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奇摄动非线性系统边值问题*

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摘　要

本文利用对角化技巧和方法讨论二阶奇摄动非线性系统边值问题

$$\varepsilon y''=f(t,y,y',\varepsilon),\ y(0,\varepsilon)=a(\varepsilon),\ y(1,\varepsilon)=b(\varepsilon)$$

当Jacobi矩阵$f_{y'}$的特征值有K个负实部和$N-K$个正实部时，解的存在性及其渐近性质.

关键词　非线性系统　边值问题　对角化　奇摄动

一、引　　言

利用对角化技巧和方法研究非线性系统

$$\varepsilon y''=f(t,y,y',\varepsilon) \tag{1.1}$$

边值问题奇异摄动已有不少研究成果[1~4]．在上述文章中都基于这样一个条件，即f关于y'的Jacobi矩阵$f_{y'}$的特征值具有非零同号实部，这显然是比较强的条件．文[5]显然放宽了这一条件，但由于它是通过微分不等式理论得以论证的，所以仅讨论了弱耦合系统，即要求$f_{y'}$是对角矩阵．本文的工作是在文[1～2]的基础上，证明了系统(1.1)在$f_{y'}$的特征值具有非零实部时的第一边值问题

$$y(0,\varepsilon)=a(\varepsilon),\quad y(1,\varepsilon)=b(\varepsilon) \tag{1.2}$$

解的存在性，并估计了余项，进一步完善了对角化技巧．这里y,f,a,b均属于R^N，我们用下标$i\,(1\leqslant i\leqslant N)$表示$N$维向量的第$i$个分量．

我们假设

(D_1) 退化问题

$$0=f(t,\bar{y},\bar{y}',0) \tag{1.3}$$

$$\left.\begin{aligned}&\bar{y}_i(1)=b_i(0), \qquad (i=1,2,\cdots,K)\\&\bar{y}_i(0)=a_i(0), \qquad (i=K+1,\cdots,N)\end{aligned}\right\} \tag{1.4}$$

存在解$\bar{y}(t)\in C^2[0,1]$

(D_2) Jacobi矩阵$f_{y'}(t,\bar{y}(t),\bar{y}'(t),0)$有$K$个特征值实部$\leqslant-8\mu<0$，有$N-K$个特征值实部$\geqslant 8\mu>0$．

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(D_3) f及其Jacobi矩阵f_y，$f_{y'}$关于(t,y,y',ε)连续，同时

$$f(t,\bar{y}(t),\bar{y}'(t),\varepsilon)=O(\varepsilon)$$

$$f_y(t,\bar{y}(t),\bar{y}'(t),\varepsilon)-f_y(t,\bar{y}(t),\bar{y}'(t),0)=O(\varepsilon)$$

$$f_{y'}(t,\bar{y}(t),\bar{y}'(t),\varepsilon)-f_{y'}(t,\bar{y}(t),\bar{y}'(t),0)=O(\varepsilon)$$

f_y关于y,y'，$f_{y'}$关于y满足Lipschitz条件，而要求$f_{y'}$关于y'下式成立：

$$|f_{y'}(t,y,y_1',\varepsilon)-f_{y'}(t,y,y_2',\varepsilon)|\leqslant k_0\varepsilon|y_1'-y_2'|$$

这里$0\leqslant t\leqslant 1$，$0<\varepsilon\leqslant\varepsilon_0$，$|y-\bar{y}(t)|\leqslant\delta_0$，$|y'-\bar{y}'(t)|\leqslant d_0$，$\varepsilon_0,\delta_0$是小常数，而$d_0$，$k_0$是足够大的常数．

(D_4) $b_i(\varepsilon)=b_i(0)+O(\varepsilon)$ $(i=1,2,\cdots,K)$，$a_i(\varepsilon)=a_i(0)+O(\varepsilon)$ $(i=K+1,\cdots,N)$；$b_i(\varepsilon)$ $(i=K+1,\cdots,N)$，$a_i(\varepsilon)$ $(i=1,2,\cdots,K)$关于$\varepsilon\geqslant 0$连续．

标准阶符号$O(\varepsilon)$当$\varepsilon\to 0^+$时关于$0\leqslant t\leqslant 1$一致地成立．此外，对$0\leqslant t\leqslant 1$上的向量或矩阵函数$A(t)=[a_{ij}(t)]$，我们规定

$$|A(t)|=\left(\sum_{i,j}a_{ij}^2(t)\right)^{\frac{1}{2}},\quad \|A(t)\|=\sup_{0\leqslant t\leqslant 1}|A(t)|$$

二、预备定理

为了证明本文的主要结果，我们必须把文[2]中的对角化技巧加以完善，为此先证明如下引理．

我们假设

(Ⅰ) $C(t,\varepsilon)=C(t,0)+O(\varepsilon)$，$D(t,\varepsilon)=D(t,0)+O(\varepsilon)$是关于$0\leqslant t\leqslant 1$及$\varepsilon>0$连续有界的$N\times N$矩阵函数；

(Ⅱ) 对于$0\leqslant t\leqslant 1$，$C(t,0)$有K个特征值实部$\geqslant 8\mu>0$，有$N-K$个特征值实部$\leqslant -8\mu<0$．

引理 如果假设条件(Ⅰ)～(Ⅱ)成立，那么

(1°) 当$N\times N$阶方阵$M_i(\varepsilon)=M_i(0)+O(\varepsilon)$ $(i=1,2)$，而且$M_1(0)P^*+M_2(0)X(1,0)(I_N-P^*)X^{-1}(1,0)$非奇异时，存在$\varepsilon_1>0$，使得当$0<\varepsilon\leqslant\varepsilon_1$和$0<t\leqslant 1$时矩阵边值问题

$$\varepsilon P'=-C(t,\varepsilon)P-\varepsilon P^2-D(t,\varepsilon) \tag{2.1}$$

$$M_1(\varepsilon)P(0)+M_2(\varepsilon)P(1)=0 \tag{2.2}$$

有解$P(t)=P(t,\varepsilon)$关于(t,ε)一致有界．这里$X(t)=X(t,\varepsilon)$是线性方程

$$\varepsilon x'=-C(t,\varepsilon)x \tag{2.3}$$

具有$X(0,\varepsilon)=I_N$的基本矩阵，$P^*=\begin{bmatrix} I_K & 0 \\ 0 & 0\end{bmatrix}$是投影矩阵，$I_N$，$I_K$分别是$N$阶和$K$阶单位矩阵．

$$X(1,0)=\lim_{\varepsilon\to 0^+}X(1,\varepsilon),\quad X^{-1}(1,0)=\lim_{\varepsilon\to 0^+}X^{-1}(1,\varepsilon)$$

(2°) 当$N\times N$阶方阵

$$N_1(\varepsilon)=\begin{bmatrix} N_{11}(\varepsilon) & 0 \\ 0 & 0\end{bmatrix},\quad N_2(\varepsilon)=\begin{bmatrix} 0 & 0 \\ 0 & N_{22}(\varepsilon)\end{bmatrix}$$

具有性质$N_{11}(\varepsilon)=N_{11}(0)+O(\varepsilon)$，$N_{22}(\varepsilon)=N_{22}(0)+O(\varepsilon)$和$N_1(0)W(0,0)+N_2(0)$非奇异时，存在$\varepsilon_2>0$，使得当$0<\varepsilon\leqslant\varepsilon_2$和$0\leqslant t\leqslant 1$时，矩阵边值问题

$$\varepsilon Q'=\varepsilon P(t,\varepsilon)Q+Q[C(t,\varepsilon)+\varepsilon P(t,\varepsilon)]-I_N \tag{2.4}$$

$$N_1(\varepsilon)Q(0)=0,\quad N_2(\varepsilon)Q(1)=0 \tag{2.5}$$

有解$Q(t)=Q(t,\varepsilon)$关于(t,ε)一致有界，这里$P(t,\varepsilon)$是问题(2.1)、(2.2)的解，$N_{11}(\varepsilon)$ 是K阶子块，$N_{22}(\varepsilon)$是$(N-K)$阶子块，$W(0,0)=\lim\limits_{\varepsilon\to0+}W(0,\varepsilon)$.

证明 (1°) 由假设条件(Ⅱ)及文[6]引理知，当$\varepsilon>0$充分小时，方程(2.3)有一个指数二分法，存在常数$l>0$，使得：

$$\left.\begin{aligned}&|X(t,\varepsilon)P^*X^{-1}(s,\varepsilon)|\leqslant l\exp[-2\mu(t-s)/\varepsilon], && t\geqslant s\\ &|X(t,\varepsilon)(I_N-P^*)X^{-1}(s,\varepsilon)|\leqslant l\exp[-2\mu(s-t)/\varepsilon], && s\geqslant t\end{aligned}\right\} \tag{2.6}$$

令
$$\begin{aligned}\Delta_1(\varepsilon)=&M_1(\varepsilon)[P^*+(I_N-P^*)X^{-1}(1,\varepsilon)]+M_2(\varepsilon)[X(1,\varepsilon)P^*\\&+X(1,\varepsilon)(I_N-P^*)X^{-1}(1,\varepsilon)]\end{aligned}$$

由(2.6)知

$$\Delta_1(0)=\lim_{\varepsilon\to0^+}\Delta_1(\varepsilon)=M_1(0)P^*+M_2(0)X(1,0)(I_N-P^*)X^{-1}(1,0)$$

非奇异，从而当$\varepsilon>0$充分小时，$\Delta_1(\varepsilon)$非奇异．我们定义映射T：

$$\begin{aligned}TP(t)=&[X(t,\varepsilon)P^*+X(t,\varepsilon)(I_N-P^*)X^{-1}(1,\varepsilon)]\Delta_1^{-1}(\varepsilon)\\&\cdot\left\{M_2(\varepsilon)\int_0^1X(1,\varepsilon)P^*X^{-1}(s,\varepsilon)[P^2(s)+\varepsilon^{-1}D(s,\varepsilon)]ds\right.\\&\left.-M_1(\varepsilon)\int_0^1(I_N-P^*)X^{-1}(s,\varepsilon)[P^2(s)+\varepsilon^{-1}D(s,\varepsilon)]ds\right\}\\&-\int_0^tX(t,\varepsilon)P^*X^{-1}(s,\varepsilon)[P^2(s)+\varepsilon^{-1}D(s,\varepsilon)]ds\\&-\int_1^tX(t,\varepsilon)(I_N-P^*)X^{-1}(s,\varepsilon)[P^2(s)+\varepsilon^{-1}D(s,\varepsilon)]ds\end{aligned} \tag{2.7}$$

取 $\rho=2l\mu^{-1}\|D(t,\varepsilon)\|[l\|\Delta_1^{-1}(\varepsilon)\|(\|M_1(\varepsilon)\|+\|M_2(\varepsilon)\|+1)]$
和充分小的$\varepsilon>0$，便得$\varepsilon\rho^2/\|D(t,\varepsilon)\|<1/2$，那么当$\|P(t)\|\leqslant\rho$时，有$\|TP(t)\|\leqslant\rho$，和当$\|P_1(t)\|\leqslant\rho$，$\|P_2(t)\|\leqslant\rho$时，有$\|TP_1(t)-TP_2(t)\|\leqslant(1/2)\|P_1(t)-P_2(t)\|$，故$T$是将球$S_\rho=\{P(t):\|P(t)\|\leqslant\rho\}$映入自身的压缩映射．由不动点原理知$T$在$S_\rho$中有唯一的不动点，即存在唯一的$P(t)\in S_\rho$，使$P(t)=TP(t)$．可以验证积分方程$P(t)=TP(t)$的唯一解$P(t)=P(t,\varepsilon)$是方程(2.1)～(2.2)的解，且关于$(t,\varepsilon)$一致有界：$\|P(t,\varepsilon)\|\leqslant\rho$.

(2°) 设$W(t)=W(t,\varepsilon)$是方程

$$\omega'=P(t,\varepsilon)\omega \tag{2.8}$$

的基本矩阵，具有$W(1,\varepsilon)=I$．由(1°)知$\|P(t,\varepsilon)\|\leqslant\rho$，所以

$$|W(t,\varepsilon)W^{-1}(s,\varepsilon)|\leqslant\exp[\rho|t-s|],\qquad 0\leqslant t,\ \varepsilon\leqslant1$$

由假设条件(Ⅱ)及文[6]同样可推得当$\varepsilon>0$充分小时，方程

$$\varepsilon z'=-[C(t,\varepsilon)+\varepsilon P(t,\varepsilon)]z \tag{2.9}$$

有一个指数二分法：存在常数$L>0$使得

$$\left.\begin{aligned}&|Z(t,\varepsilon)P^*Z^{-1}(s,\varepsilon)|\leqslant L\exp[\mu(s-t)/\varepsilon], && t\geqslant s\\ &|Z(t,\varepsilon)(I_N-P^*)Z^{-1}(s,\varepsilon)|\leqslant L\exp[\mu(t-s)/\varepsilon], && s\geqslant t\end{aligned}\right\} \tag{2.10}$$

这里$Z(t)=Z(t,\varepsilon)$是方程(2.9)的基本矩阵，具有$Z(0)=I_N$．令$\Delta_2(\varepsilon)=N_1(\varepsilon)W(0,\varepsilon)$

$+N_2(\varepsilon)$，因为

$$\Delta_2(0)=\lim_{\varepsilon\to 0^+}\Delta_2(\varepsilon)=N_1(0)W(0,0)+N_2(0)$$

非奇异，从而当$\varepsilon>0$充分小时$\Delta_2(\varepsilon)$非奇异。通过微分容易验证

$$\begin{aligned}Q(t)=&W(t)\|\Delta_2^{-1}(\varepsilon)\|\Big\{\Big[N_2(\varepsilon)\varepsilon^{-1}\int_0^1 W^{-1}(s)Z(s)(I_N-P^*)Z^{-1}(1)ds\Big]\\&\cdot[Z(1)P^*Z^{-1}(1)+Z(0)(I_N-P^*)Z^{-1}(1)]^{-1}\\&-\Big[N_1(\varepsilon)\varepsilon^{-1}\int_0^1 W(0)W^{-1}(s)Z(s)P^*ds\Big][Z(1)P^*\\&+(I^N-P^*)]^{-1}\Big\}[Z(1)P^*Z^{-1}(t)+Z(0)(I_N-P^*)Z^{-1}(t)]\\&-\varepsilon^{-1}\int_0^t W(t)W^{-1}(s)Z(s)(I_N-P^*)Z^{-1}(t)ds\\&-\varepsilon^{-1}\int_1^t W(t)W^{-1}(s)Z(s)P^*Z^{-1}(t)ds\end{aligned}$$

是方程(2.4)～(2.5)的解，对于

$$\begin{aligned}q=&4L^2\mu^{-1}\{L\|\Delta_2^{-1}(\varepsilon)\|[\|N_1(\varepsilon)\|\cdot\|[Z(1)P^*+(I_N-P^*)]^{-1}\|\\&+\|N_2(\varepsilon)\|\cdot\|[Z(1)P^*Z^{-1}(1)+Z(0)(I_N-P^*)Z^{-1}(1)]^{-1}\|+1\}\end{aligned}$$

如果取$\varepsilon>0$充分小使得$\rho\varepsilon\mu^{-1}\leqslant 1/2$，我们有$Q(t,\varepsilon)$ 关于(t,ε)一致有界：$\|Q(t,\varepsilon)\|\leqslant q$。引理得证。

三、主要结果

定理 如果假设条件(D_1)～(D_4)成立，那么存在与ε无关的常数 $\delta>0$，使得当$|a_i(0)-\bar{y}_i(0)|\leqslant\delta$ $(i=1,2,\cdots,K)$，$|b_i(0)-\bar{y}_i(1)|\leqslant\delta$，$(i=K+1,\cdots,N)$和 $\varepsilon>0$ 充分小时，边值问题(1.1)、(1.2)在$[0,1]$上有解$y(t,\varepsilon)$，并且满足

$$y(t,\varepsilon)=\bar{y}(t)+O(\varepsilon)+O(\exp[-\mu t/\varepsilon]+\exp[\mu(t-1)/\varepsilon])$$

$$y'(t,\varepsilon)=\bar{y}'(t)+O(\varepsilon)+O[\varepsilon^{-1}(\exp[-\mu t/\varepsilon]+\exp[\mu(t-1)/\varepsilon])]$$

上述表达式中的最后一项表示边界层的估计。

证明 令 $y-\bar{y}(t)=u(t)+v(t)$ (3.1)

要求满足：

$$\varepsilon u''=f(t,\bar{y}+u,\bar{y}'+u',\varepsilon)-\varepsilon\bar{y}'' \tag{3.2}$$

$$\left.\begin{aligned}&u_i(1)=b_i(\varepsilon)-b_i(0),\ u_i'(0)=0, \quad (i=1,2,\cdots,K)\\&u_i(0)=a_i(\varepsilon)-a_i(0),\ u_i'(1)=0, \quad (i=K+1,\cdots,N)\end{aligned}\right\} \tag{3.3}$$

$v(t)$满足

$$\varepsilon v''=f(t,\bar{y}+u+v,\bar{y}'+u'+v',\varepsilon)-f(t,\bar{y}+u,\bar{y}'+u',\varepsilon) \tag{3.4}$$

$$\left.\begin{aligned}&v_i(1)=0,\ v_i(0)=a_i(\varepsilon)-\bar{y}_i(0)-u_i(0), \quad (i=1,2,\cdots,K)\\&v_i(0)=0,\ v_i(1)=b_i(\varepsilon)-\bar{y}_i(1)-u_i(1), \quad (i=K+1,\cdots,N)\end{aligned}\right\} \tag{3.5}$$

首先讨论(3.2)～(3.3)，把(3.2)改写为

$$\varepsilon u''-f_{y'}(t,\bar{y}(t),\bar{y}'(t),\varepsilon)u'-f_y(t,\bar{y}(t),\bar{y}'(t),\varepsilon)u=g(t,\varepsilon) \tag{3.6}$$

其中

$$g(t,\varepsilon)=g(t,u,u',\varepsilon)=f(t,\bar{y}+u,\bar{y}'+u',\varepsilon)-\varepsilon y''$$
$$-f_{y'}(t,\bar{y}(t),\bar{y}'(t),\varepsilon)u'-f_y(t,\bar{y}(t),\bar{y}'(t),\varepsilon)u$$

(3.3)相应地写成

$$\left.\begin{array}{l}\begin{bmatrix}0 & 0\\ 0 & I_{N-K}\end{bmatrix}u(0)+\begin{bmatrix}I_k & 0\\ 0 & 0\end{bmatrix}u(1)=\theta_1(\varepsilon)\\ \begin{bmatrix}I_K & 0\\ 0 & 0\end{bmatrix}u'(0)+\begin{bmatrix}0 & 0\\ 0 & I_{N-K}\end{bmatrix}u'(1)=0\end{array}\right\}\tag{3.7}$$

这里

$$\theta_1(\varepsilon)=(b_1(\varepsilon)-b_1(0),\cdots,b_K(\varepsilon)-b_K(0),a_{K+1}(\varepsilon)-a_{K+1}(0),\cdots,a_N(\varepsilon)-a_N(0))^T$$

令 $M_1(\varepsilon)=N_2(\varepsilon)=\begin{bmatrix}I_K & 0\\ 0 & 0\end{bmatrix}$, $M_2(\varepsilon)=N_1(\varepsilon)=\begin{bmatrix}0 & 0\\ 0 & I_{N-K}\end{bmatrix}$

$$C(t,\varepsilon)=-f_{y'}(t,\bar{y}(t),\bar{y}'(t),\varepsilon),\quad D(t,\varepsilon)=-f_y(t,\bar{y}(t),\bar{y}'(t),\varepsilon)$$

根据假设条件(D_2)、(D_3)知$C(t,\varepsilon)$，$D(t,\varepsilon)$满足引理的条件(Ⅰ)、(Ⅱ)，而且由$M_i(\varepsilon)$，$N_i(\varepsilon)$ $(i=1,2)$的结构知

$$M_1(0)P^*+M_2(0)X(1,0)(I_N-P^*)X^{-1}(1,0),N_1(0)W(0,0)+N_2(0)$$

显然非奇异．于是，由引理推得关于$(t,\varepsilon)\in[0,1]\times[0,\varepsilon_0]$存在一致有界的矩阵函数$P(t)=P(t,\varepsilon)$，$Q(t)=Q(t,\varepsilon)$，分别满足(2.1)、(2.2)和(2.4)、(2.5)同时又由(3.7)中$M_i(\varepsilon)$的特殊结构，我们可推得

$$M_1(\varepsilon)P(0)+M_2(\varepsilon)P(1)=0$$

等价于

$$M_1(\varepsilon)P(0)=0,\quad M_2(\varepsilon)P(1)=0$$

利用变量替换

$$\begin{bmatrix}u\\ u'\end{bmatrix}=\begin{bmatrix}I_N & -\varepsilon Q(t)\\ P(t) & I_N-\varepsilon P(t)Q(t)\end{bmatrix}\begin{bmatrix}\omega\\ z\end{bmatrix}\tag{3.8}$$

把(3.6)～(3.7)变换为

$$\left.\begin{array}{l}\omega'=P(t,\varepsilon)\omega+Q(t,\varepsilon)\tilde{g}(t,\varepsilon)\\ \varepsilon z'=-[C(t,\varepsilon)+\varepsilon P(t,\varepsilon)]z+\tilde{g}(t,\varepsilon)\end{array}\right\}\tag{3.9}$$

$$\left.\begin{array}{l}\begin{bmatrix}0 & 0\\ 0 & I_{N-K}\end{bmatrix}\omega(0)+\begin{bmatrix}I_K & 0\\ 0 & 0\end{bmatrix}\omega(1)=\theta_1(\varepsilon)\\ \begin{bmatrix}I_K & 0\\ 0 & 0\end{bmatrix}z(0)+\begin{bmatrix}0 & 0\\ 0 & I_{N-K}\end{bmatrix}z(1)=0\end{array}\right\}\tag{3.10}$$

这里$\tilde{g}(t,\varepsilon)=\tilde{g}(t,\omega(t),z(t),\varepsilon)=g(t,u(t),v(t),\varepsilon)$，为简便起见，本段中采用的$W(t)=W(t,\varepsilon)$，$Z(t)=Z(t,\varepsilon)$等符号与引理中的符号具有相同的意义．易知

$$\Delta_3(\varepsilon)=\begin{bmatrix}0 & 0\\ 0 & I_{N-K}\end{bmatrix}W(0,\varepsilon)+\begin{bmatrix}I_K & 0\\ 0 & 0\end{bmatrix}$$

$$\Delta_4(\varepsilon)=\begin{bmatrix}0 & 0\\ 0 & I_{N-K}\end{bmatrix}[Z(1,\varepsilon)P^*+Z(1,\varepsilon)(I_N-P^*)Z^{-1}(1,\varepsilon)]$$

$$+\begin{bmatrix}I_K & 0\\ 0 & 0\end{bmatrix}[P^*+(I_N-P^*)Z^{-1}(1,\varepsilon)]$$

非奇异．定义映射$(\xi(t),\eta(t))=T^*(\omega(t),z(t))$如下：

$$\xi(t)=W(t)\Delta_3^{-1}(\varepsilon)\left\{\theta_1(\varepsilon)-\begin{bmatrix}I_K & 0\\ 0 & 0\end{bmatrix}\int_0^1 W^{-1}(s)Q(s)\tilde{g}(s,\omega(s),z(s),\varepsilon)ds\right\}$$

$$+\int_0^t W(t)W^{-1}(s)Q(s)\tilde{g}(s,\omega(s),z(s),\varepsilon)ds \qquad (3.11)$$

$$\eta(t)=[Z(t)P^*+Z(t)(I_N-P^*)Z^{-1}(1)]\Delta_4^{-1}(\varepsilon)\left\{\begin{bmatrix}I_K & 0\\ 0 & 0\end{bmatrix}\right.$$

$$\cdot\int_0^1\varepsilon^{-1}(I_N-P^*)Z^{-1}(s)\tilde{g}(s,\omega(s),z(s),\varepsilon)ds-\begin{bmatrix}0 & 0\\ 0 & I_{N-K}\end{bmatrix}$$

$$\left.\cdot\int_0^1\varepsilon^{-1}Z(1)P^*Z^{-1}(s)\tilde{g}(s,\omega(s),z(s),\varepsilon)ds\right\}$$

$$+\varepsilon^{-1}\int_0^t Z(t)P^*Z^{-1}(s)\tilde{g}(s,\omega(s),z(s),\varepsilon)ds$$

$$+\varepsilon^{-1}\int_1^t Z(t)(I_N-P^*)Z^{-1}(s)\tilde{g}(s,\omega(s),z(s),\varepsilon)ds \qquad (3.12)$$

我们规定向量函数对(a,b)的范数为$\|(a,b)\|=\|a\|+\|b\|$，由假设条件(D_3)利用中值定理我们推得存在正常数K，使得

$$|\tilde{g}(t,\omega_1(t),z_1(t),\varepsilon)-\tilde{g}(t,\omega_2(t),z_2(t),\varepsilon)|\leqslant K\pi(\omega_1,\omega_2,z_1,z_2) \qquad (3.13)$$

其中$\pi(\omega_1,\omega_2,z_1,z_2)$见下面三个值中最大者．

$$|\omega_1-\omega_2|\max(|\omega_1|,|\omega_2|,|z_1|,|z_2|)$$

$$\varepsilon|z_1-z_2|\max(|\omega_1|,|\omega_2|,|z_1|,|z_2|)$$

$$|z_1-z_2|\max(|\omega_1|,|\omega_2|,\varepsilon|z_1|,\varepsilon|z_2|)$$

注意到$\tilde{g}(t,0,0,\varepsilon)=f(t,\bar{y}(t),\bar{y}'(t),\varepsilon)-\varepsilon\bar{y}''=O(\varepsilon)$及由假设条件$(D_4)$知$\theta_1(\varepsilon)=O(\varepsilon)$，故存在$m>0$使得

$$\|\theta_1(\varepsilon)\|\leqslant m\varepsilon,\quad \|\tilde{g}(t,0,0,\varepsilon)\|\leqslant m\varepsilon \qquad (3.14)$$

如果(2.10)中取足够大的$L>1$，则同时还有

$$|W(t)W^{-1}(s)|\leqslant L \qquad (3.15)$$

当$\varepsilon>0$充分小，使得

$$LK(q+2\mu^{-1}+Lq\|\Delta_3^{-1}(\varepsilon)\|+4L\|\Delta_4^{-1}(\varepsilon)\|\mu^{-1})r\varepsilon<1/2$$

其中 $r=2Lm(\|\Delta_3^{-1}(\varepsilon)\|+q+2\mu^{-1}+Lq\|\Delta_3^{-1}(\varepsilon)\|+4L\|\Delta_4^{-1}(\varepsilon)\|\mu^{-1})$

我们可以证明当$\|\omega(t,\varepsilon)\|,\|\omega^*(t,\varepsilon)\|,\|z(t,\varepsilon)\|,\|z^*(t,\varepsilon)\|\leqslant r\varepsilon$时，有

$$\|T^*(\omega(t),z(t))=\|\xi(t)\|+\|\eta(t)\|\leqslant r\varepsilon$$

$$\|T^*(\omega(t),z(t))-T^*(\omega^*(t),z^*(t))\|$$

$$=\|(\xi(t),\eta(t))-(\xi^*(t),\eta^*(t))\|$$

$$\leqslant(\|\omega(t)-\omega^*(t)\|+\|z(t)-z^*(t)\|)/2$$

因此 T^* 是球$S_r=\{(\omega(t),z(t)):\|(\omega(t),z(t))\|\leqslant 2r\varepsilon\}$中的一个压缩映射，从而$T^*$在$S_r$中有唯一的不动点，即在$S_r$中存在唯一的$(\omega(t),z(t))$使得

$$(\omega(t),z(t))=T^*(\omega(t),z(t))$$

通过微分可以验证$\omega(t,\varepsilon),z(t,\varepsilon)$是微分方程(3.9)、(3.10)的唯一解，满足

$$\|\omega(t,\varepsilon)\|\leqslant r\varepsilon,\|z(t,\varepsilon)\|\leqslant r\varepsilon \tag{3.16}$$

利用(3.8)返回到原来的变量，得知(3.2)～(3.3)存在解$u(t,\varepsilon)$满足

$$u(t,\varepsilon)=O(\varepsilon),\ u'(t,\varepsilon)=O(\varepsilon) \tag{3.17}$$

其次我们讨论(3.4)～(3.5)的解，其中$u(t)$是前面已得到的，借助于改写(3.4)～(3.5)可化为如下的形式：

$$\varepsilon v''+\tilde{C}(t,\varepsilon)v'+\tilde{D}(t,\varepsilon)v=h(t,v,v',\varepsilon) \tag{3.18}$$

$$\left.\begin{aligned}&\begin{bmatrix}0&0\\0&I_{N-K}\end{bmatrix}v(0)+\begin{bmatrix}I_K&0\\0&0\end{bmatrix}v(1)=0\\&\begin{bmatrix}I_K&0\\0&0\end{bmatrix}v(0)+\begin{bmatrix}0&0\\0&I_{N-K}\end{bmatrix}v(1)=\theta_2(\varepsilon)\end{aligned}\right\} \tag{3.19}$$

其中

$$\tilde{C}(t,\varepsilon)=-f_{y'}(t,\bar{y}+u,\bar{y}'+u',\varepsilon),\tilde{D}(t,\varepsilon)=-f_y(t,\bar{y}+u,\bar{y}'+u',\varepsilon)$$

$$\begin{aligned}h(t,v,v',\varepsilon)=&f(t,\bar{y}+u+v,\bar{y}'+u'+v',\varepsilon)-f(t,\bar{y}+u,\bar{y}'+u',\varepsilon)\\&-f_{y'}(t,\bar{y}+u,\bar{y}'+u',\varepsilon)v'-f_y(t,\bar{y}+u,\bar{y}'+u',\varepsilon)v\end{aligned}$$

$$\begin{aligned}\theta_2(\varepsilon)=&(a_1(\varepsilon)-\bar{y}_1(0)-u_1(0),\cdots,a_K(\varepsilon)-\bar{y}_K(0)-u_K(0),b_{K+1}(\varepsilon)\\&-\bar{y}_{K+1}(1)-u_{K+1}(1),\cdots,b_N(\varepsilon)-\bar{y}_N(1)-u_N(1))^T\end{aligned}$$

显然$\tilde{C}(t,\varepsilon),\tilde{D}(t,\varepsilon)$满足引理的条件，用$\tilde{P}(t)=\tilde{P}(t,\varepsilon),\tilde{Q}(t)=\tilde{Q}(t,\varepsilon)$表示以$\tilde{C}(t,\varepsilon)$，$\tilde{D}(t,\varepsilon)$代替$C(t,\varepsilon),D(t,\varepsilon)$的(2.1)～(2.2)和(2.4)～(2.5)的有界解，并设$|\tilde{Q}(t,\varepsilon)|\leqslant\tilde{q}$，利用变换

$$v=\tilde{\omega}-\varepsilon\tilde{Q}(t)\tilde{z},\ v'=\tilde{P}(t)v+\tilde{z} \tag{3.20}$$

把(3.18)～(3.19)化为对角线系统

$$\tilde{\omega}'=\tilde{P}(t,\varepsilon)\tilde{\omega}+\tilde{Q}(t,\varepsilon)\tilde{h}(t,\tilde{\omega},\tilde{z},\varepsilon) \tag{3.21}$$

$$\begin{bmatrix}0&0\\0&I_{N-K}\end{bmatrix}\tilde{\omega}(0)+\begin{bmatrix}I_K&0\\0&0\end{bmatrix}\tilde{\omega}(1)=0 \tag{3.22}$$

$$\varepsilon\tilde{z}'=-[\tilde{C}(t,\varepsilon)+\varepsilon\tilde{P}(t,\varepsilon)]\tilde{z}+\tilde{h}(t,\tilde{\omega},\tilde{z},\varepsilon) \tag{3.23}$$

$$\begin{aligned}&\varepsilon\begin{bmatrix}I_K&0\\0&0\end{bmatrix}\tilde{Q}(0)\tilde{z}(0)+\varepsilon\begin{bmatrix}0&0\\0&I_{N-K}\end{bmatrix}\tilde{Q}(1)\tilde{z}(1)\\&=\begin{bmatrix}I_K&0\\0&0\end{bmatrix}\tilde{\omega}(0)+\begin{bmatrix}0&0\\0&I_{N-K}\end{bmatrix}\tilde{\omega}(1)-\theta_2(\varepsilon)\end{aligned} \tag{3.24}$$

容易证明与(3.21)～(3.24)等价的积分方程为

$$\begin{aligned}\tilde{\omega}(t)=&\tilde{W}(t)\Delta_5^{-1}(\varepsilon)\begin{bmatrix}I_K&0\\0&0\end{bmatrix}\int_0^1\tilde{W}^{-1}(s)\tilde{Q}(s)\tilde{h}(s,\tilde{\omega}(s),\tilde{z}(s),\varepsilon)ds\\&+\int_0^t\tilde{W}(t)\tilde{W}^{-1}(s)\tilde{Q}(s)\tilde{h}(s,\tilde{\omega}(s),z(s),\varepsilon)ds\end{aligned} \tag{3.25}$$

$$\varepsilon\tilde{z}(t)=[\tilde{Z}(t)P^*+\tilde{Z}(t)(I_N-P^*)\tilde{Z}^{-1}(1)]\Delta_6^{-1}(\varepsilon)\left[\begin{matrix}I_K&0\\0&0\end{matrix}\right]\tilde{\omega}(0)+\left[\begin{matrix}0&0\\0&I_{N-K}\end{matrix}\right]\tilde{\omega}(1)$$

$$-\theta_2(\varepsilon)+\left[\begin{matrix}I_K&0\\0&0\end{matrix}\right]\tilde{Q}(0)\int_0^1(I_N-P^*)\tilde{Z}^{-1}(s)\tilde{h}(s,\tilde{\omega}(s),\tilde{z}(s),\varepsilon)ds$$

$$-\left[\begin{matrix}0&0\\0&I_{N-K}\end{matrix}\right]\tilde{Q}(1)\int_0^1\tilde{Z}(1)P^*\tilde{Z}^{-1}(s)\tilde{h}(s,\tilde{\omega}(s),\tilde{z}(s),\varepsilon)ds$$

$$+\int_0^t\tilde{Z}(t)P^*Z^{-1}(s)\tilde{h}(s,\tilde{\omega}(s),\tilde{z}(s),\varepsilon)ds$$

$$+\int_1^t\tilde{Z}(t)(I_N-P^*)\tilde{Z}^{-1}(s)\tilde{h}(s,\tilde{\omega}(s),\tilde{z}(s),\varepsilon)ds \tag{3.26}$$

这里$\tilde{W}(t)=\tilde{W}(t,\varepsilon),\tilde{Z}(t)=\tilde{Z}(t,\varepsilon)$是从$\tilde{C},\tilde{P}$代替 $C.P$ 时方程(2.8)、(2.9) 的基本矩阵，具有$\tilde{W}(1)=I,\tilde{Z}(0)=I$并且满足(3.15)和(2.10),

$$\Delta_5(\varepsilon)=\left[\begin{matrix}0&0\\0&I_{N-K}\end{matrix}\right]\tilde{W}(0)+\left[\begin{matrix}I_K&0\\0&0\end{matrix}\right]$$

非奇异是显然的，至于

$$\Delta_6(\varepsilon)=\left[\begin{matrix}I_K&0\\0&0\end{matrix}\right]\tilde{Q}(0)\Big[P^*+(I_N-P^*)\tilde{Z}^{-1}(1)+\left[\begin{matrix}0&0\\0&I_{N-K}\end{matrix}\right]\tilde{Q}(1)$$

$$\cdot[\tilde{Z}(1)P^*+\tilde{Z}(1)(I_N-P^*)\tilde{Z}^{-1}(1)\Big]$$

的非奇异性则要求$\tilde{Q}^{-1}(0,\varepsilon),\tilde{Q}^{-1}(1,\varepsilon)$对充分小的$\varepsilon>0$存在．此可参阅文[1～2].

注意到$\tilde{h}(t,0,0,\varepsilon)=0$，并且以$\tilde{\omega},\tilde{z}$代替$\omega,z,\tilde{h}$也满足不等式(3.13)，为方便起见，采用同样的常数K，虽然现在它是根据 $\tilde{P}$，$\tilde{Q}$ 而定的．现在我们采用逐步逼近法证明积分方程(3.25)～(3.26)有一个解．置$(\tilde{\omega}_0,\tilde{z}_0)=(0,0)$并作迭代：$(\tilde{\omega}_n,\tilde{z}_n)=T_1(\tilde{\omega}_{n-1},\tilde{z}_{n-1})$ $(n=1,2,\cdots)$如下：

$$\tilde{\omega}_n(t)=\tilde{W}(t)\Delta_5^{-1}(\varepsilon)\left[\begin{matrix}I_K&0\\0&0\end{matrix}\right]\int_1^0\tilde{W}^{-1}(s)\tilde{Q}(s)\tilde{h}(s,\tilde{\omega}_{n-1}(s),\tilde{z}_{n-1}(s),\varepsilon)ds$$

$$+\int_0^t\tilde{W}(t)\tilde{W}^{-1}(s)\tilde{Q}(s)\tilde{h}(s,\tilde{\omega}_{n-1}(s),\tilde{z}_{n-1}(s),\varepsilon)ds \tag{3.27}$$

$$\varepsilon\tilde{z}_n(t)=[\tilde{Z}(t)P^*+\tilde{Z}(t)(I_N-P^*)\tilde{Z}^{-1}(1)]\Delta_6^{-1}(\varepsilon)\Big\{\left[\begin{matrix}I_K&0\\0&0\end{matrix}\right]\tilde{\omega}_n(0)$$

$$+\left[\begin{matrix}0&0\\0&I_{N-K}\end{matrix}\right]\tilde{\omega}_n(1)-\theta_2(\varepsilon)+\left[\begin{matrix}I_K&0\\0&0\end{matrix}\right]\tilde{Q}(0)\int_0^1(I_N$$

$$-P^*)\tilde{Z}^{-1}(s)\tilde{h}(s,\tilde{\omega}_{n-1}(s),\tilde{z}_{n-1}(s),\varepsilon)ds$$

$$-\left[\begin{matrix}0&0\\0&I_{N-K}\end{matrix}\right]\tilde{Q}(1)\int_0^1\tilde{Z}(1)P^*\tilde{Z}^{-1}(s)\tilde{h}(s,\tilde{\omega}_{n-1}(s),\tilde{z}_{n-1}(s),\varepsilon)ds\Big\}$$

$$+\int_0^t\tilde{Z}(t)P^*\tilde{Z}^{-1}(s)\tilde{h}(s,\tilde{\omega}_{n-1}(s),\tilde{z}_{n-1}(s),\varepsilon)ds$$

$$+\int_1^t \tilde{Z}(t)(I_N-P^*)\tilde{Z}^{-1}(s)\tilde{h}(s,\tilde{\omega}_{n-1}(s),\tilde{z}_{n-1}(s),\varepsilon)ds \tag{3.28}$$

由$u(t,\varepsilon)=O(\varepsilon)$及假设条件$(D_4)$知，存在与$\varepsilon$无关的$\delta=1/4N\lambda$当

$$|a_i(0)-\bar{y}_i(0)|\leqslant 1/8N\lambda \qquad (i=1,2,\cdots,K)$$

$$|b_i(0)-\tilde{y}_i(1)|\leqslant 1/8N\lambda \qquad (i=K+1,\cdots,N)$$

取充分小的 $0<\varepsilon<1$使得$|u_i(t)|\leqslant 1/4N\lambda\ (i=1,2,\cdots,N)$

和 $4L\|\Delta_6^{-1}(\varepsilon)\|\varepsilon^{-1}(e^{-\mu t/\varepsilon}+e^{-\mu(1-t)/\varepsilon})\leqslant 1$

可推得$\|\theta_2(\varepsilon)\|\leqslant 1/2\lambda$，这里

$$\lambda=LK[2\mu^{-1}(\tilde{q}/4+1)+\tilde{q}(L\|\Delta_5^{-1}(\varepsilon)\|+1)]$$

利用数学归纳法可证得

$$|\tilde{\omega}_n(t)|,\varepsilon|\tilde{z}_n(t)|\leqslant 2L\|\Delta_6^{-1}(\varepsilon)\|\|\theta_2(\varepsilon)\|(e^{-\mu t/\varepsilon}+e^{-\mu(1-t)/\varepsilon})$$

$$|\tilde{\omega}_{n-1}(t)-\tilde{\omega}_{n-1}(t)|,\varepsilon|\tilde{z}_n(t)-\tilde{z}_{n-1}(t)|$$

$$\leqslant L\|\Delta_6^{-1}(\varepsilon)\|\|\theta_2(\varepsilon)\|(\lambda\|\theta_2(\varepsilon)\|)^{n-1}(e^{-\mu t/\varepsilon}+e^{-\mu(1-t)/\varepsilon})$$

因此级数

$$\sum_{n=1}^{\infty}[\tilde{\omega}_n(t)-\tilde{\omega}_{n-1}(t)],\quad \sum_{n=1}^{\infty}[\tilde{z}_n(t)-\tilde{z}_{n-1}(t)]$$

从而序列$\{\tilde{\omega}_n(t)\}_{n=1}^{\infty},\{\tilde{z}_n(t)\}_{n=1}^{\infty}$关于$(t,\varepsilon)$一致收敛于(3.25)～(3.26)的解$\tilde{\omega}(t,\varepsilon),\tilde{z}(t,\varepsilon)$，此亦为方程(3.21)～(3.24)的解，由(3.20)返回到原来的变量．得边值问题(3.4)～(3.5)有解$v(t,\varepsilon)$，并且满足

$$v(t,\varepsilon)=O[\|\theta_2(\varepsilon)\|(e^{-\mu t/\varepsilon}+e^{-\mu(1-t)/\varepsilon})]$$

$$v'(t,\varepsilon)=O[\|\theta_2(\varepsilon)\|\varepsilon^{-1}(e^{-\mu t/\varepsilon}+e^{-\mu(1-t)/\varepsilon})]$$

于是定理得证．

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Boundary Value Problem for a Singularly Perturbed Nonlinear System

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Abstract

In this paper, by the technique and the method of diagonalization, the boundary value problem for second order singularly perturbed nonlinear system as follows is dealt with:

$\varepsilon y''=f(t,y,y',\varepsilon)$, $y(0,\varepsilon)=a(\varepsilon)$, $y(1,\varepsilon)=b(\varepsilon)$

The existance of the solution and its asymptotic properties are discussed when the eigenvalues of Jacobi matrix $f_{y'}$ has K negative real parts and $N-K$ positive real parts.

Key words nonlinear system, boundary value problem, diagonalization, singular perturbation