{\sqrt{2a^3}}{3\sqrt{\pi}\,r}\cos\theta$$

$$\tau_{rr}=\frac{T}{\sqrt{2\pi a}}\left[\frac{4+\eta}{4}\,\frac{r}{a}\arcsin\frac{a}{r}-\frac{4-\eta}{4}\left(1-\frac{a^2}{r^2}\right)^{1/2}-\frac{\eta}{2}\left(1-\frac{a^2}{r^2}\right)^{3/2}\right]\cos\theta$$

$$\tau_{\theta\theta}=-\frac{T\eta}{\sqrt{2\pi a}}\left[\frac{5r}{4a}\arcsin\frac{a}{r}-\frac{3}{4}\left(1-\frac{a^2}{r^2}\right)^{1/2}-\frac{1}{2}\left(1-\frac{a^2}{r^2}\right)^{3/2}\right]\cos\theta$$

$$\tau_{zz}=\tau_{rz}=\tau_{\theta z}=0$$

$$\tau_{r\theta}=\frac{T}{\sqrt{2\pi a}}\left[\frac{2+\eta}{4}\left(1-\frac{a^2}{r^2}\right)^{1/2}-\frac{2-\eta}{4}\,\frac{r}{a}\arcsin\frac{a}{r}-\frac{\eta}{2}\left(1-\frac{a^2}{r^2}\right)^{3/2}\right]\sin\theta$$

为了求得二阶解答，首先需要求 $L(n,\ s)$．余下的计算包括积分和代数变换．首先列出所需的$L(n,\ s)$值如下：

$$L\left(0,\ -\frac{3}{2}\right)=\begin{cases}\left(\dfrac{\pi a}{8}\right)^{1/2}\left(1-\dfrac{1}{2}\,\dfrac{r^2}{a^2}\right), & r\leqslant a\\[2ex] \left(\dfrac{a}{8\pi}\right)^{1/2}\left[\dfrac{(r^2-a^2)^{1/2}}{a}+\left(2-\dfrac{r^2}{a^2}\right)\arcsin\dfrac{a}{r}\right], & r>a\end{cases}$$

$$L\left(0,\ -\frac{1}{2}\right)=\begin{cases}\left(\dfrac{2}{\pi a}\right)^{1/2}\left(1-\dfrac{r^2}{a^2}\right)^{1/2}, & r\leqslant a\\[2ex] 0, & r>a\end{cases}$$

$$L\left(0,\ \frac{1}{2}\right)=\begin{cases}\left(\dfrac{\pi}{2a^3}\right)^{1/2}, & r\leqslant a\\[2ex] \left(\dfrac{2}{\pi a}\right)^{1/2}\left[\dfrac{r}{a}\arcsin\dfrac{a}{r}-\left(\dfrac{1}{r^2-a^2}\right)^{1/2}\right], & r>a\end{cases}$$

$$L\left(1,\ -\frac{3}{2}\right)=\begin{cases}\left(\dfrac{2a^3}{9\pi r^2}\right)^{1/2}\left[1-\left(1-\dfrac{r^2}{a^2}\right)^{3/2}\right], & r\leqslant a\\[2ex] \left(\dfrac{2a^3}{9\pi r^2}\right)^{1/2}, & r>a\end{cases}$$

$$L\left(1,\ -\frac{1}{2}\right)=\begin{cases}\left(\dfrac{\pi r^2}{8a^3}\right)^{1/2}, & r\leqslant a\\[2ex] \left(\dfrac{1}{2\pi a}\right)^{1/2}\left[\dfrac{r}{a}\arcsin\dfrac{a}{r}-\left(1-\dfrac{a^2}{r^2}\right)^{1/2}\right], & r>a\end{cases}$$

$$
\left.\begin{aligned}
&L\left(1,\ \frac{1}{2}\right)=\begin{cases}\left(\dfrac{2r^2}{\pi a^5}\right)^{1/2}\left(1-\dfrac{r^2}{a^2}\right)^{-1/2}, & r<a\\ 0, & r>a\end{cases}\\
&L\left(2,\ \frac{3}{2}\right)=\begin{cases}\left(\dfrac{\pi r^4}{128a^3}\right)^{1/2}, & r\leqslant a\\ \left(\dfrac{r}{32\pi a}\right)^{1/2}\left[\left(1-\dfrac{a^2}{r^2}\right)^{1/2}+\dfrac{r}{a}\arcsin\dfrac{a}{r}-2\left(1-\dfrac{a^2}{r^2}\right)^{3/2}\right], & r>a\end{cases}\\
&L\left(2,\ -\frac{1}{2}\right)=\begin{cases}\left(\dfrac{2}{9\pi a}\right)^{1/2}\left[2\dfrac{a^2}{r^2}\left(1-\left(1-\dfrac{r^2}{a^2}\right)^{1/2}\right)-\left(1-\dfrac{r^2}{a^2}\right)^{1/2}\right], & r\leqslant a\\ \left(\dfrac{8a^3}{9\pi r^4}\right)^{1/2}, & r>a\end{cases}\\
&L\left(2,\ \frac{1}{2}\right)=\begin{cases}0, & r<a\\ \left(\dfrac{2a^3}{\pi r^6}\right)^{1/2}\left(1-\dfrac{a^2}{r^2}\right)^{-1/2}, & r>a\end{cases}
\end{aligned}\right\}\quad(1.1)
$$

将这些值及从文献[1]中(3.41)至(3.43)的其它值代入式(3.40)，我们得到二阶位移分量为

$$
\left.\begin{aligned}
\frac{16\mu^2 w_r}{T^2}=&-2(1-\eta)M_1(r)+(1-2\eta)M_2(r)+\frac{\cos2\theta}{4-5\eta+2\eta^2}\Big[(1+\eta-2\eta^2)M_3(r)\\
&+\frac{2\eta}{r}M_4(r)-2(1-\eta)^2-\frac{1(1-2\eta)^2}{r}M_6(r)-(4+9\eta+4\eta^2)M_7(r)\\
&+\frac{2(4+5\eta-8\eta)^2}{r}M_8(r)+\frac{\mu(1-\eta)(4+9\eta+4\eta^2)}{2}M_9(r)\\
&+\frac{\mu(1-\eta)(4+5\eta-8\eta^2)}{r}M_{10}(r)\Big]\\
\frac{16\mu^2 w_\theta}{T^2}=&\frac{2\sin2\theta}{4-5\eta+2\eta^2}\Big[(1-\eta)^2M_3(r)-\frac{3(1-\eta)}{r}M_4(r)+(1-\eta)M_5(r)\\
&-\frac{2\eta(1-\eta)}{r}M_6(r)+2\eta(1-\eta)M_7(r)+\frac{4-13\eta+8\eta^2}{r}M_8(r)\\
&-\mu\eta(1-\eta)^2M_9(r)-\frac{\mu(1-\eta)(4-13\eta+8\eta^2)}{2r}M_{10}(r)\Big]\\
\frac{16\mu^2 w_z}{T^2}=&(1-2\eta)M_{11}(r)-2(1-\eta)M_{12}(r)\\
&+\frac{\cos2\theta}{4-5\eta+2\eta^2}\Big[\frac{10-21\eta+8\eta^2}{2}M_{13}(r)+(5\eta-6\eta^2)M_{14}(r)\\
&-(8-34\eta+24\eta^2)M_{15}(r)+\mu(1-\eta)(8-34\eta+24\eta^2)M_{16}(r)\Big]
\end{aligned}\right\}
$$

式中 (1.2)

$$
\begin{aligned}
M_1(r)=&B_1\int_0^a x^2\left(1-\frac{x^2}{a^2}\right)^{1/2}K_{11}(0,\ x)dx+B_2\int_0^a\left[1-\left(1-\frac{x^2}{a^2}\right)^{3/2}\right]K_{11}(0,\ x)dx\\
&+B_3\int_a^\infty P(x)K_{11}(0,\ x)\frac{dx}{x}-\frac{4b_2+b_6}{24}\int_0^a\left[1-\left(1-\frac{x^2}{a^2}\right)^{1/2}\right]K_{11}(0,\ x)dx\\
&+B_4\int_a^\infty\left(1-\frac{a^2}{x^2}\right)^{1/2}K_{11}(0,\ x)\frac{dx}{x^3}
\end{aligned}
$$

$$M_2(r)=B_5\int_0^a x\left(1-\frac{x^2}{a^2}\right)K_{10}(0,\ x)dx-B_6\int_0^a x^3K_{10}(0,\ x)dx$$

$$-\frac{a^3b_{13}}{9\pi}\int_0^a\left[1-\left(1-\frac{x^2}{a^2}\right)^{1/2}\right]^2 K_{10}(0,\ x)\frac{dx}{x^3}$$

$$-\frac{2a^3b_{18}}{9\pi}\int_0^a\left[1-\left(1-\frac{x^2}{a^2}\right)^{3/2}\right]\left[1-\left(1-\frac{x^2}{a^2}\right)^{1/2}\right]K_{10}(0,\ x)\frac{dx}{x^3}$$

$$-\frac{4a^3b_{17}}{9\pi}\int_0^a\left[1-\left(1-\frac{x^2}{a^2}\right)^{1/2}\right]^2 K_{10}(0,\ x)\frac{dx}{x^3}$$

$$-\frac{a(3b_{15}-b_{18})}{9\pi}\int_0^a\left(1-\frac{x^2}{a^2}\right)^{1/2}\left[1-\left(1-\frac{x^2}{a^2}\right)^{3/2}\right]K_{10}(0,\ x)\frac{dx}{x}$$

$$-\frac{a(6b_{19}-4b_{17})}{9\pi}\int_0^a\left(1-\frac{x^2}{a^2}\right)^{1/2}\left[1-\left(1-\frac{x^2}{a^2}\right)^{1/2}\right]K_{10}(0,\ x)\frac{dx}{x}$$

$$-B_7\int_a^\infty K_{10}(0,\ x)\frac{dx}{x^3}-B_8\int_a^\infty P^2(x)K_{10}(0,\ x)dx$$

$$-\frac{a(b_{14}+2b_{16})}{16\pi}\int_a^\infty\left(1-\frac{a^2}{x^2}\right)^{1/2}P(x)K_{10}(0,\ x)\frac{dx}{x}$$

$$-\frac{b_{14}a^3}{16\pi}\int_a^\infty\left(1-\frac{a^2}{x^2}\right)K_{10}(x)\frac{dx}{x^3}$$

$$M_3(r)=B_9\int_0^a\left[1-\left(1-\frac{x^2}{a^2}\right)^{3/2}\right]K_{13}(0,\ x)dx$$

$$-\frac{b_8+b_{27}}{8}\int_0^a\left[1-\left(1-\frac{x^2}{a^2}\right)^{1/2}\right]K_{13}(0,\ x)dx$$

$$+B_{10}\int_0^a x^2\left(1-\frac{x^2}{a^2}\right)^{1/2}K_{13}(0,x)dx+B_{11}\int_a^\infty P(x)K_{13}(0,\ x)\frac{dx}{x}$$

$$-\frac{a^3b_{29}}{6\pi}\int_a^\infty\left(1-\frac{a^2}{x^2}\right)^{1/2}K_{13}(0,\ x)\frac{dx}{x^3}$$

$$M_5(r)=B_{12}\int_0^a\left[1-\left(1-\frac{x^2}{a^2}\right)^{3/2}\right]K_{11}(0,\ x)dx$$

$$+\frac{b_{27}-b_8}{3}\int_0^a\left[1-\left(1-\frac{x^2}{a^2}\right)^{1/2}\right]K_{11}(0,\ x)dx$$

$$-B_{13}\int_0^a x^2\left(1-\frac{x^2}{a^2}\right)^{1/2}K_{11}(0,x)dx+B_{14}\int_a^\infty P(x)K_{11}(0,\ x)\frac{dx}{x}$$

$$+\frac{a^3b_{29}}{6\pi}\int_a^\infty\left(1-\frac{a^2}{x^2}\right)^{1/2}K_{11}(0,\ x)\frac{dx}{x^3}$$

$$M_7(r)=-\frac{\pi(4b_{20}+b_{23})}{64a^3}\int_0^a x^3K_{12}(0,\ x)dx+B_{15}\int_0^a x\left(1-\frac{x^2}{a^2}\right)K_{12}(0,\ x)dx$$

$$-\frac{a(2\mu(1-2\eta)+b_{22})}{3\pi}\int_0^a\left[\left(1-\frac{x^2}{a^2}\right)^{1/2}-\left(1-\frac{x^2}{a^2}\right)^2\right]K_{12}(0,\ x)\frac{dx}{x}$$

$$-\frac{2ab_{24}}{3\pi}\int_0^a\left(1-\frac{x^2}{a^2}\right)^{1/2}\left[1-\left(1-\frac{x^2}{a^2}\right)^{1/2}\right]K_{12}(0,\ x)\frac{dx}{x}$$

$$-\frac{4b_{20}+b_{23}}{16\pi a}\int_a^\infty P^2(x)K_{12}(0,\ x)xdx$$

$$-\frac{ab_{23}}{8\pi}\int_a^\infty\left(1-\frac{a^2}{x^2}\right)^{1/2}P(x)K_{12}(0,\ x)\frac{dx}{x}$$

$$M_9(r)=B_{16}\int_0^a x^2K_{13}(-1,\ x)dx-\frac{4(b_{35}+b_{45})}{3\pi a}\int_0^a\left[1-\left(1-\frac{x^2}{a^2}\right)^{1/2}\right]K_{13}(-1,\ x)dx$$

$$+B_{17}\int_0^a\left(1-\frac{x^2}{a^2}\right)K_{13}(-1,\ x)dx+\frac{2b_{36}}{3\pi a}\int_0^a\left[\left(1-\frac{x^2}{a^2}\right)^{-1/2}\right.$$

$$\left.-\left(1-\frac{x^2}{a^2}\right)\right]K_{13}(-1,\ x)dx$$

$$+\frac{4a(b_{27}+b_{51})}{3\pi}\int_0^a\left[\left(1-\frac{x^2}{a^2}\right)^{1/2}-\left(1-\frac{x^2}{a^2}\right)\right]K_{13}(-1,\ x)\frac{dx}{x^2}$$

$$+B_{18}\int_0^a\left[\left(1-\frac{x^2}{a^2}\right)^{1/2}-\left(1-\frac{x^2}{a^2}\right)^2\right]K_{13}(-1,\ x)\frac{dx}{x^2}$$

$$+\frac{4a^3(b_{40}+b_{42})}{9\pi}\int_0^a\left[1-\left(1-\frac{x^2}{a^2}\right)^{3/2}\right]\left[1-\left(1-\frac{x^2}{a^2}\right)^{1/2}\right]K_{13}(-1,\ x)\frac{dx}{x^4}$$

$$+\frac{2b_{46}}{9\pi a}\int_0^a R(x)K_{13}(-1,\ x)dx+\frac{b_{50}}{6a^2}\int_0^a x^2R(x)K_{13}(-1,\ x)dx$$

$$+B_{19}\int_a^\infty P(x)Q(x)K_{13}(-1,\ x)\frac{dx}{x^4}+B_{20}\int_a^\infty P^2(x)K_{13}(-1,\ x)dx$$

$$+\frac{a(b_{33}+b_{48})}{4\pi}\int_a^\infty\left(1-\frac{a^2}{x^2}\right)^{1/2}P(x)K_{13}(-1,\ x)\frac{dx}{x^2}$$

$$+B_{21}\int_a^\infty\frac{K_{13}(-1,\ x)}{x^4}dx+\frac{a(b_{33}+b_{47})}{2\pi}\int_a^\infty\left(1-\frac{a^2}{x^2}\right)^{1/2}Q(x)K_{13}(-1,\ x)dx$$

$$+\frac{ab_{39}}{\pi}\int_a^\infty\left(1-\frac{a^2}{x^2}\right)^{-1/2}K_{13}(-1,\ x)\frac{dx}{x^2}+\frac{2ab_{50}}{3\pi}\int_a^\infty P(x)K_{13}(-1,\ x)\frac{dx}{x}$$

且式中

$$\left.\begin{aligned}P(x)&=\frac{x}{a}\arcsin\frac{a}{x}-\left(1-\frac{a^2}{x^2}\right)^{1/2}\\Q(x)&=\frac{x}{a}\arcsin\frac{a}{x}-\left(1-\frac{a^2}{x^2}\right)^{-1/2}\\R(x)&=\frac{2a^2}{x^2}\left[1-\left(1-\frac{x^2}{a^2}\right)^{1/2}\right]-\left(1-\frac{x^2}{a^2}\right)^{1/2}\end{aligned}\right\}\tag{1.3}$$

注意到，在M_3中用$K_{23}(-1,x)$代替$K_{13}(0,x)$，得到M_4；用$K_{21}(-1,x)$代替M_5中的 $K_{11}(0,x)$，得到 M_6；类似地用 $K_{22}(-1,\ x)$，$K_{23}(-2,\ x)$ 分别代替 M_7，M_9 中的$K_{12}(0,\ x)$ 和 $K_{13}(-1,\ x)$，得到 M_8和M_{10}。再者，用 $K_{01}(0,\ x)$，$K_{00}(0,\ x)$，$K_{23}(0,\ x)$，$K_{21}(0,\ x)$，$K_{22}(0,\ x)$，$K_{23}(-1,\ x)$分别代替M_1，M_2，M_3，M_5，M_7，M_9中的 $K_{11}(0,\ x)$，$K_{10}(0,$

x), $K_{13}(0,\ x)$, $K_{11}(0,\ x)$, $K_{12}(0,\ x)$, $K_{13}(0,\ x)$，我们得到M_{11}, M_{12}, M_{13}, M_{14}, M_{15}, M_{16}。

对于应力分量，这里仅给出分量τ''_{zz}与τ''_{zr}，而其它分量可用类似方法计算．我们求得

(i) 当$r\leqslant a$时

$$\frac{8\mu^2\tau''_{zz}}{T^2}=B_5\left(1-\frac{r^2}{a^2}\right)-B_6r^2-\frac{a^3b_{13}}{9\pi r^4}\left[1-\left(1-\frac{r^2}{a^2}\right)^{3/2}\right]^2-\frac{4a^2b_{17}}{9\pi r^4}\left[1-\left(1-\frac{r^2}{a^2}\right)^{1/2}\right]^2$$
$$-\frac{2a^3b_{18}}{9\pi r^4}\left[1-\left(1-\frac{r^2}{a^2}\right)^{3/2}\right]\left[1-\left(1-\frac{r^2}{a^2}\right)^{1/2}\right]$$
$$-\frac{a(3b_{15}-b_{18})}{9\pi r^2}\left(1-\frac{r^2}{a^2}\right)^{1/2}\left[1-\left(1-\frac{r^2}{a^2}\right)^{3/2}\right]-\frac{a(6b_{19}-4b_{17})}{9\pi r^2}\left(1-\frac{r^2}{a^2}\right)^{1/2}\left[1\right.$$
$$\left.-\left(1-\frac{r^2}{a^2}\right)^{1/2}\right]+\cos 2\theta\left\{B_{15}\left(1-\frac{r^2}{a^2}\right)-\frac{4(b_{20}+b_{23})\pi r^2}{64a^3}\right.$$
$$-\frac{a(2\mu(1-2\eta)+b_{22})}{3\pi r^2}\left[\left(1-\frac{r^2}{a^2}\right)^{1/2}-\left(1-\frac{r^2}{a^2}\right)^2\right]$$
$$\left.-\frac{2ab_{24}}{3\pi r^2}\left(1-\frac{r^2}{a^2}\right)^{1/2}\left[1-\left(1-\frac{r^2}{a^2}\right)^{1/2}\right]\right\}$$

$$\frac{8\mu^2\tau''_{rz}}{T^2}=B_1r\left(1-\frac{r^2}{a^2}\right)^{1/2}+\frac{B_2}{r}\left[1-\left(1-\frac{r^2}{a^2}\right)^{3/2}\right]-\frac{4b_2+b_6}{24r}\left[1-\left(1-\frac{r^2}{a^2}\right)^{1/2}\right]$$
$$+\cos 2\theta\left\{-B_{22}r\left(1-\frac{r^2}{a^2}\right)^{1/2}+\frac{3\mu(1-2\eta)-b_9}{12r}\left[1-\left(1-\frac{r^2}{a^2}\right)^{3/2}\right]\right.$$
$$\left.-\frac{b_8}{6r}\left[1-\left(1-\frac{r^2}{a^2}\right)^{1/2}\right]\right\}$$

(ii) 当$r>a$时

$$\frac{8\mu^2\tau''_{zz}}{T^2}=-\frac{B_7}{r^4}-B_3P^2(r)-\frac{a^3b_{14}}{16\pi r^4}\left(1-\frac{a^2}{r^2}\right)-\frac{a(b_{14}+2b_{16})}{16\pi r^2}P(r)\left(1-\frac{a^2}{r^2}\right)^{1/2}$$
$$-\cos 2\theta\left[\frac{4b_{20}+b_{23}}{16\pi r^2}P^2(r)+\frac{ab_{23}}{8\pi r^2}P(r)\left(1-\frac{a^2}{r^2}\right)^{1/2}\right]$$

$$\frac{8\mu^2\tau''_{rz}}{T^2}=\frac{B_3P(r)}{r^2}+\frac{B_4}{r^2}\left[P(r)+\frac{2a^2}{r^2}\left(1-\frac{a^2}{r^2}\right)^{1/2}\right]$$
$$+\left[\frac{a\mu(1-2\eta)}{\pi}-\frac{a(2b_8+b_9)}{6\pi}\right]\frac{\cos 2\theta P(r)}{r^2}$$

将式(1.2)代入式(1.1)，我们可得到τ'_{zz}与τ'_{rz}．上式中，B_{ij}可在文[1]中找到，$P(r)$由式(1.3)定义，而T由文[1](3.14)式给出．

二、数 值 结 果

为了显示二阶效应，我们提供一些数值计算结果．在下面的计算中，主项是线弹性解，而余项表示二阶效应．我们仅计算z方向的位移和应力．

对于可压缩材料，应变不变量J_1, J_2, J_3可写为

$$J_1=2e_{rr},\ J_2=2(e_{rr}e_{ss}-e_{rs}e_{rs}),\ J_3=8\det(e_{rs}) \tag{2.1}$$

式中 $e_{rs}=\frac{1}{2}\left(\frac{\partial u_r}{\partial x_s}+\frac{\partial u_s}{\partial x_r}+\frac{\partial u_k}{\partial x_r}\frac{\partial u_k}{\partial x_s}\right)$

采用由Murnaghan[2]提供的三个其它应变不变量$\bar{I}_1$，$\bar{I}_2$和$\bar{I}_3$，我们有

$$J_1=2\bar{I}_1,\ J_2=4\bar{I}_2,\ J_3=8\bar{I}_3 \tag{2.2}$$

式中 $\bar{I}_1=e_{rr},\ \bar{I}_2=(e_{rr}e_{ss}-e_{rs}e_{rs})/2,\ \bar{I}_3=\det(e_{rs})$

由 Murnaghan[2] 采用的5个弹性系数是λ，μ，l，m，n. Murnaghan与Rivlin采用的弹性系数之间的关系由Truesdell与Noll[4]给出：

$$\left.\begin{aligned}&a_1=-\mu/2,\ a_2=(\lambda+2\mu)/8\\&a_3=m+\mu,\ a_4=-\mu/3+l,\ a_5=n-\mu\end{aligned}\right\} \tag{2.3}$$

Foux[3]对于钢，给出了如下实验数据

$$\left.\begin{aligned}&\mu=8.26\times10^3\text{kg/mm}^2\\&K=\lambda+2\mu/3=17.0\times10^3\text{kg/mm}^2\end{aligned}\right\} \tag{2.4}$$

$$l/\mu=-1.6,\ m/\mu=-10.1,\ n/\mu=-22.7 \tag{2.5}$$

由式(2.2)，(2.3)和(2.4)得

$$\left.\begin{aligned}&a_1/\mu=-0.5,\ a_2/\mu=\frac{1}{6}+\frac{17}{8.26},\ a_3/\mu=-9.1\\&a_4/\mu=-(1.6+1/3),\ a_5/\mu=-23.7\end{aligned}\right\} \tag{2.6}$$

且 $\eta=862/2963$

采用上述数值并注意到$\bar{r}=r/a$，对于z方向的位移和应力t_{rz}获得如下数值结果：

$\bar{r}$	0.2	0.5
u_z/a	$-0.0099\cos\theta\epsilon-0.0214\epsilon^2+0.0201\cos2\theta\epsilon^2$	$-0.0301\cos\theta\epsilon-0.9575\epsilon^2+0.9408\cos2\theta\epsilon^2$
t_{rz}/μ	$0.2292\cos\theta\epsilon-0.0047\epsilon^2-0.0001\cos2\theta\epsilon^2$	$0.1528\cos\theta\epsilon-0.0090\epsilon^2+0.0007\cos2\theta\epsilon^2$
$\bar{r}$	0.8	1.0−
u_z/a	$-0.0326\cos\theta\epsilon-1.6888\epsilon^2+1.4137\cos2\theta\epsilon^2$	$-0.0333\cos\theta\epsilon-1.7216\epsilon^2+1.6805\cos2\theta\epsilon^2$
t_{rz}/μ	$0.0859\cos\theta\epsilon-0.0071\epsilon^2+0.0045\cos2\theta\epsilon^2$	$0.0081\epsilon^2+0.0187\cos2\theta\epsilon^2$
$\bar{r}$	1.0+	2.0
u_z/a	$-0.0333\cos\theta\epsilon-1.3259\epsilon^2+1.3874\cos2\theta\epsilon^2$	$-0.0042\cos\theta\epsilon-0.0095\epsilon^2+0.0087\cos2\theta\epsilon^2$
t_{rz}/μ	$0.0212\epsilon^2+0.0187\cos2\theta\epsilon^2$	$0.0022\epsilon^2+0.0005\cos2\theta\epsilon^2$

式中 $\epsilon=P/(\mu a^2)$

由此，我们发现在整个区域，二阶效应使z方向的位移减小. 而应力分量t_{rz}的二阶效应的变化在圆内与圆外不同.例如，在圆内t_{rz}的值由于二阶效应而减小，而在圆外其值增加.二阶效应的大小也随θ值而变化. 我们还发现二阶位移和应力在$r=a$处发生间断. 最后，我们还指出，参数ϵ也影响二阶弹性效应的大小，即施力总量P越大，则二阶效应越大；而弹性常数μ越大，则二阶效应越小.

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The Illustration Calculations of Second Order Effects in Elastic Half-Space Acted upon By A Non-Uniform Shear Load

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Abstract

This paper is a continuation of [1]. An example is discussed in detail to illustrate the second order effects. Numerical calculations for the second order elastic material for the z-direction displacement and the stress t_{rz} are carried out. It is found that the second order effect is to reduce z-direction displacement and to decrease t_{rz} inside the circle but to increase its value outside the circle.

Key words elastic half-space, shear load, second order elasticity effects, integral transform