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正交各向异性弹性力学正交关系的研究*

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(钟万勰推荐)

摘要: 新的正交关系被推广到正交各向异性三维弹性力学• 将弹性力学新正交关系中构造对偶向量的思路推广到正交各向异性问题• 将弹性力学求解辛体系的对偶向量重新排序后, 提出了一种新的对偶向量• 由混合变量求解法直接得到对偶微分方程• 所导出的对偶微分矩阵具有主对角子矩阵为零矩阵的特点• 由于对偶微分矩阵的这一特点, 对于正交各向异性三维弹性力学发现了 2 个独立的、对称的正交关系• 采用分离变量法求解对偶微分方程• 从正交各向异性弹性力学求解体系的积分形式出发, 利用一些恒等式证明了新的正交关系• 新的正交关系不但包含原有的辛正交关系, 而且比原有的关系简洁• 新正交关系的物理意义是对偶方程的解关于 z 坐标的对称性的体现• 辛正交关系是一个广义关系, 但辛正交关系可以在一定的条件下以狭义的强形式出现• 新的研究成果将为研究正交各向异性三维弹性力学的解析解和有限元解提供新的有效工具•

关 键 词: 弹性力学; 对偶向量; 正交关系
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引 言

钟万勰建立了弹性力学求解辛体系[1~3], 创造性地提出了对偶向量和辛正交关系, 展开了一个与传统弹性力学平行的工作平台[4~9]• 在二维弹性力学求解辛体系中, 罗建辉提出了一种新的对偶向量和对偶微分矩阵, 对于各向同性平面问题发现了一种新的正交关系[4]• 本文将这种新的正交关系推广到正交各向异性三维弹性力学• 从正交各向异性三维弹性力学求解的微分形式出发, 由混合变量求解法直接得到以新对偶向量表示的对偶微分矩阵 $\boldsymbol{L}$• $\boldsymbol{L}$ 的主对角子矩阵为零矩阵• 由于 $\boldsymbol{L}$ 的这种特点, 对于正交各向异性弹性力学问题, 我们发现辛正交关系可以分解为 2 个子正交关系• 文中从弹性力学求解体系的积分形式[10]出发, 证明了新的正交关系• 辛正交关系是一个广义关系• 本文的研究表明辛正交关系可以在正交各向异性的条件下以狭义的强形式出现•

1 新的对偶向量

从三维弹性力学混合求解法出发, 直接得到对偶微分方程为:

$$\dot{\boldsymbol{v}} = \boldsymbol{L}\boldsymbol{v} + \boldsymbol{f} \quad (\text{在域 } V \text{ 内}), \tag{1}$$

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式中

$$\dot{\boldsymbol{v}} = \frac{\partial \boldsymbol{v}}{\partial z},\ \boldsymbol{L} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{B} \\ \boldsymbol{D} & \boldsymbol{0} \end{bmatrix},\ \boldsymbol{D} = \begin{bmatrix} 0 & -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial x} & \frac{1}{G_{31}} & 0 \\ -\frac{\partial}{\partial y} & 0 & \frac{1}{G_{23}} \end{bmatrix}, \tag{2}$$

$$\boldsymbol{B} = \begin{bmatrix} \frac{1-\mu_{31}b_1-\mu_{32}b_2}{E_3} & -b_1\frac{\partial}{\partial x} & -b_2\frac{\partial}{\partial y} \\ -b_1\frac{\partial}{\partial x} & -\frac{E_1}{b_0}\frac{\partial^2}{\partial x^2}-G_{12}\frac{\partial^2}{\partial y^2} & -\left(\mu_{12}\frac{E_2}{b_0}+G_{12}\right)\frac{\partial^2}{\partial x\partial y} \\ -b_2\frac{\partial}{\partial y} & -\left(\mu_{12}\frac{E_2}{b_0}+G_{12}\right)\frac{\partial^2}{\partial x\partial y} & -\frac{E_2}{b_0}\frac{\partial^2}{\partial y^2}-G_{12}\frac{\partial^2}{\partial x^2} \end{bmatrix}, \tag{3}$$

$$b_0 = 1-\mu_{12}\mu_{21},\quad b_1 = \frac{\mu_{13}+\mu_{12}\mu_{23}}{b_0},\quad b_2 = \frac{\mu_{23}+\mu_{21}\mu_{13}}{b_0}. \tag{4}$$

(1) 中采用了正交各向异性、线性的弹性方程. 新的对偶向量为

$$\boldsymbol{v} = \left[\boldsymbol{v}_{\mathrm{d}}^{\mathrm{T}}\quad \boldsymbol{v}_{\mathrm{b}}^{\mathrm{T}}\right]^{\mathrm{T}},\ \boldsymbol{v}_{\mathrm{b}} = \left[w\quad \tau_{xz}\quad \tau_{yz}\right]^{\mathrm{T}},\ \boldsymbol{v}_{\mathrm{d}} = \left[\sigma_z\quad u\quad v\right]^{\mathrm{T}}. \tag{5}$$

2 新的正交关系

由(1) 得对偶方程组的齐次解

$$\dot{\boldsymbol{v}}_{\mathrm{b}} = \boldsymbol{B}\boldsymbol{v}_{\mathrm{d}},\ \dot{\boldsymbol{v}}_{\mathrm{d}} = \boldsymbol{D}\boldsymbol{v}_{\mathrm{b}}. \tag{6}$$

采用分离变量法求解, 设

$$\begin{cases} \boldsymbol{v}_{\mathrm{b}} = \phi_{\mathrm{b}}(x, y)\mathrm{e}^{\lambda z}, \\ \boldsymbol{v}_{\mathrm{d}} = \phi_{\mathrm{d}}(x, y)\mathrm{e}^{\lambda z}, \end{cases} \tag{7}$$

式中, λ 是特征值, $\phi = [\phi_{\mathrm{b}}^{\mathrm{T}}\quad \phi_{\mathrm{d}}^{\mathrm{T}}]^{\mathrm{T}}$ 是特征向量.

考虑柱形物体, 选其母线方向为 z 轴, x 轴和 y 轴位于横截面 S 上. S 具有边界 ∂S. ∂S 的外法线方向余弦为 l、m、$n(n = 0)$. 定义

$$\begin{cases} \boldsymbol{J}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \\ \langle \boldsymbol{v}, \boldsymbol{J}_1, \boldsymbol{u} \rangle = \iint_S \boldsymbol{v}\boldsymbol{J}_1\boldsymbol{u}\,\mathrm{d}S. \end{cases} \tag{8}$$

易于验证(9) 和(10) 为恒等式.

$$\begin{aligned} &\langle \boldsymbol{v}_{\mathrm{d}}^{*\mathrm{T}}, \boldsymbol{J}_1, \boldsymbol{B}\boldsymbol{v}_{\mathrm{d}} \rangle - \langle \boldsymbol{v}_{\mathrm{d}}^{\mathrm{T}}, \boldsymbol{J}_1, \boldsymbol{B}\boldsymbol{v}_{\mathrm{d}}^{*} \rangle = \\ &\quad \int_{\partial S}[(l\sigma_x + m\tau_{xy})u^* + (l\tau_{xy} + m\sigma_y)v^*]\mathrm{d}(\partial S) - \\ &\quad \int_{\partial S}[(l\sigma_x^* + m\tau_{xy}^*)u + (l\tau_{xy}^* + m\sigma_y^*)v]\mathrm{d}(\partial S), \end{aligned} \tag{9}$$

$$\begin{aligned} &\langle \boldsymbol{v}_{\mathrm{b}}^{*\mathrm{T}}, \boldsymbol{J}_1, \boldsymbol{D}\boldsymbol{v}_{\mathrm{b}} \rangle - \langle \boldsymbol{v}_{\mathrm{b}}^{\mathrm{T}}, \boldsymbol{J}_1, \boldsymbol{D}\boldsymbol{v}_{\mathrm{b}}^{*} \rangle = \\ &\quad \int_{\partial S}(l\tau_{xz}^* + m\tau_{yz}^*)w\,\mathrm{d}(\partial S) - \int_{\partial S}(l\tau_{xy} + m\tau_{yz})w^*\,\mathrm{d}(\partial S). \end{aligned} \tag{10}$$

若 $\boldsymbol{v}_{\mathrm{b}}$、$\boldsymbol{v}_{\mathrm{d}}$ 和 $\boldsymbol{v}_{\mathrm{b}}^*$、$\boldsymbol{v}_{\mathrm{d}}^*$ 均满足(6) 和边界条件(11)

$$\begin{cases} l\sigma_x + m\tau_{xy} = 0 \quad \text{或} \quad u = 0, \\ l\tau_{xy} + m\sigma_y = 0 \quad \text{或} \quad v = 0, \\ l\tau_{xz} + m\tau_{yz} = 0 \quad \text{或} \quad w = 0. \end{cases} \tag{11}$$

由(9)、(10)得

$$\begin{cases} \langle \boldsymbol{v}_d^{*\mathrm{T}}, \boldsymbol{J}_1, \dot{\boldsymbol{v}}_b \rangle = \langle \boldsymbol{v}_d^{\mathrm{T}}, \boldsymbol{J}_1, \dot{\boldsymbol{v}}_b^{*} \rangle, \\ \langle \boldsymbol{v}_b^{*\mathrm{T}}, \boldsymbol{J}_1, \dot{\boldsymbol{v}}_d \rangle = \langle \boldsymbol{v}_b^{\mathrm{T}}, \boldsymbol{J}_1, \dot{\boldsymbol{v}}_d^{*} \rangle. \end{cases} \tag{12}$$

将(7)代入(12)得

$$\begin{cases} \lambda\langle \phi_d^{*\mathrm{T}}, \boldsymbol{J}_1, \phi_b \rangle - \lambda^{*} \langle \phi_d^{\mathrm{T}}, \boldsymbol{J}_1, \phi_b^{*} \rangle = 0, \\ -\lambda^{*} \langle \phi_d^{*\mathrm{T}}, \boldsymbol{J}_1, \phi_b \rangle + \lambda\langle \phi_d^{\mathrm{T}}, \boldsymbol{J}_1, \phi_b^{*} \rangle = 0. \end{cases} \tag{13}$$

若特征根 λ 和 λ^* 满足 $\lambda^2 - \lambda^{*2} \neq 0$, 由上式得

$$\langle \phi_d^{*\mathrm{T}}, \boldsymbol{J}_1, \phi_b \rangle = 0, \quad \langle \phi_d^{\mathrm{T}}, \boldsymbol{J}_1, \phi_b^{*} \rangle = 0. \tag{14}$$

由(14)可得辛正交关系[3]

$$\langle \phi_d^{\mathrm{T}}, \boldsymbol{J}_1, \phi_b^{*} \rangle = \langle \phi_d^{*\mathrm{T}}, \boldsymbol{J}_1, \phi_b \rangle. \tag{15}$$

对于正交各向异性问题, 新的正交关系(14)包含辛正交关系(15).

3 结 论

对于正交各向异性问题, 新的正交关系不但包含原有的辛正交关系, 而且比原有的关系简洁. 新正交关系的条件是 $\lambda^2 - \lambda^{*2} \neq 0$. 其物理意义是对偶方程组的解关于 z 坐标的对称性. 对于各向异性体这种对称性将不再成立, 所以新正交关系也不成立. 而文献[3]的正交关系对于各向异性体仍成立. 希望本文的工作能对正交各向异性问题的特征函数直接解法的研究有所帮助.

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Research on an Orthogonal Relationship for Orthotropic Elasticity

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Abstract: The new orthogonal relationship is generalized for orthotropic elasticity of three_dimensions. The thought of how dual vectors are constructed in a new orthogonal relationship for theory of elasticity is generalized into orthotropic problems. A new dual vector is presented by the dual vector of the symplectic systematic methodology for elasticity that is over again sorted. A dual differential equation is directly obtained by using a mixed variables method. A dual differential matrix to be derived possesses a peculiarity of which principal diagonal sub_matrixes are zero matrixes. As a result of the peculiarity of the dual differential matrix, two independently and symmetrically orthogonal sub_relationships are discovered for orthotropic elasticity of three_dimensions. The dual differential equation is solved by a method of separation of variable. Based on the integral form of orthotropic elasticity a new orthogonal relationship is proved by using some identical equations. The new orthogonal relationship not only includes the symplectic orthogonal relationship but is also simpler. The physical significance of the new orthogonal relationship is the symmetry representation about an axis z for solutions of the dual equation. The symplectic orthogonal relationship is a generalized relationship but it may be appeared in a strong form with narrow sense in certain condition. This theoretical achievement will provide new effective tools for the research on analytical and finite element solutions to orthotropic elasticity of three_dimensions.

Key words: elasticity; dual vector; orthogonal relationship