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# 压电材料椭圆夹杂界面局部 脱粘问题的分析\*

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摘要: 利用复变函数方法, 研究在反平面剪切和面内电场共同作用下压电材料椭圆夹杂的界面脱粘问题。假定夹杂界面脱粘导致了界面电绝缘型裂纹的产生。通过保角变换和解析延拓, 将原问题化为两个黎曼-希尔伯特问题, 获得了夹杂和基体复势的级数解, 进而求得应力变形场以及夹杂-基体界面脱粘的能量释放率的一般表达式。通过理想粘结的椭圆夹杂、完全脱粘的椭圆夹杂、局部脱粘的刚性导体椭圆夹杂、局部脱粘的圆形夹杂等特例的分析说明了该解的有效性和通用性。

关键词: 压电材料; 椭圆夹杂; 脱粘; 能量释放率

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## 引 言

压电材料可以做成传感器或作动器来检测和控制结构的响应, 因而受到了高度重视。尽管压电材料的研制方面已经取得了许多进展, 但是与传统材料相比, 压电材料的力学分析工作还是相当有限。因此研究压电复合材料中的夹杂(作动器和传感器也可以当作夹杂来处理)对复合材料机械和电学性能的影响就显得尤为重要。

压电材料中的夹杂问题已有不少研究成果, 包括理想粘结的椭球或椭圆夹杂问题<sup>[1~5]</sup>和部分脱粘的圆形夹杂问题<sup>[6]</sup>。文[7]得到了无限大压电材料基体中刚性导体椭圆夹杂的界面脱粘问题的显式解。文[8]则研究了压电材料基体中一般椭圆夹杂界面裂纹的电弹性场强度因子, 但是并未给出能量释放率这一对界面裂纹问题更为有效的断裂参量。

本文则重点研究压电材料中椭圆夹杂界面局部脱粘问题的能量释放率。通过复变函数保角变换和解析延拓, 将原问题化为两个黎曼-希尔伯特问题, 进而获得夹杂和基体的复势的级数解, 并确定夹杂-基体界面脱粘的能量释放率。

## 1 问题的描述

考虑如图 1 所示的无限大压电材料基体中局部脱粘的椭圆夹杂, 其长轴和短轴分别沿  $x$

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轴和  $y$  轴, 长径和短径分别为  $2a$  和  $2b$ 。假定夹杂和基体在  $x, y$  平面内都是横观各向同性的, 但具有不同的材料性质。基体在远场受面内电场和反平面剪切的共同作用。将基体和夹杂所占的区域分别记为 1 区和 2 区, 与 1 区和 2 区有关的量则用相应的上标或下标表示。基体和夹杂的界面记为  $L (= L_c + L_b)$ , 其中  $L_c$  表示脱粘部分 (可看作绝缘和自由的界面曲线裂纹),  $L_b$  则表示界面未脱粘部分。

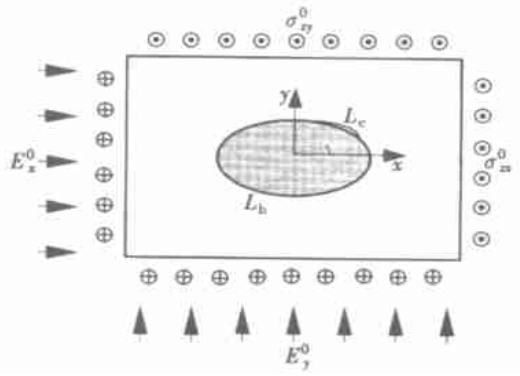


图 1 局部脱粘椭圆夹杂示意图

对于上述问题, 离面位移  $w$  和电势  $\phi$  只是坐标  $x$  和  $y$  的函数, 满足如下控制方程:

$$\nabla^2 w = 0, \quad \nabla^2 \phi = 0, \quad (1)$$

这里  $\nabla^2$  是拉普拉斯算子。

剪应变  $\gamma_{zx}$  和  $\gamma_{zy}$ 、电场  $E_x$  和  $E_y$  由下式计算:

$$\gamma_{zx} = \frac{\partial w}{\partial x}, \quad \gamma_{zy} = \frac{\partial w}{\partial y}, \quad E_x = -\frac{\partial \phi}{\partial x}, \quad E_y = -\frac{\partial \phi}{\partial y}. \quad (2)$$

剪应力  $\alpha_{zx}$  和  $\alpha_{zy}$ 、电位移  $D_x$  和  $D_y$  满足以下本构关系

$$\begin{cases} \alpha_{zx} = G_L \gamma_{zx} - e_{15} E_x, & \alpha_{zy} = G_L \gamma_{zy} - e_{15} E_y, \\ D_x = e_{15} \gamma_{zx} + k_{11} E_x, & D_y = e_{15} \gamma_{zy} + k_{11} E_y, \end{cases} \quad (3)$$

式中  $G_L$  是纵向剪切模量,  $e_{15}$  是压电模量,  $k_{11}$  是介电常数。

夹杂和基体的界面条件可表示为

$$\begin{cases} \alpha_{zn1} = \alpha_{zn2} = 0, & D_{n1} = D_{n2} = 0, & \text{在 } L_c \text{ 上,} \\ \alpha_{zn1} = \alpha_{zn2}, & w_1 = w_2, & D_{n1} = D_{n2}, \quad \phi_1 = \phi_2, & \text{在 } L_b \text{ 上,} \end{cases} \quad (4)$$

这里  $n$  是界面外法线方向。(4) 式的第一个方程描述了界面裂纹的绝缘和自由边界条件。裂纹面电位移的法线分量为零隐含了两个假设: 1) 裂纹面上无自由电荷; 2) 裂纹空隙中的电位移可忽略。

## 2 保角变换

方程 (1) 表明  $w$  和  $\phi$  为调和函数, 可以写成

$$w = \frac{1}{2G_L} [\Psi(z) + \overline{\Psi(z)}], \quad \phi = \frac{1}{2k_{11}} [\Phi(z) + \overline{\Phi(z)}], \quad (5)$$

这里  $\Psi(z)$  和  $\Phi(z)$  是解析函数, 其共轭为  $\overline{\Psi(z)}$  和  $\overline{\Phi(z)}$ , 而  $z = x + iy$  表示复变量。引入如下映射函数

$$z = \Omega(\zeta) = \frac{c}{2} \left[ R\zeta + \frac{1}{R\zeta} \right], \quad (6)$$

这里

$$\begin{aligned} \zeta &= \xi + i\eta = \rho e^{i\theta}, \quad c = (a^2 - b^2)^{1/2} = a(1 - \varepsilon^2)^{1/2}, \\ R &= \left[ \frac{a+b}{a-b} \right]^{1/2} = \left[ \frac{1+\varepsilon}{1-\varepsilon} \right]^{1/2}, \quad \varepsilon = \frac{b}{a}, \end{aligned}$$

(6) 式的函数将  $z$  平面的 1 区映射成  $\zeta$  平面的单位圆周  $\Gamma_1$  (半径  $\rho = 1$ ) 的外部区域,  $z$  平面的

2 区映射成  $\zeta$  平面的单位圆周  $\Gamma_1$  与另一圆周  $\Gamma_2$ (半径  $\rho = 1/R$ ) 之间的圆环域, 圆周  $\Gamma_2$  对应于  $z$  平面从  $-c$  到  $+c$  的割线, 如图 2 所示.  $\Gamma_1$  由  $\Gamma_c$  和  $\Gamma_b$  两部分组成, 分别对应于  $z$  平面的  $L_c$  和  $L_b$ . 界面裂纹尖端  $t_1 = r_1 e^{i\theta_1}$  和  $t_2 = r_2 e^{i\theta_2}$  则被映射成  $\zeta$  平面单位圆周  $\Gamma_1$  上的点  $\sigma_1 = e^{i\beta_1}$  和  $\sigma_2 = e^{i\beta_2}$ .

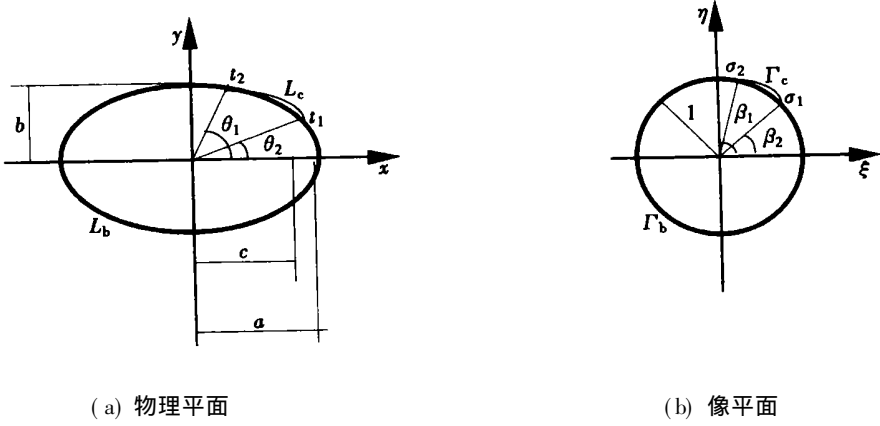


图 2 保角变换

映射之后的界面边界条件(4)可以表示为

$$\left\{ \begin{aligned} & \sigma \psi_1'(\sigma) + \overline{\sigma \psi_1'(\sigma)} + \frac{e_{15}^1}{k_{11}^1} [\sigma \phi_1'(\sigma) + \overline{\sigma \phi_1'(\sigma)}] = 0, & \sigma \in \Gamma_c, \\ & \sigma \psi_2'(\sigma) + \overline{\sigma \psi_2'(\sigma)} + \frac{e_{15}^2}{k_{11}^2} [\sigma \phi_2'(\sigma) + \overline{\sigma \phi_2'(\sigma)}] = 0, & \sigma \in \Gamma_c, \\ & \frac{e_{15}^1}{G_L^1} [\sigma \psi_1'(\sigma) + \overline{\sigma \psi_1'(\sigma)}] - [\sigma \phi_1'(\sigma) + \overline{\sigma \phi_1'(\sigma)}] = 0, & \sigma \in \Gamma_c, \\ & \frac{e_{15}^2}{G_L^2} [\sigma \psi_2'(\sigma) + \overline{\sigma \psi_2'(\sigma)}] - [\sigma \phi_2'(\sigma) + \overline{\sigma \phi_2'(\sigma)}] = 0, & \sigma \in \Gamma_c, \\ & \sigma \psi_1'(\sigma) + \overline{\sigma \psi_1'(\sigma)} + \frac{e_{15}^1}{k_{11}^1} [\sigma \phi_1'(\sigma) + \overline{\sigma \phi_1'(\sigma)}] = \\ & \quad \sigma \psi_2'(\sigma) + \overline{\sigma \psi_2'(\sigma)} + \frac{e_{15}^2}{k_{11}^2} [\sigma \phi_2'(\sigma) + \overline{\sigma \phi_2'(\sigma)}], & \sigma \in \Gamma_b, \\ & \frac{e_{15}^1}{G_L^1} [\sigma \psi_1'(\sigma) + \overline{\sigma \psi_1'(\sigma)}] - [\sigma \phi_1'(\sigma) + \overline{\sigma \phi_1'(\sigma)}] = \\ & \quad \frac{e_{15}^2}{G_L^2} [\sigma \psi_2'(\sigma) + \overline{\sigma \psi_2'(\sigma)}] - [\sigma \phi_2'(\sigma) + \overline{\sigma \phi_2'(\sigma)}], & \sigma \in \Gamma_b, \\ & \mu_1 [\sigma \psi_1'(\sigma) - \overline{\sigma \psi_1'(\sigma)}] = \sigma \psi_2'(\sigma) - \overline{\sigma \psi_2'(\sigma)}, & \sigma \in \Gamma_b, \\ & \mu_2 [\sigma \phi_1'(\sigma) - \overline{\sigma \phi_1'(\sigma)}] = \sigma \phi_2'(\sigma) - \overline{\sigma \phi_2'(\sigma)}, & \sigma \in \Gamma_b, \end{aligned} \right. \quad (7)$$

这里  $\mu_1 = G_L^2/G_L^1$ ,  $\mu_2 = k_{11}^2/k_{11}^1$ . 另外, 夹杂内部  $\psi_2(z)$  和  $\phi_2(z)$  的连续性条件要求

$$\psi_2\left(\frac{\sigma}{R}\right) = \psi_2\left(\frac{\sigma}{R}\right), \quad \phi_2\left(\frac{\sigma}{R}\right) = \phi_2\left(\frac{\sigma}{R}\right), \quad \sigma \in \Gamma_2, \quad (8)$$

这是因为点  $\sigma/R$  和  $\sigma/R$  对应于  $z$  平面从  $-c$  到  $+c$  的切割线上的同一点.

### 3 问题的求解

本节将得到满足界面条件(7)和连续性条件(8)的复势的一般解。为方便起见,在基体区域和夹杂区域引入新函数

$$\begin{cases} u_1(\zeta) = \zeta \dot{\Psi}_1(\zeta), & |\zeta| > 1, \\ u_2(\zeta) = \zeta \dot{\Psi}_2(\zeta), & 1/R < |\zeta| < 1, \\ v_1(\zeta) = \zeta \dot{\Phi}_1(\zeta), & |\zeta| > 1, \\ v_2(\zeta) = \zeta \dot{\Phi}_2(\zeta), & 1/R < |\zeta| < 1. \end{cases} \quad (9)$$

进一步将上述函数进行延拓,使之满足

$$\begin{cases} u_1(\zeta) = -\frac{1}{\zeta} \overline{\Psi_1} \left\{ \frac{1}{\zeta} \right\}, & 0 < |\zeta| < 1, \\ u_2(\zeta) = -\frac{1}{\zeta} \overline{\Psi_2} \left\{ \frac{1}{\zeta} \right\}, & 1 < |\zeta| < R, \\ v_1(\zeta) = -\frac{1}{\zeta} \overline{\Phi_1} \left\{ \frac{1}{\zeta} \right\}, & 0 < |\zeta| < 1, \\ v_2(\zeta) = -\frac{1}{\zeta} \overline{\Phi_2} \left\{ \frac{1}{\zeta} \right\}, & 1 < |\zeta| < R. \end{cases} \quad (10)$$

这样,由边界条件(7)可得到下述两个黎曼-希尔伯特问题:

$$\begin{cases} u_1^+(\sigma) - u_1^-(\sigma) = 0, & \sigma \in \Gamma_c, \\ u_1^+(\sigma) + u_1^-(\sigma) = \sum_{k=0}^{\infty} (p_k \sigma^{k+1} + q_k \sigma^{-(k+1)}), & \sigma \in \Gamma_b; \end{cases} \quad (11)$$

$$\begin{cases} v_1^+(\sigma) - v_1^-(\sigma) = 0, & \sigma \in \Gamma_c, \\ v_1^+(\sigma) + v_1^-(\sigma) = \sum_{k=0}^{\infty} (r_k \sigma^{k+1} + s_k \sigma^{-(k+1)}), & \sigma \in \Gamma_b, \end{cases} \quad (12)$$

其中

$$\begin{cases} p_k = \frac{2}{\Delta_2(K_1K_4 + K_2K_3)} [k_{11}^2(G_L^2K_4 + e_{15}^2K_2)e_k + G_L^2(e_{15}^2K_4 - k_{11}^2K_2)g_k], \\ q_k = \frac{2}{\Delta_2(K_1K_4 + K_2K_3)} [k_{11}^2(G_L^2K_4 + e_{15}^2K_2)f_k + G_L^2(e_{15}^2K_4 - k_{11}^2K_2)h_k], \\ r_k = \frac{-2}{\Delta_2(K_1K_4 + K_2K_3)} [k_{11}^2(G_L^2K_3 - e_{15}^2K_1)e_k + G_L^2(e_{15}^2K_3 + k_{11}^2K_1)g_k], \\ s_k = \frac{-2}{\Delta_2(K_1K_4 + K_2K_3)} [k_{11}^2(G_L^2K_3 - e_{15}^2K_1)f_k + G_L^2(e_{15}^2K_3 + k_{11}^2K_1)h_k], \end{cases} \quad (13)$$

这里

$$\Delta_2 = G_L^2 k_{11}^2 + (e_{15}^2)^2,$$

$$K_1 = \mu_1 + (G_L^2 k_{11}^2 + \mu_1 e_{15}^2) / \Delta_2, \quad K_2 = G_L^2 (e_{15}^2 - \mu_2 e_{15}) / \Delta_2,$$

$$K_3 = k_{11}^2 (e_{15}^2 - \mu_1 e_{15}) / \Delta_2, \quad K_4 = \mu_2 + (G_L^2 k_{11}^2 + \mu_2 e_{15}^2) / \Delta_2,$$

而  $e_k, f_k, g_k, h_k$  是下列罗伦级数的系数:

$$\omega_1(\zeta) = u_1(\zeta) + u_2(\zeta) + \frac{e_{15}^2}{k_{11}} v_1(\zeta) + \frac{e_{15}^2}{k_{11}} v_2(\zeta) =$$

$$\sum_{k=0}^{\infty} (e_k \zeta^{k+1} + f_k \zeta^{-(k+1)}), \quad (14a)$$

$$\omega_2(\zeta) = \frac{e_{15}^1}{G_L^1} u_1(\zeta) + \frac{e_{15}^2}{G_L^2} u_2(\zeta) - v_1(\zeta) - v_2(\zeta) =$$

$$\sum_{k=0}^{\infty} (g_k \zeta^{k+1} + h_k \zeta^{-(k+1)}), \quad (14b)$$

这里  $\omega_1(\zeta)$  和  $\omega_2(\zeta)$  是区域  $|\zeta| > 0$  内的解析函数。

因为  $\Psi_1(\zeta)$ 、 $\Psi_2(\zeta)$ 、 $\Phi_1(\zeta)$ 、 $\Phi_2(\zeta)$  是各自定义域内的解析函数, 所以它们可以被分别展开成如下的罗伦级数:

$$\left\{ \begin{array}{l} \Psi_1(\zeta) = \sum_{k=0}^{\infty} (a_k^1 \zeta^{k+1} + b_k^1 \zeta^{-(k+1)}), \quad |\zeta| > 1, \\ \Psi_2(\zeta) = \sum_{k=0}^{\infty} (a_k^2 \zeta^{k+1} + b_k^2 \zeta^{-(k+1)}), \quad 1/R < |\zeta| < 1, \\ \Phi_1(\zeta) = \sum_{k=0}^{\infty} (c_k^1 \zeta^{k+1} + d_k^1 \zeta^{-(k+1)}), \quad |\zeta| > 1, \\ \Phi_2(\zeta) = \sum_{k=0}^{\infty} (c_k^2 \zeta^{k+1} + d_k^2 \zeta^{-(k+1)}), \quad 1/R < |\zeta| < 1, \end{array} \right. \quad (15)$$

式中对应于刚体位移的常数项省略不写。系数  $a_k^1$  和  $c_k^1$  应当由远场边界条件确定, 而  $b_k^1$ 、 $d_k^1$ 、 $a_k^2$ 、 $b_k^2$ 、 $c_k^2$ 、 $d_k^2$  则是需要求解的未知量。相应地, 可以得到

$$\left\{ \begin{array}{l} u_1(\zeta) = \sum_{k=0}^{\infty} (k+1)(a_k^1 \zeta^{k+1} - b_k^1 \zeta^{-(k+1)}), \quad |\zeta| > 1, \\ u_1(\zeta) = \sum_{k=0}^{\infty} (k+1)(b_k^1 \zeta^{k+1} - a_k^1 \zeta^{-(k+1)}), \quad |\zeta| < 1, \\ v_1(\zeta) = \sum_{k=0}^{\infty} (k+1)(c_k^1 \zeta^{k+1} - d_k^1 \zeta^{-(k+1)}), \quad |\zeta| > 1, \\ v_1(\zeta) = \sum_{k=0}^{\infty} (k+1)(d_k^1 \zeta^{k+1} - c_k^1 \zeta^{-(k+1)}), \quad |\zeta| < 1, \\ u_2(\zeta) = \sum_{k=0}^{\infty} (k+1)(a_k^2 \zeta^{k+1} - b_k^2 \zeta^{-(k+1)}), \quad 1/R < |\zeta| < 1, \\ u_2(\zeta) = \sum_{k=0}^{\infty} (k+1)(b_k^2 \zeta^{k+1} - a_k^2 \zeta^{-(k+1)}), \quad 1 < |\zeta| < R, \\ v_2(\zeta) = \sum_{k=0}^{\infty} (k+1)(c_k^2 \zeta^{k+1} - d_k^2 \zeta^{-(k+1)}), \quad 1/R < |\zeta| < 1, \\ v_2(\zeta) = \sum_{k=0}^{\infty} (k+1)(d_k^2 \zeta^{k+1} - c_k^2 \zeta^{-(k+1)}), \quad 1 < |\zeta| < R. \end{array} \right. \quad (16)$$

(11) 和(12)式的解可以由下式给出

$$u_1(\zeta) = \frac{1}{2} \left\{ \sum_{k=0}^{\infty} (p_k \zeta^{k+1} + q_k \zeta^{-(k+1)}) - X(\zeta) [U_{\infty}(\zeta) + U_0(\zeta)] \right\} + X(\zeta) R(\zeta), \quad (18)$$

$$v_1(\zeta) = \frac{1}{2} \left\{ \sum_{k=0}^{\infty} (r_k \zeta^{k+1} + s_k \zeta^{-(k+1)}) - X(\zeta) [V_{\infty}(\zeta) + V_0(\zeta)] \right\} + X(\zeta) R(\zeta), \quad (19)$$

这里  $X(z)$  是 Plemelj 函数, 其定义为

$$X(\zeta) = [(\zeta - \sigma_1)(\zeta - \sigma_2)]^{-1/2}.$$

该函数在整个  $\zeta$  平面内, 除了沿未脱粘圆弧  $\Gamma_b$  所做的切割线外, 都是解析的; 而在圆弧  $\Gamma_b$  上有  $X^+(\sigma) = -X^-(\sigma)$ . 选择满足  $\lim_{\zeta \rightarrow \infty} X(\zeta) = 1$  的  $X(\zeta)$  的分支, 这样在点  $\zeta = \infty$  和  $\zeta = 0$ , 函数  $X(z)$  可以被展开成

$$X(\zeta) = \sum_{n=0}^{\infty} \alpha_n^* \zeta^{-(n+1)}, \quad \frac{1}{X(\zeta)} = \sum_{n=0}^{\infty} \alpha_n \zeta^{1-n} \quad (\zeta \rightarrow \infty),$$

$$X(\zeta) = X(0) \sum_{n=0}^{\infty} \beta_n^* \zeta^{-n}, \quad \frac{1}{X(\zeta)} = \frac{1}{X(0)} \sum_{n=0}^{\infty} \beta_n \zeta^n \quad (\zeta \rightarrow 0),$$

其中  $X(0) = (\sigma_1 \sigma_2)^{-1/2}$ , 而

$$\alpha_n^* = \sum_{m=0}^n \gamma_m^* \gamma_{n-m}^* \sigma_1^m \sigma_2^{n-m}, \quad \beta_n^* = \sum_{m=0}^n \gamma_m^* \gamma_{n-m}^* \sigma_1^m \sigma_2^{(n-m)},$$

$$\alpha_n = \sum_{m=0}^n \gamma_m \gamma_{n-m} \sigma_1^m \sigma_2^{n-m}, \quad \beta_n = \sum_{m=0}^n \gamma_m \gamma_{n-m} \sigma_1^m \sigma_2^{(n-m)},$$

并且

$$\gamma_m = \begin{cases} 1, & m = 0, \\ -\frac{\gamma_{m-1}^*}{2m}, & m \geq 1, \end{cases} \quad \gamma_m^* = \frac{(2m)!}{2^{2m}(m!)^2}.$$

而(18)和(19)式中的函数  $U_{\infty}(\zeta)$ ,  $U_0(\zeta)$ ,  $V_{\infty}(\zeta)$ ,  $V_0(\zeta)$ ,  $R(\zeta)$ ,  $R(\zeta)$  由下述各式给定

$$U_{\infty}(\zeta) = \sum_{n=0}^{\infty} p_n \sum_{m=0}^{n+1} \alpha_m \zeta^{n+2-m}, \quad U_0(\zeta) = \frac{1}{X(0)} \sum_{n=0}^{\infty} q_n \sum_{m=0}^{n+1} \beta_m \zeta^{m-n-1},$$

$$V_{\infty}(\zeta) = \sum_{n=0}^{\infty} r_n \sum_{m=0}^{n+1} \alpha_m \zeta^{n+2-m}, \quad V_0(\zeta) = \frac{1}{X(0)} \sum_{n=0}^{\infty} s_n \sum_{m=0}^{n+1} \beta_m \zeta^{m-n-1},$$

$$R(\zeta) = h^* + \sum_{k=0}^{\infty} l_k \zeta^{k+1} + \sum_{k=0}^{\infty} h_k \zeta^{-k-1},$$

$$R(\zeta) = h^* + \sum_{k=0}^{\infty} l_k \zeta^{k+1} + \sum_{k=0}^{\infty} h_k \zeta^{-k-1},$$

这里

$$h^* = -\frac{1}{X(0)} \sum_{m=0}^{\infty} (m+1) \beta_{m+1} a_m^1,$$

$$h_k = -\frac{1}{X(0)} \sum_{m=0}^{\infty} (m+k+1) \beta_{m+1} a_{m+k}^1 \quad (k \geq 0),$$

$$l_0 = \sum_{m=0}^{\infty} (m+1) \alpha_{m+1} a_m^1, \quad l_k = \sum_{m=0}^{\infty} (m+k) \alpha_{m+1} a_{m+k-1}^1 \quad (k \geq 1),$$

$$h^* = -\frac{1}{X(0)} \sum_{m=0}^{\infty} (m+1) \beta_{m+1} c_m^1,$$

$$h_k = -\frac{1}{X(0)} \sum_{m=0}^{\infty} (m+k+1) \beta_{m+1} c_{m+k}^1 \quad (k \geq 0),$$

$$l_0 = \sum_{m=0}^{\infty} (m+1) \alpha_{m+1} c_m^1, \quad l_k = \sum_{m=0}^{\infty} (m+k) \alpha_m c_{m+k-1}^1 \quad (k \geq 1).$$

由(14)式,可以得到

$$u_2(\zeta) = \frac{G_L^2}{\Delta_2} \left\{ k_{11}^2 \left[ \omega_1(\zeta) - u_1(\zeta) - \frac{e_{15}^1}{k_{11}^1} v_1(\zeta) \right] + e_{15}^2 \left[ \omega_2(\zeta) - \frac{e_{15}^1}{G_L^1} u_1(\zeta) + v_1(\zeta) \right] \right\}, \quad (20)$$

$$v_2(\zeta) = \frac{k_{11}^2}{\Delta_2} \left\{ e_{15}^2 \left[ \omega_1(\zeta) - u_1(\zeta) - \frac{e_{15}^1}{k_{11}^1} v_1(\zeta) \right] - G_L^2 \left[ \omega_2(\zeta) - \frac{e_{15}^1}{G_L^1} u_1(\zeta) + v_1(\zeta) \right] \right\}. \quad (21)$$

一旦未知量  $e_k, f_k, g_k, h_k$  确定以后, (18)~(21)式构成了所研究问题的一般解。由此可进一步确定夹杂和基体中的应力场和电场。

由(14)和(16)式,可以获得下列方程

$$\begin{cases} e_k = (k+1) \left[ (a_k^1 + b_k^2) + \frac{e_{15}^1}{k_{11}^1} c_k^1 + \frac{e_{15}^2}{k_{11}^2} d_k^2 \right], \\ f_k = -(k+1) \left[ (b_k^1 + a_k^2) + \frac{e_{15}^1}{k_{11}^1} d_k^1 + \frac{e_{15}^2}{k_{11}^2} c_k^2 \right], \\ e_k = -f_k, \\ g_k = (k+1) \left[ \frac{e_{15}^1}{G_L^1} a_k^1 + \frac{e_{15}^2}{G_L^2} b_k^2 - c_k^1 - d_k^2 \right], \\ f_k = -(k+1) \left[ \frac{e_{15}^1}{G_L^1} b_k^1 + \frac{e_{15}^2}{G_L^2} a_k^2 - d_k^1 - c_k^2 \right], \\ g_k = -h_k. \end{cases} \quad (22)$$

将(18)和(19)式在区域  $|\zeta| > 1$  中进行罗伦奇数展开, 并且和(16)式的各级数项进行对比, 可得

$$\begin{cases} b_k^1 = -\frac{1}{2(k+1)} \left[ q_k - \sum_{n=0}^{\infty} p_n \sum_{m=0}^{n+1} \alpha_m^* \alpha_{n+k+2-m} - \frac{1}{X(0)} \sum_{n=0}^{\infty} q_n \sum_{m=0}^{\min(k, n+1)} \beta_{n+1-m} \alpha_{k-m}^* \right] + \alpha_k^* h^* + \sum_{m=0}^{\infty} \alpha_{k+m+1}^* l_m + \sum_{m=0}^{k-1} \alpha_m^* h_{k-m-1}, \\ d_k^1 = -\frac{1}{2(k+1)} \left[ s_k - \sum_{n=0}^{\infty} J_n \sum_{m=0}^{n+1} \alpha_m^* \alpha_{n+k+2-m} - \frac{1}{X(0)} \sum_{n=0}^{\infty} s_n \sum_{m=0}^{\min(k, n+1)} \beta_{n+1-m} \alpha_{k-m}^* \right] + \alpha_k^* h^* + \sum_{m=0}^{\infty} \alpha_{k+m+1}^* l_m + \sum_{m=0}^{k-1} \alpha_m^* h_{k-m-1}. \end{cases} \quad (23)$$

另外, 由(8)式可以得到如下关系

$$a_k^2 = R^{2k+2} b_k^2, \quad c_k^2 = R^{2k+2} d_k^2. \quad (24)$$

因此,未知量  $b_k^1, d_k^1, a_k^2, b_k^2, c_k^2, d_k^2$  和  $e_k, f_k, g_k, h_k$  可通过联立求解方程(22)~(24)来获得。

## 4 能量释放率

横截面面积为  $D$  的单位厚度压电固体的内能  $W$  可以写成<sup>[6]</sup>

$$W = \frac{1}{2} \iint_D (\alpha_{zx} \gamma_{zx} + \alpha_{zy} \gamma_{zy} + D_x E_x + D_y E_y) dS. \quad (25)$$

上式可进一步分解为

$$W = W_0 + \Delta W, \quad (26)$$

这里  $W_0$  是没有夹杂时的内能,而  $\Delta W$  则表示由于局部脱粘的夹杂引起的内能变化。这样,能量释放率可以表示为

$$\begin{cases} g_1 = \frac{\partial(\Delta W)}{\partial S_1} = -\frac{\partial(\Delta W)}{\lambda_1 \partial \beta_1}, \\ g_2 = \frac{\partial(\Delta W)}{\partial S_2} = \frac{\partial(\Delta W)}{\lambda_2 \partial \beta_2}, \end{cases} \quad (27)$$

其中  $g_1$  和  $g_2$  分别是裂尖  $t_1$  和  $t_2$  的能量释放率,而

$$\lambda_\alpha = \frac{c}{2} \left[ \left( R - \frac{1}{R} \right)^2 \cos^2 \beta_\alpha + \left( R + \frac{1}{R} \right)^2 \sin^2 \beta_\alpha \right]^{1/2} \quad (\alpha = 1, 2), \quad (28)$$

这里

$$\begin{aligned} \Delta W = & \frac{\pi}{2G_L^1} (a_0^1 b_0^1 + \overline{a_0^1 b_0^1}) + \frac{\pi}{2k_{11}^1} (c_0^1 d_0^1 + \overline{c_0^1 d_0^1}) + \\ & \frac{\pi e_{15}^1}{2G_L^1 k_{11}^1} (c_0^1 b_0^1 + \overline{c_0^1 b_0^1} - a_0^1 d_0^1 - \overline{a_0^1 d_0^1}). \end{aligned} \quad (29)$$

## 5 特例及讨论

下面我们将通过若干特例来说明所获得一般解的有效性和通用性。

### 5.1 理想粘结的椭圆夹杂

理想粘结的椭圆夹杂问题的解可以作为上面所获得的一般解的极限情况:令界面裂纹长度  $L_c$  为零(即  $\sigma_2 = \sigma_1 = \sigma$  和  $\beta_2 = \beta_1$ )。在这种情况下,有

$$\begin{cases} X(0) = \sigma^1, \quad \alpha_n^* = \sigma^n, \quad \beta_n^* = \sigma^{-n} & (n \geq 0), \\ \alpha_0 = \beta_0 = 1, \quad \alpha_1 = -\sigma, \quad \beta_1 = -\sigma^{-1}, \quad \alpha_n = \beta_n = 0 & (n \geq 2). \end{cases} \quad (30)$$

将(30)代入(23)可得

$$b_k^1 = a_k^1 - \frac{1}{k+1} q_k, \quad d_k^1 = c_k^1 - \frac{1}{k+1} s_k. \quad (31)$$

由(31)和(22)、(24)式,可得

$$\begin{cases} \mu_1(a_k^1 + b_k^1) = a_k^2 + b_k^2, \quad \mu_2(c_k^1 + d_k^1) = c_k^2 + d_k^2, \\ a_k^1 - b_k^1 + \frac{e_{15}^1}{k_{11}^1} (c_k^1 - d_k^1) = a_k^2 - b_k^2 + \frac{e_{15}^2}{k_{11}^2} (c_k^2 - d_k^2), \\ \frac{e_{15}^1}{G_L^1} (a_k^1 - b_k^1) - (c_k^1 - d_k^1) = \frac{e_{15}^2}{G_L^2} (a_k^2 - b_k^2) - (c_k^2 - d_k^2), \\ a_k^2 = R^{2(k+1)} b_k^2, \quad c_k^2 = R^{2(k+1)} d_k^2, \end{cases} \quad (32)$$



这里  $b_k^1, d_k^1, a_k^2, b_k^2, c_k^2, d_k^2$  可以通过  $a_k^1$  和  $c_k^1$  显式表示。这样, 由(15) 式可求出  $\zeta$  平面的复势, 由于这个特例简单, 还可以进一步获得复势在物理平面的显式表达式。这个结果和文[5] 中压电材料理想粘结椭圆夹杂的结果一致。

### 5.2 完全脱粘的椭圆夹杂

完全脱粘的椭圆夹杂问题的解也可作为上述一般解的特例获得: 令界面未脱粘部分  $L_b$  长度为零(即  $\sigma_1 = \sigma_2 = \sigma$  和  $\beta_2 = \beta_1 + 2\pi$ )。此时, 有

$$\begin{cases} X(0) = -\sigma^{-1}, \alpha_n^* = \sigma, \beta_n^* = \sigma^n & (n \geq 0), \\ \alpha_0 = \beta_0 = 1, \alpha_1 = -\sigma, \beta_1 = -\sigma^{-1}, \alpha_n = \beta_n = 0 & (n \geq 2). \end{cases} \quad (33)$$

将(33) 代入(23) 可得

$$b_k^1 = a_k^1, \quad d_k^1 = c_k^1. \quad (34)$$

由(22)、(24) 和(34) 式, 可求得

$$a_k^2 = b_k^2 = c_k^2 = d_k^2 = 0. \quad (35)$$

这样  $\zeta$  平面的复势可由(15) 式求出, 而  $z$  平面的复势则可以由(6) 式的逆变换求得。结果表明完全脱粘的椭圆夹杂作用相当于绝缘和自由的空洞。

### 5.3 局部脱粘的刚性导体椭圆夹杂

考虑在远场均匀力电场作用下, 局部对称脱粘的刚性导体椭圆夹杂问题。在这种情况下, 令  $G_L^2 \rightarrow \infty, k_{11}^2 \rightarrow \infty$  和  $\beta_2 = -\beta_1$ , 可求得

$$\begin{cases} a_0^1 = \frac{1}{2} a(1 + \varepsilon)(A^r + iA^i), \\ c_0^1 = \frac{1}{2} a(1 + \varepsilon)(C^r + iC^i), \\ a_k^1 = c_k^1 = 0 \quad (k \geq 1), \end{cases} \quad (36)$$

这里  $A^r, A^i, C^r$  和  $C^i$  由远场力电载荷确定如下:

第一种情况: 基体在远场受均匀应变  $\gamma_{zx}^0$  和  $\gamma_{zy}^0$ 、均匀电场强度  $E_x^0$  和  $E_y^0$  的共同作用。这时有

$$A^r = G_L^1 \gamma_{zx}^0, \quad A^i = -G_L^1 \gamma_{zy}^0, \quad C^r = -k_{11}^1 E_x^0, \quad C^i = k_{11}^1 E_y^0. \quad (37)$$

第二种情况: 基体在远场受均匀应力  $\tau_{zx}^0$  和  $\tau_{zy}^0$ 、均匀电位移  $D_x^0$  和  $D_y^0$  的共同作用。这时有

$$\begin{cases} A^r = \frac{G_L^1(k_{11}^1 \sigma_{zx}^0 + e_{15}^1 D_x^0)}{G_L^1 k_{11}^1 + (e_{15}^1)^2}, \quad A^i = -\frac{G_L^1(k_{11}^1 \sigma_{zy}^0 + e_{15}^1 D_y^0)}{G_L^1 k_{11}^1 + (e_{15}^1)^2}, \\ C^r = \frac{k_{11}^1(e_{15}^1 \sigma_{zx}^0 - G_L^1 D_x^0)}{G_L^1 k_{11}^1 + (e_{15}^1)^2}, \quad C^i = -\frac{k_{11}^1(e_{15}^1 \sigma_{zy}^0 - G_L^1 D_y^0)}{G_L^1 k_{11}^1 + (e_{15}^1)^2}. \end{cases} \quad (38)$$

第三种情况: 基体在远场受均匀应变  $\gamma_{zx}^0$  和  $\gamma_{zy}^0$ 、均匀电位移  $D_x^0$  和  $D_y^0$  的共同作用。这时有

$$A^r = G_L^1 \gamma_{zx}^0, \quad A^i = -G_L^1 \gamma_{zy}^0, \quad C^r = e_{15}^1 \gamma_{zx}^0 - D_x^0, \quad C^i = e_{15}^1 \gamma_{zy}^0 - D_y^0. \quad (39)$$

第四种情况: 基体在远场受均匀应力  $\tau_{zx}^0$  和  $\tau_{zy}^0$ 、均匀电场强度  $E_x^0$  和  $E_y^0$  的共同作用。这时有

$$A^r = \sigma_{zx}^0 + e_{15}^1 E_x^0, \quad A^i = -\sigma_{zy}^0 - e_{15}^1 E_y^0, \quad C^r = -k_{11}^1 E_x^0, \quad C^i = k_{11}^1 E_y^0. \quad (40)$$

将  $G_L^2 \rightarrow \infty$  和  $k_{11}^2 \rightarrow \infty$  代入(13), 得到

$$p_k = q_k = r_k = s_k = 0. \quad (41)$$

利用(36)和(41)式,可以由(23)式得到 $b_k^1$ 和 $d_k^1$ ,由(15)式得到复势。至此,基体的弹性场和静电场就可以完全确定。由(27)和(29)式还可求出界面裂纹尖端的能量释放率,但需用到如下关系

$$\begin{cases} b_0^1 = -a_0^1 \cos \left\{ \frac{\beta_2 - \beta_1}{2} \right\} - \frac{1}{8} a_0^1 [e^{2i\beta_1} + e^{2i\beta_2} - 2e^{i(\beta_1 + \beta_2)}], \\ d_0^1 = -c_0^1 \cos \left\{ \frac{\beta_2 - \beta_1}{2} \right\} - \frac{1}{8} d_0^1 [e^{2i\beta_1} + e^{2i\beta_2} - 2e^{i(\beta_1 + \beta_2)}]. \end{cases} \quad (42)$$

#### 5.4 局部脱粘的圆形夹杂问题

考虑局部脱粘的圆形夹杂问题( $b = a$ 和 $R \rightarrow \infty$ ),假设无限大基体在远场受均匀反平面剪切和均匀面内电场的共同作用。不失一般性,假定界面圆弧裂纹 $L_c$ 关于 $x$ 轴对称(即 $\theta_2 = -\theta_1 = \theta$ , $\beta_2 = -\beta_1 = \beta$ )。利用远场载荷条件,可以确定

$$a_0^1 = a(A^r + iA^i), \quad c_0^1 = a(C^r + iC^i), \quad a_k^1 = c_k^1 = 0 \quad (k \geq 1), \quad (43)$$

这里 $A^r, A^i, C^r$ 和 $C^i$ 在4种远场条件下的表达式如(37)~(40)式所示。因为 $R \rightarrow \infty$ ,由(24)得

$$b_k^2 = d_k^2 = 0. \quad (44)$$

将(43)和(44)代入(22),可得

$$\begin{cases} e_0 = -f_0 = a_0^1 + \frac{e_{15}^1}{k_{11}^1} c_0^1, \\ g_0 = -h_0 = \frac{e_{15}^1}{G_L^1} a_0^1 - c_0^1, \\ e_k = f_k = g_k = h_k = 0 \quad (k \geq 1). \end{cases} \quad (45)$$

再由(13)式得

$$\begin{cases} p_0 = \frac{2}{\Delta_2(K_1K_{4+} - K_2K_3)} [k_{11}^2(G_L^2K_{4+} - e_{15}^2K_2)e_0 + G_L^2(e_{15}^2K_{4-} - k_{11}^2K_2)g_0], \\ q_0 = \frac{2}{\Delta_2(K_1K_{4+} - K_2K_3)} [k_{11}^2(G_L^2K_{4+} - e_{15}^2K_2)f_0 + G_L^2(e_{15}^2K_{4-} - k_{11}^2K_2)h_0], \\ r_0 = \frac{-2}{\Delta_2(K_1K_{4+} - K_2K_3)} [k_{11}^2(G_L^2K_{3-} - e_{15}^2K_1)e_0 + G_L^2(e_{15}^2K_{3+} + k_{11}^2K_1)g_0], \\ s_0 = \frac{-2}{\Delta_2(K_1K_{4+} - K_2K_3)} [k_{11}^2(G_L^2K_{3-} - e_{15}^2K_1)f_0 + G_L^2(e_{15}^2K_{3+} + k_{11}^2K_1)h_0], \\ p_0 = q_0 = r_0 = s_0 = 0 \quad (k \geq 1), \end{cases} \quad (46)$$

而 $b_k^1$ 和 $d_k^1$ 则由(23)式得到,再由(15)式可获得复势。这样夹杂和基体的弹性场和静电场就完全确定了。界面裂纹尖端的能量释放率可以表示成

$$\begin{aligned} g_1 = & \frac{\pi a \sin \beta}{4G_L^1} [A^r(2A^r - p_0^r)(1 + \cos \beta) + A^i(2A^i - p_0^i)(1 - \cos \beta)] + \\ & \frac{\pi a \sin \beta}{4k_{11}^1} [C^r(2C^r - r_0^r)(1 + \cos \beta) + C^i(2C^i - r_0^i)(1 - \cos \beta)] + \\ & \frac{\pi a \sin^2 \beta}{4G_L^1} [A^r(2A^i - p_0^i) + A^i(2A^r - p_0^r)] + \\ & \frac{\pi a \sin^2 \beta}{4G_L^1} [C^r(2C^i - r_0^i) + C^i(2C^r - r_0^r)] + \end{aligned}$$

$$\begin{aligned}
& \frac{\pi a e_{155} \sin \beta}{4 G_L^1 k_{11}^1} [(A^r r_0^r - C^r p_0^r)(1 - \cos \beta) + (A^i r_0^i - C^i p_0^i)(1 + \cos \beta)] - \\
& \frac{\pi a e_{155} \sin^2 \beta}{4 G_L^1 k_{11}^1} (A^r r_0^i + A^i r_0^r - C^r p_0^i - C^i p_0^r), \quad (47) \\
g_2 = & \frac{\pi a \sin \beta}{4 G_L^1} [A^r (2A^r - p_0^r)(1 + \cos \beta) + A^i (2A^i - p_0^i)(1 - \cos \beta)] + \\
& \frac{\pi a \sin \beta}{4 k_{11}^1} [C^r (2C^r - r_0^r)(1 + \cos \beta) + C^i (2C^i - r_0^i)(1 - \cos \beta)] - \\
& \frac{\pi a \sin^2 \beta}{4 G_L^1} [A^r (2A^i - p_0^i) + A^i (2A^r - p_0^r)] - \\
& \frac{\pi a \sin^2 \beta}{4 G_L^1} [C^r (2C^i - r_0^i) + C^i (2C^r - r_0^r)] + \\
& \frac{\pi a e_{155} \sin \beta}{4 G_L^1 k_{11}^1} [(A^r r_0^r - C^r p_0^r)(1 + \cos \beta) + (A^i r_0^i - C^i p_0^i)(1 - \cos \beta)] - \\
& \frac{\pi a e_{155} \sin^2 \beta}{4 G_L^1 k_{11}^1} (A^r r_0^i + A^i r_0^r - C^r p_0^i - C^i p_0^r), \quad (48)
\end{aligned}$$

$g_1$  和  $g_2$  分别是裂尖  $t_1$  和  $t_2$  的能量释放率。

## 5 结 论

本文利用复变函数法获得在反平面剪切和面内电场共同作用下, 压电材料椭圆夹杂局部脱粘问题的一般解。利用解析延拓和复级数展开法, 可将原问题化为黎曼-希尔伯特问题来求解。由此可以确定夹杂和基体的复势。利用这个解, 可以求得夹杂界面裂纹的能量释放率。给出的一系列算例说明了这个解的有效性和正确性。

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# Analysis of a Partially Debonded Elliptic Inhomogeneity in Piezoelectric Materials

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**Abstract:** A generalized solution was obtained for the partially debonded elliptic inhomogeneity problem in piezoelectric materials under antiplane shear and inplane electric loading using the complex variable method. It was assumed that the interfacial debonding induced an electrically impermeable crack at the interface. The principle of conformal transformation and analytical continuation were employed to reduce the formulation into two Riemann-Hilbert problems. This enabled the determination of the complex potentials in the inhomogeneity and the matrix by means of series of expressions. The resulting solution was then used to obtain the electroelastic fields and the energy release rate involving the debonding at the inhomogeneity-matrix interface. The validity and versatility of the current general solution have been demonstrated through some specific examples such as the problems of perfectly bonded elliptic inhomogeneity, totally debonded elliptic inhomogeneity, partially debonded rigid and conducting elliptic inhomogeneity, and partially debonded circular inhomogeneity.

**Key words:** piezoelectric material; elliptic inhomogeneity; debonding; energy release rate