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奇异摄动偏微分方程的周期边界问题*

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摘　要

本文讨论奇异摄动椭圆抛物型偏微分方程的周期边界问题，构造一个差分格式，利用分离解的奇性项的方法，结合问题的渐近展开，证明所构造的差分格式具有$O(\tau+h^2)$一致收敛阶.

关键词　椭圆抛物型偏微分方程　奇异摄动问题　周期边界　差分格式　一致收敛性

一、引　言

近年来，在高阶导数项含小参数ε的二阶常微和偏微分方程的第一和第三边值问题的数值解法研究得较为充分，而对周期边值问题的研究较少见．А.А.Печенкина 在文[1]中对二阶常微分方程周期边界奇异摄动问题，构造了一个差分格式，利用古典和非古典的估计方法，证明格式达到$O(h^{1/2})$一致收敛阶．本文作者在[2]中利用分离解的奇性项方法，结合问题的渐近展开，证明文[1]的格式具有$O(h)$一致收敛阶．本文讨论椭圆抛物型偏微分方程的周期边值问题，构造一个差分格式，证明所构造的格式具有$O(\tau+h^2)$一致收敛阶．

二、微分方程的一些性质

考虑如下偏微分方程周期边值问题：

$$\begin{cases} Lu(x,y)\equiv\varepsilon\dfrac{\partial^2(x,y)}{\partial y^2}+\dfrac{\partial^2u(x,y)}{\partial x^2}-p(x,y)\dfrac{\partial u(x,y)}{\partial y}-q(x,y)u(x,y) \\ \qquad =f(x,y) \qquad\qquad (x,y)\in D=(0,1)\times(0,1) & (2.1) \\ u(0,y)=u(1,y)=0 \qquad y\in[0,\ 1] & (2.2) \\ u(x,1)=u(x,0),\ lu\equiv\dfrac{\partial u}{\partial y}\Big|_{y=1}-\dfrac{\partial u}{\partial y}\Big|_{y=0}=\dfrac{e(x)}{\varepsilon} \qquad (x\in[0,\ 1]) & (2.3) \end{cases}$$

其中　ε为小参数且假定：

（1）　$p(x,y)$，$q(x,y)$，$f(x,y)$，$e(x)$为充分光滑函数且满足$A\geqslant p(x,\ y)$，$q(x,\ y)$

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$\geqslant \alpha \geqslant 2\alpha_0 > 0$，$|e(x)| \leqslant A$，$(x, y) \in [0, 1] \times [0, 1]$，这里$A$，$\alpha$，$\alpha_0$为给定的常数。

（2） 在区域D的四个角点T上满足相容性条件：

$$p_x(x, y)|_T = 0, \quad q_x(x, y)|_T = 0$$

$$\begin{cases} f(0, 1) = f(0, 0) \\ f(1, 1) = f(1, 0) \end{cases} \quad 和 \quad \begin{cases} f_y(0, 1) - f_y(0, 0) = \dfrac{e''(0)}{\varepsilon} \\ f_y(1, 1) - f_y(1, 0) = \dfrac{e''(1)}{\varepsilon} \end{cases}$$

及

$$\begin{cases} F(0, 1) = F(0, 0) \\ F(1, 1) = F(1, 0) \end{cases}$$

其中记$f_y = \partial f/\partial y$等等，而

$$F(x, y) = \varepsilon f_{yy}(x, y) - f_{xx}(x, y) - p(x, y) f_y(x, y) - q(x, y) f(x, y)$$

下面讨论(2.1)～(2.3)的一些性质。

引理2.1 若$Lu \leqslant 0$，$u(0, y) \geqslant 0$，$u(1, y) \geqslant 0$且$u(x, 1) = u(x, 0)$，$lu \geqslant 0$，则

$$u(x, y) \geqslant 0 \qquad ((x, y) \in \bar{D}).$$

证明 类似于[2]中引理2.1的讨论，可证引理成立。

引理2.2 设 $|Lu| \leqslant K\left\{1 + \dfrac{1}{\varepsilon}\exp\left(-\dfrac{\alpha(1-y)}{2\varepsilon}\right)\right\}$，$u(0, y) = 0$，$u(1, y) = 0$ 且

$u(x, 0) = u(x, 1)$，$|lu| \leqslant \dfrac{|e(x)|}{\varepsilon}$，则

$$|u(x, y)| \leqslant M\{K + \max_{x \in [0,1]} |e(x)|\} \qquad ((x, y) \in \bar{D}) \tag{2.4}$$

证明 令 $\Phi(x, y) = k_0 \exp(-\alpha(1-y)/2\varepsilon) + k_1(1-y) + k_2$，与[2]的引理2.2一样，可适当选取只与$\alpha$，$A$，$\max\limits_{x \in [0,1]}|e(x)|$有关的正常数$k_0$，$k_1$，$k_2$使得

$$|L\Phi(x, y)| \leqslant -K\left\{1 + \frac{1}{\varepsilon}\exp\left(-\frac{\alpha(1-y)}{2\varepsilon}\right)\right\}.$$

$$\Phi(0, y) \geqslant 0, \quad \Phi(1, y) \geqslant 0$$

$$\Phi(x, 1) = \Phi(x, 0), \quad l\Phi > \max_{x \in [0,1]} \frac{|e(x)|}{\varepsilon}$$

成立，这样再把引理2.1应用于函数$\Phi(x, y) \pm u(x, y)$就可推出此引理的(2.4)式。

引理2.3 问题(2.1)～(2.3)的解$u(x, y)$可表示成

$$u(x, y) = w(x, y) + v(x, y) + R(x, y) \tag{2.5}$$

其中边界层函数

$$v(x,y) = \frac{e(x) \cdot \exp\left(-\dfrac{p(x,1)(1-y)}{\varepsilon}\right)}{p(x,1)\left[1 - \exp\left(-\dfrac{p(x,1)}{\varepsilon}\right)\right]}$$

满足

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} + p(x, 1)\dfrac{\partial v}{\partial t} = 0 \\ \dfrac{\partial v}{\partial y}\Big|_{y=1} - \dfrac{\partial v}{\partial y}\Big|_{y=0} = \dfrac{e(x)}{\varepsilon}, \qquad t = \dfrac{1-y}{\varepsilon} \end{cases}$$

而$w(x, y)$满足

$$\begin{cases} w_{xx}-p(x,y)w_y-q(x,y)w=f(x,y) \\ w(0,y)=-v|_{x=0},\ \ w(1,y)=-v|_{x=1} \\ w(x,1)-w(x,0)=v(x,1)-v(x,0) \end{cases}$$

余项

$$R(x,y)=O(\varepsilon).$$

证明 当$w(x,y)$，$v(x,y)$取如上定义时，令

$$R(x,y)=u(x,y)-w(x,y)-v(x,y)$$

则有

$$\begin{cases} |LR|\leqslant M\varepsilon\{1+\varepsilon^{-1}\exp\left(-\dfrac{\alpha_0(1-y)}{\varepsilon}\right) \\ R(0,\ y)=0,\ \ R(1,\ y)=0 \\ R(x,1)=R(x,0)\quad lR\equiv\dfrac{\partial R}{\partial y}\Big|_{y=1}-\dfrac{\partial R}{\partial y}\Big|_{y=0}=O(1) \end{cases}$$

由引理2.2推出

$$|R(x,y)|\leqslant M\varepsilon$$

由于微分方程(2.1)～(2.3)满足相容性条件，再结合解的渐近展开式(2.5)，可证明如下引理[6]

引理2.4 设$u(x,y)$是(2.1)～(2.3)的解，则

$$\left|\frac{\partial^i u}{\partial x^i}\right|\leqslant M,\quad \left|\frac{\partial^i u}{\partial y^i}\right|\leqslant M\left\{1+\varepsilon^{-i}\exp\left(-\alpha_0\frac{(1-y)}{\varepsilon}\right)\right\}$$

$$(i=0,1,\cdots,4) \qquad (2.6)$$

跟常微分方程情况一样，也可以把方程(2.1)～(2.3)的解的奇异性分离出来：

引理2.5 设$u(x,y)$为(2.1)～(2.3)的解，满足

$$u(x,y)=r(x)\bar{v}(x,y)+z(x,y)\qquad((x,y)\in\bar{D}) \qquad (2.7)$$

其中

$$|r(x)|\leqslant M,\qquad \bar{v}(x,y)=\exp\left(-\frac{p(x,1)(1-y)}{\varepsilon}\right)$$

而

$$\left|\frac{\partial^j z}{\partial x^j}\right|\leqslant M,\qquad \left|\frac{\partial^j z}{\partial y^j}\right|\leqslant M\left\{1+\varepsilon^{-j+1}\exp\left(-\frac{\alpha_0(1-y)}{\varepsilon}\right)\right\}\quad (j=0,1,\cdots,4)$$

证明 取 $r(x)=\varepsilon\dfrac{\partial u(x,1)}{\partial y}\Big/p(x,\ 1)$并令

$$z(x,y)=u(x,y)-r(x)\bar{v}(x,y)$$

$$\tilde{L}u\equiv\varepsilon\frac{\partial^2 u}{\partial y^2}-p(x,y)\frac{\partial u}{\partial y}-g(x,y)u$$

由引理2.4，得

$$|r(x)|\leqslant M$$

而

$$\left|\frac{\partial^j z}{\partial x^j}\right|=\left|\frac{\partial^j u}{\partial x^j}-\frac{\partial^j}{\partial x^j}\left(r(x)\cdot\bar{v}(x,y)\right)\right|\leqslant M \qquad (2.8)$$

$$\tilde{L}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}(Lz)+\frac{\partial p}{\partial y}\frac{\partial z}{\partial y}+\frac{\partial q}{\partial y}z-\frac{\partial^3 z}{\partial x^2\partial y}$$

$$=\frac{\partial}{\partial y}(f-L(r\bar{v}))+\frac{\partial p}{\partial y}\frac{\partial}{\partial y}(u-r\bar{v})+\frac{\partial q}{\partial y}(u-r\bar{v})-\frac{\partial^3(u-r\bar{v})}{\partial x^2\partial y}$$

$$\triangleq g(x,\ y) \tag{2.9}$$

其中

$$|g(x,y)|\leqslant M\left\{1+\frac{1}{\varepsilon}\exp\left(-\frac{\alpha_0(1-y)}{\varepsilon}\right)\right\}$$

从引理 2.4 知

$$\left|\frac{\partial^j}{\partial y^j}g(x,y)\right|\leqslant M\left\{1+\varepsilon^{-j+1}\exp\left(-\frac{\alpha_0(1-y)}{\varepsilon}\right)\right\}$$

$$\left.\frac{\partial z}{\partial y}\right|_{y=0}=\left.\frac{\partial u}{\partial y}\right|_{y=0}-r\left.\frac{\partial\bar{v}}{\partial y}\right|_{y=0}=\left.\frac{\partial u}{\partial y}\right|_{y=0}-\frac{\partial u(x,1)}{\partial y}\exp\left(-\frac{p(x,1)}{\varepsilon}\right)$$

$$\therefore\quad \left|\left.\frac{\partial z}{\partial y}\right|_{y=0}\right|\leqslant M\cdot\frac{1}{\varepsilon}\exp\left(-\frac{\alpha_0}{\varepsilon}\right)\leqslant M$$

$$\left.\frac{\partial z}{\partial y}\right|_{y=1}=\left.\frac{\partial u}{\partial y}\right|_{y=1}-r\left.\frac{\partial\bar{v}}{\partial y}\right|_{y=1}=0 \tag{2.10}$$

利用 Kellogg[3]的结论于(2.8)～(2.10)，得

$$\left|\frac{\partial^j z}{\partial y^j}\right|\leqslant M\left\{1+\varepsilon^{-j+1}\exp\left(-\frac{\alpha_0(1-y)}{\varepsilon}\right)\right\}\qquad(j=0,1,\cdots,4) \tag{2.11}$$

注 引理2.4和引理 2.5的结论可进一步推广为：

$$\left|\frac{\partial^{i+j}u}{\partial x^i\partial y^j}\right|\leqslant M\left\{1+\varepsilon^{-j}\exp\left(-\frac{\alpha_0(1-y)}{\varepsilon}\right)\right\}\qquad(0\leqslant i+j\leqslant 4) \tag{2.12}$$

$$\left|\frac{\partial^{i+j}z}{\partial x^i\partial x^j}\right|\leqslant M\left\{1+\varepsilon^{-j+1}\exp\left(-\frac{\alpha_0(1-y)}{\varepsilon}\right)\right\}\qquad(0\leqslant i+j\leqslant 4) \tag{2.13}$$

三、差分格式及其性质

对问题(2.1)～(2.3)构造差分格式如下：

$$\begin{cases} L^{\tau,h}u_{ij}\equiv\varepsilon\sigma_{ij}\delta_y^2u_{ij}+\delta_x^2u_{ij}-p_{ij}D_{0y}u_{ij}-q_{ij}u_{ij}=f_{ij} \\ \qquad (i=1,2,\cdots,\ m-1;\ \ j=1,2,\cdots,\ N-1) & (3.1)\\ u_{0j}=0,\ u_{mj}=0\quad(j=0,1,2,\cdots,N) & (3.2)\\ u_{iN}-u_{i0}=0,\qquad l^{\tau,h}u_{ij}\equiv\bar{A}_i\dfrac{u_{iN}-u_{iN-1}}{\tau}-\dfrac{u_{i1}-u_{i0}}{\tau}=\dfrac{e_i}{\varepsilon} \\ \qquad (i=0,\ 1,\ 2,\ \cdots,\ m) & (3.3)\end{cases}$$

其中

$$x_i=ih\qquad(i=0,\ 1,\ 2,\ \cdots,\ m;\ h=1/m)$$

$$y_j=j\tau\qquad(j=0,\ 1,\ 2,\ \cdots,\ N;\ \tau=1/N)$$

$$\delta_y^2u_{ij}=\frac{u_{i,j+1}-2u_{ij}+u_{i,j-1}}{\tau^2},\qquad \delta_x^2u_{ij}=\frac{u_{i+1,j}-2u_{ij}+u_{i-1,j}}{h^2},$$

$$D_{0y}=\frac{u_{i,j+1}-u_{i,j-1}}{2\tau},\ \ p(x_i,\ y_j)=p_{ij},\qquad q(x_i,\ y_j)=q_{ij}$$

而

$$\sigma_{ij}=\frac{p_{ij}}{2}\rho\ \coth\frac{p_{ij}}{2}\rho,\quad \rho=\frac{\tau}{\varepsilon}$$

$$\bar{A}_i=\frac{p_{iN}\rho}{1-\exp(-p_{iN}\rho)}$$

下面讨论(3.1)～(3.3)解的性质

引理3.1 设$L^{\tau,h}u_{ij}\leqslant 0$，$u_{0j}$、$u_{mj}$为给定的非负数($j=0,1,\cdots,N$)且$u_{i0}=u_{iN}$ ($i=0,1,\cdots,m$)，$l^{\tau,h}u_{ij}\geqslant 0$ 则 $u_{ij}\geqslant 0$ ($i=0,1,\cdots,m$; $j=0,1,\cdots,N$)

证明 类似于[2]中引理3.1的证明。

引理3.2 设$|L^{\tau,h}u_{ij}|\leqslant K\left\{1+\frac{1}{\max(h,\varepsilon)}\exp\left(-\frac{\alpha_0(1-y_j)}{\varepsilon}\right)\right\}$，且 $u_{0j}=0$，$u_{mj}=0$ 及$u_{iN}=u_{i0}$，$|l^{\tau,h}u_{ij}|\leqslant\frac{|e_i|}{\varepsilon}$，则

$$|u_{ij}|\leqslant M\{K+|e_i|\}\qquad (i=0,1,2,\cdots m;\ j=0,1,2,\cdots,N)\tag{3.4}$$

证明 令$\Phi_{ij}=k_0\exp(-\alpha_0(1-y_j)/\varepsilon)+k_1(1-y_j)+k_2$，适当选取只与$K$，$|e_i|$有关的非负常数$k_0$，$k_1$，$k_2$，有

$$\begin{cases}L^{\tau,h}\Phi_{i,j}\leqslant -K\left\{1+\dfrac{1}{\max(h,\varepsilon)}\exp\left(-\dfrac{\alpha_0(1-y_j)}{\varepsilon}\right)\right\}\\ \Phi_{0j}\geqslant 0,\ \Phi_{mj}\geqslant 0,\\ \Phi_{iN}=\Phi_{i0},\quad l^{h,\tau}\Phi_{ij}\geqslant\dfrac{|e_i|}{\varepsilon}\end{cases}$$

然后把引理3.1应用于$\Phi_{ij}\pm u_{ij}$立即推出

$$|u_{ij}|\leqslant M\{K+|e_i|\}\qquad (i=0,1,2,\cdots,m;\ j=0,1,2,\cdots,N)$$

四、误 差 估 计

下面对(2.1)～(2.3)的解$u(x_i,y_j)$和(3.1)～(3.3)的解u_{ij}的差$u(x_i,y_j)-u_{ij}$进行估计。为此，利用引理2.5的(2.7)，把差分格式写成另一形式

$$\begin{cases}L^{\tau,h}(r_i\tilde{v}_{ij}+z_{ij})=L[r(x_i)\tilde{v}(x_i,y_j)+z(x_i,y_j)] & (4.1)\\ r_0\tilde{v}_{0j}+z_{0j}=0,\ r_m\tilde{v}_{mj}+z_{mj}=0\qquad (j=0,1,\cdots,N) & (4.2)\\ r_i\tilde{v}_{iN}+z_{iN}=r_i\tilde{v}_{i0}+z_{i0},\ l^{\tau,h}(r_i\tilde{v}_{ij}+z_{ij})=l(r(x_i)\tilde{v}(x_i,y_j)+z(x_i,y_j)) & (4.3)\end{cases}$$

这样，

$$u_{ij}=r_i\tilde{v}_{ij}+z_{ij}\qquad (i=0,1,\cdots,m;\ j=0,1,\cdots N)$$

$$\begin{cases}u_{0j}-u(x_0,y_j)=r_0\tilde{v}_{0j}+z_{0j}-r(x_0)\tilde{v}(x_0,y_j)-z(x_0,y_j)=0 & (4.4)\\ u_{mj}-u(x_m,y_j)=0\qquad (j=0,1,\cdots,N)\\ [u_{iN}-u(x_i,y_N)]-[u_{i0}-u(x_i,y_0)]=0 & (4.5a)\\ l^{\tau,h}[u_{ij}-u(x_i,y_j)]=l^{\tau,h}[r_i\tilde{v}_{ij}+z_{ij}-(r(x_i)\tilde{v}(x_i,y_j)+z(x_i,y_j))]\end{cases}$$

$$=\frac{e(x_i)}{\varepsilon}-l^{\tau,h}[r(x_i)\tilde{v}(x_i,y_j)+z(x_i,y_j)]$$

$$=\frac{e(x_i)}{\varepsilon}-\left[r(x_i)\bar{A}_i\frac{\tilde{v}(x_i,y_N)-\tilde{v}(x_i,y_{N-1})}{\tau}\right.$$

$$-r(x_i)\frac{\tilde{v}(x_i,y_1)-\tilde{v}(x_i,y_0)}{\tau}+\bar{A}_i\frac{z(x_i,y_N)-z(x_i,y_{N-1})}{\tau}-\frac{z(x_i,y_1)-z(x_i,y_0)}{\tau}\Big]$$

再用Taylor展开及 $\tilde{v}(x,y)=\exp(-p(x,1)(1-y)/\varepsilon$ 和关于 $z(x,y)$ 的估计式(2.13)，推出

$$l^{\tau,h}[u_{ij}-u(x_i,y_j)]=O\left(\frac{\tau}{\varepsilon}\right)\tag{4.5b}$$

仿照[2]§4关于 $|L^h(y(x_i)-y_i)|$ 的估计，易见当 $\tau\leqslant\varepsilon$ 时，

$$\begin{aligned}|L^{\tau,h}(u_{ij}-u(x_i,y_j))|&=|L^{\tau,h}(r_i\tilde{v}_{ij}+z_{ij})-L^{\tau,h}(r(x_i)\tilde{v}(x_i,y_j)+z(x_i,y_j))|\\&=|L(r(x_i)\tilde{v}(x_i,y_j)+z(x_i,y_j))-L^{\tau,h}(r(x_i)\tilde{v}(x_i,y_j)+z(x_i,y_j))|\\&\leqslant M(h^2+\tau)\left\{1+\frac{1}{\varepsilon}\exp\left(-\alpha_0\frac{1-y}{\varepsilon}\right)\right\}\end{aligned}\tag{4.6}$$

注. 与[2]§4一样，(6.6)是在 $\tau<\tau_0$ 时成立.

对(4.4)～(4.6)利用引理 3.2 得

引理4.1　设 u_{ij} 是差分格式(3.1)～(3.3)的解，而 $u(x_i,y_j)$ 是(2.1)～(2.3)的解，则当 $\tau\leqslant\varepsilon$ 和 $\tau\leqslant\tau_0$ 时有

$$|u_{ij}-u(x_i,y_j)|\leqslant M(h^2+\tau)\tag{4.7}$$

令

$$R_{ij}=u_{ij}-w(x_i,y_j)-v(x_i,y_j)$$

其中 u_{ij} 是差分格式(3.1)～(3.3)的解，$w(x,y)$ 和 $v(x,y)$ 是引理2.3所定义的函数。

仿照[2]中§4.2的证明，通过计算得，当 $\tau\geqslant\varepsilon$ 时，

$$|L^{\tau,h}R_{ij}|\leqslant M(h^2+\tau+\varepsilon)\left\{1+\frac{1}{\max(\tau,\ \varepsilon)}\exp\left(-\alpha_0\frac{1-y_j}{\varepsilon}\right)\right\}\quad(\tau\leqslant\tau_1)\tag{4.8}$$

$$R_{0j}=0,\ R_{mj}=0\tag{4.9}$$

$$R_{iN}=R_{i0},\ |l^{\tau,h}R_{ij}|\leqslant M\frac{h^2+\tau+\varepsilon}{\varepsilon}\tag{4.10}$$

把引理3.2应用于(4.8)～(4.10)，得

$$|R_{ij}|\leqslant M(h^2+\tau+\varepsilon)\tag{4.11}$$

再利用引理2.3的结论(2.5)式.

$$\begin{aligned}|u_{ij}-u(x_i,y_j)|&=|u_{ij}-v(x_i,y_j)-w(x_i,y_j)-R(x_i,y_j)|\\&=|R_{ij}-R(x_i,\ y_j)|\leqslant|R_{ij}|+|R(x_i,\ y_j)|\\&\leqslant(M(h^2+\tau+\varepsilon)\end{aligned}\tag{4.12}$$

这样就得到如下引理.

引理4.2　设 u_{ij} 是(3.1)～(3.3)的解，而 $u(x_i,y_j)$ 为(2.1)～(2.3)的解，当 $\tau\geqslant\varepsilon$ 且 $\tau\leqslant\tau_1$ 时有

$$|u_{ij}-u(x_i,\ y_j)|\leqslant M(h^2+\tau+\varepsilon)$$

综合引理4.1和引理4.2，令 $\bar{\tau}=\min\{\tau_0,\ \tau_1\}$，则得

定理4.1　设 u_{ij} 和 $u(x_i,y_j)$ 分别为(3.1)～(3.3)和(2.1)～(2.3)的解，则当 $\tau\leqslant\bar{\tau}$ 时有

$$|u_{ij}-u(x_i,y_j)|\leqslant M(h^2+\tau)\tag{4.13}$$

注　定理中 $\tau\leqslant\bar{\tau}$ 条件可去掉，事实上，对于给定的 $\bar{\tau}>0$，当 $\tau\geqslant\bar{\tau}$ 时，由引理2.2和引理3.2有 $|u_{ij}|\leqslant M_1$，

$|u(x_i,y_j)|\leqslant M_1$.

$$|u_{ij}-u(x_i,y_j)|\leqslant|u_{ij}|+|u(x_i,y_j)|\leqslant 2M_1\leqslant M_2\bar{\tau}\leqslant M(\tau+h^2)$$

故当$\tau\geqslant\bar{\tau}$时(4.13)亦成立.

五、数 值 例 子

考虑下列奇异摄动偏微分方程周期边界问题

$$\begin{cases}\varepsilon u_{yy}+u_{xx}-u_y-u=10\sin\pi x\Big\{(\pi^2+1)\Big[\Big(1-\exp\Big(-\dfrac{1}{\varepsilon}\Big)y-\exp\Big(-\dfrac{1-y}{\varepsilon}\Big)\Big]\\ \qquad\qquad +1-\exp\Big(-\dfrac{1}{\varepsilon}\Big)\Big\}\triangleq 10\sin\pi x\ g(y)\qquad ((x,y)\in(0,1)\times(0,1))\\ u(0,\ y)=0,\ u(1,\ y)=0\quad (y\in[0,\ 1])\\ u(x,1)=u(x,0),\ \dfrac{\partial u}{\partial y}\Big|_{y=1}-\dfrac{\partial u}{\partial y}\Big|_{y=0}=\dfrac{10\Big(1-\exp\Big(-\dfrac{1}{\varepsilon}\Big)\Big)\sin\pi x}{\varepsilon}\quad (x\in[0,\ 1])\end{cases}\tag{5.1}$$

易见这个问题满足相容性条件。其精确解为

$$u(x,\ y)=10\sin\pi x\Big\{\exp\Big(-\frac{1-y}{\varepsilon}\Big)-\Big(1-\exp\Big(-\frac{1}{\varepsilon}\Big)\Big)y\Big\}$$

渐近展开到一阶：

$$u(x,y)=10\sin\pi x\Big\{\exp\Big(-\frac{1-y}{\varepsilon}\Big)+\exp(-(\pi^2+1)y)\Big[c_1-\int_0^y\exp((\pi^2+1)t)g(t)dt\Big]\Big\}+O(\varepsilon)$$

其中常数

$$c_1=\frac{\exp\Big(-\dfrac{1}{\varepsilon}\Big)-1+\exp(-\pi^2-1)\displaystyle\int_0^1\exp((\pi^2+1)t)g(t)dt}{\exp(-\pi^2-1)-1}$$

采用正方形网格：$h=\tau=1/N$.

差分格式为

$$\begin{cases}\dfrac{1}{h^2}(u_{i-1,j}+u_{i+1,j})+au_{ij}+cu_{i,j-1}+du_{i,j+1}=10\sin\pi x_i\cdot g(y_j)\triangleq f_{ij}\\ \qquad\qquad (i,\ j=1,\ 2,\ \cdots,\ N-1)\\ u_{0j}=0,\ u_{Nj}=0\qquad (j=0,\ 1,\ 2,\ \cdots,\ N)\\ u_{iN}=u_{i0},\ \dfrac{b}{h}(u_{iN}-u_{i,N-1})-\dfrac{1}{h}(u_{i1}-u_{i0})=\dfrac{10}{\varepsilon}(1-\exp(-1/\varepsilon))\sin\pi x_i\triangleq e_i\\ \qquad\qquad (i=0,\ 1,\ 2,\ \cdots,\ N)\end{cases}\tag{5.2}$$

其中 $\rho=\dfrac{h}{\varepsilon}$, $SI=\dfrac{1}{2h}+\coth\dfrac{\rho}{2}$, $a=-2\Big(SI+\dfrac{1}{h^2}+0.5\Big)$, $c=SI+\dfrac{1}{2h}$

$d=SI-\dfrac{1}{2h}$, $b=\dfrac{\rho}{1-\exp(-\rho)}$

方程组(5.2)在IBM—PC机上进行求解，数值结果列举于表1～5．计算结果表明数值实验跟理论分析相符合．

表 1 $\varepsilon=0.001,\ N=16$

坐标	精确解	数值解	误差
$\left(\frac{1}{16},\frac{1}{16}\right)$	−0.121932	−0.121068	8.63499 E-4
$\left(\frac{1}{16},\frac{15}{16}\right)$	−1.828972	−1.829370	3.986459E-4
$\left(\frac{3}{16},\frac{4}{16}\right)$	−1.388926	−1.388874	5.114079E-5
$\left(\frac{6}{16},\frac{15}{16}\right)$	−8.661370	−8.664071	2.700806E-3
$\left(\frac{7}{16},\frac{4}{16}\right)$	−2.451965	−2.452000	3.671646E-5
$\left(\frac{7}{16},\frac{12}{16}\right)$	−7.355890	−7.358262	2.372742E-3
$\left(\frac{8}{16},\frac{15}{16}\right)$	−9.375000	−9.378411	3.141403E-3
$\left(\frac{13}{16},\frac{4}{16}\right)$	−1.388925	−1.389074	1.486540E-4
$\left(\frac{13}{16},\frac{8}{16}\right)$	−2.777851	−2.778993	1.142502E-3
$\left(\frac{15}{16},\frac{4}{16}\right)$	−0.487726	−0.487818	9.253621E-5
$\left(\frac{15}{16},\frac{15}{16}\right)$	−1.828972	−1.829917	9.449720E-4

$$\max_{\substack{0\le i\le 16\\ 0\le j\le 16}} |u(x_i,\ y_j)-u_{ij}|=3.903984\text{E-}3$$

表 2 $\varepsilon=0.0001,\ N=32$

坐标	精确解	数值解	误差
$\left(\frac{1}{32},\frac{24}{32}\right)$	−0.735129	−0.735147	1.806021E-5
$\left(\frac{2}{32},\frac{31}{32}\right)$	−1.889937	−1.889978	4.041195E-5
$\left(\frac{9}{32},\frac{8}{32}\right)$	−1.932526	−1.932021	5.055666E-4
$\left(\frac{11}{32},\frac{31}{32}\right)$	−8.543612	−8.543855	2.422333E-4
$\left(\frac{17}{32},\frac{1}{32}\right)$	−0.310995	−0.304480	6.514877E-3
$\left(\frac{17}{32},\frac{24}{32}\right)$	−9.640853	−9.641143	2.202988E-4
$\left(\frac{31}{32},\frac{16}{32}\right)$	−0.490086	−0.490103	1.776218E-5

$|u(x_i,\ y_j)-u_{ij}|\le 6.556630\text{E-}3.\quad (i,\ j=0,\ 1,\ \cdots,\ 32)$

表 3 $\varepsilon=0.0001$, $N=64$

坐标	精确解	数值解	误差
$(\frac{1}{64}, \frac{1}{64})$	−7.666825E-3	−7.231882E-3	4.349430E-4
$(\frac{1}{64}, \frac{63}{64})$	−0.483010	−0.483010	4.470349E-7
$(\frac{10}{64}, \frac{63}{64})$	−4.640312	−4.640317	4.245209E-6
$(\frac{40}{64}, \frac{63}{64})$	−8.898644	−8.898658	1.621246E-5
$(\frac{15}{64}, \frac{30}{64})$	−3.147933	−3.147892	4.053116E-5
$(\frac{63}{64}, \frac{10}{64})$	−7.666796E-2	−7.657078E-2	9.717792E-5
$(\frac{63}{64}, \frac{60}{64})$	−0.460008	−0.460010	2.443791E-6
$\lvert u(x_i, y_j)-u_{ij}\rvert \leq 8.677557E-3$ $(i, j=0, 1, \cdots, 64)$			

表 4 $\varepsilon=0.001$, $N=32$

坐标	精确解	数值解	误差
$(\frac{1}{32}, \frac{8}{32})$	−0.245043	−0.244349	6.941557E-4
$(\frac{1}{32}, \frac{24}{32})$	−0.735129	−0.735153	2.402067E-5
$(\frac{7}{32}, \frac{31}{32})$	−6.145685	−6.145966	2.808571E-4
$(\frac{9}{32}, \frac{16}{32})$	−3.865053	−3.864869	1 831055E-4
$(\frac{14}{32}, \frac{31}{32})$	−9.501357	−9.501833	4.758835E-4
$(\frac{25}{32}, \frac{24}{32})$	−4.757950	−4.758203	2.536774E-4
$(\frac{31}{32}, \frac{31}{32})$	−0.949541	−0.949609	6.800890E-5
$\lvert u(x_i, y_j)-u_{ij}\rvert \leq 5.749557E-2$ $(i, j=0, 1, \cdots, 32)$			

表 5 $\varepsilon=0.00001$, $N=32$

坐标	精确解	数值解	误差
$(\frac{1}{32}, \frac{1}{32})$	−3.063036E-2	−3.056944E-2	6.091409E-5
$(\frac{3}{32}, \frac{8}{32})$	−0.725712	−0.725720	8.285046E-6
$(\frac{4}{32}, \frac{31}{32})$	−3.707246	−3.707404	1.578331E-4
$(\frac{6}{32}, \frac{31}{32})$	−5.382087	−5.382330	2.427101E-4
$(\frac{17}{32}, \frac{24}{32})$	−7.463887	−7.464272	3.852844E-4
$(\frac{25}{32}, \frac{16}{32})$	−3.171966	−3.172145	1.785755E-4
$(\frac{31}{32}, \frac{16}{32})$	−0.490086	−0.490120	3.460050E-5
$\lvert u(x_i, y_j)-u_{ij}\rvert \leq 5.670488E-4$, $(i, j=0, 1, \cdots, 32)$			

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A Singular Perturbation Problem for Periodic Bounday Partial Differential Equation

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Abstract

In this paper, we consider a singular perturbation elliptic-parabolic partial differential equation for periodic boundary value problem, and construct a difference scheme. Using the method of decomposing the singular term from its solution and combining an asymptotic expansion of the equation, we prove that the scheme constructed by this paper converges uniformly to the solution of its original problem with $O(\tau+h^2)$.

Key words elliptic-parabolic partial differential equation, singular perturbation problem, periodic boundary, difference scheme, uniformly convergence