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定常的磁流体力学方程的非线性 Galerkin 混合元法*

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摘要: 提出了定常的磁流体动力学方程的一种非线性 Galerkin 混合元法, 并导出非线性 Galerkin 混合元解的存在性和误差估计•

关 键 词: 磁流体力学方程; 非线性 Galerkin 混合元法; 误差估计
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引 言

由于电磁场问题是复杂的耦合系统(参见文献[1]), 因此求其解析解几乎是不可能的, 有效的方法是求其数值近似解• 有限元法是一种数值近似解的有效方法, 但是其计算量也较大, 为了简化计算, 我们用非线性 Galerkin 混合有限元法处理该问题• 非线性 Galerkin 方法是一种求解带有耗散项的发展型偏微分方程的近似解的多重水平方法• 该方法是将未知量分裂成两项(或多项), 它们分别属于不同网格尺度的离散空间, 在计算过程中, 对于“小尺度”的分量引入简化逼近, 使该方法变得很便利• 这些方法原先主要是在 Fourier 谱离散化时提出的(参见文献[2]~[5]以及当中的文献)• 关于非线性 Galerkin 方法的有限元逼近是 Marion_Temam[6] 首先提出的• Ait Ou Ammi_Marion[7] 将非线性 Galerkin 方法应用于混合元法中, 用来处理非定常的 Navier_Stokes 问题• 李开泰等人[8] 又将此方法用于处理加罚的 Navier_Stokes 方程• 罗振东等人[9~10] 处理了热传导对流问题• 据我们所知, 目前尚未见用非线性 Galerkin 混合元法去处理磁流体力学问题•

本文的安排如下, 第 1 节简单回顾经典的混合有限元法; 第 2 节给出非线形 Galerkin 混合元法的存在性和收敛性; 最后第 3 节分析非线形 Galerkin 混合元解的误差•

1 经典的混合有限元法

本文用到的 Sobolev 空间及性质是熟知的(可参见文献[11])• 设 $\Omega \subset R^3$ 是适当光滑的有

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界凸区域, $W^{k,p}(\Omega)$ 和 $H^k(\Omega)$ 分别是标准 Sobolev 空间, 其范数分别是 $\|\boldsymbol{v}\|_{k,p,\Omega}$, $\|\boldsymbol{v}\|_{k,\Omega}$, 半范分别为 $|\boldsymbol{v}|_{k,p,\Omega}$, $|\boldsymbol{v}|_{k,\Omega}$ (见文献[11])· 记

$$H^1_{\boldsymbol{n}}(\Omega)^3 := \left\{\Psi \in H^1(\Omega)^3 : (\Psi, \boldsymbol{n})|_\Gamma = \boldsymbol{0}\right\}\cdot$$

$$\mathscr{W}(\Omega) := H^1(\Omega)^3 \times H^1(\Omega)^3,\quad \mathscr{W}_{0\boldsymbol{n}}(\Omega) := H^1_0(\Omega)^3 \times H^1_{\boldsymbol{n}}(\Omega)^3,$$

$$\mathscr{W}_{gq}(\Omega) := \left\{\boldsymbol{v} \in H^1(\Omega)^3 : \boldsymbol{v}|_\Gamma = \boldsymbol{g}\right\} \times \left\{\Psi \in H^1(\Omega)^3 : \Psi\cdot\boldsymbol{n}|_\Gamma = q\right\} := X \times Y,$$

其中 $\boldsymbol{n}$ 为 $\partial\Omega$ 上的单位外法向量, 其范数为

$$\|(\boldsymbol{v}, \Psi)\|_{\mathscr{W}} := \left(\|\boldsymbol{v}\|^2_{1,\Omega} + \|\Psi\|^2_{1,\Omega}\right)^{1/2}\cdot$$

定义

$$Z(\Omega) := \left\{\boldsymbol{v} \in H^1_0(\Omega)^3 : \nabla\cdot\boldsymbol{v} = 0\right\},\quad L^2_0(\Omega) := \left\{q \in L^2(\Omega) : \int_\Omega q = 0\right\}\cdot$$

记 $H^{-1}(\Omega)$ 为 $H^1_0(\Omega)$ 的对偶空间· $H^{-1}(\Omega)^3$ 的模定义为

$$\|\boldsymbol{f}\|_{-1,\Omega} := \sup_{\boldsymbol{v}\in H^1_0(\Omega)^3,\ \boldsymbol{v}\neq\boldsymbol{0}} \frac{(\boldsymbol{f}, \boldsymbol{v})_\Omega}{|\boldsymbol{v}|_{1,\Omega}}\cdot$$

对于边界 $(\Gamma = \partial\Omega)$ 我们用迹空间 $H^{1/2}(\Gamma) := \left\{\boldsymbol{v}|_\Gamma : \boldsymbol{v} \in H^1(\Omega)\right\}$, 其对偶空间记为 $H^{-1/2}(\Gamma)$, 模的定义可参见文献[12]· 另外本文使用的 C, $C_i\,(i = 0, 1, 2, \cdots)$, γ_1、γ_2 和 γ_3 均表示与剖分参数 h 无关的一般常数, 不同的地方出现可以不等·

考虑下面定常的磁流体力学方程:

问题(Ⅰ)　求 $\boldsymbol{u}$, $\boldsymbol{B}$, p 满足

$$\begin{cases} -\dfrac{1}{M^2}\Delta\boldsymbol{u} + \dfrac{1}{N}(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p - \dfrac{1}{Re_{\mathrm{m}}}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = \boldsymbol{f}, & \text{在 } \Omega \text{ 中}, \\ \dfrac{1}{Re_{\mathrm{m}}}\nabla\times(\nabla\times\boldsymbol{B}) - \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) = 0, & \text{在 } \Omega \text{ 中}, \\ \nabla\cdot\boldsymbol{u} = 0,\quad \nabla\cdot\boldsymbol{B} = 0, & \text{在 } \Omega \text{ 中}, \\ \boldsymbol{u} = \boldsymbol{g},\quad \boldsymbol{B}\cdot\boldsymbol{n} = q,\quad \dfrac{1}{Re_{\mathrm{m}}}[(\nabla\times\boldsymbol{B})\times\boldsymbol{n}] - [(\boldsymbol{u}\times\boldsymbol{B})\times\boldsymbol{n}] = \boldsymbol{k}, & \text{在 } \Gamma \text{ 上}, \end{cases} \tag{1}$$

其中 $\boldsymbol{u}$ 表示速度向量, $\boldsymbol{B}$ 表示磁场, p 表示压力, 所有变量都是无量纲的; 而 M、N 和 Re_{m} 分别是 Hartmann 数、交互参数和磁场 Reynolds 数, 及 $\boldsymbol{f} \in H^{-1}(\Omega)^3$ 均为已知· 为了满足方程(1)解的正则性和相容性条件, 我们需要限定边值满足

$$\boldsymbol{g} \in H^{1/2}(\Gamma)^3,\quad \int_\Gamma \boldsymbol{g}\cdot\boldsymbol{n} = 0, \tag{2}$$

$$q \in H^{1/2}(\Gamma),\quad \int_\Gamma q = 0, \tag{3}$$

$$\boldsymbol{k} \in H^{-1/2}(\Gamma)^3,\quad \boldsymbol{k}\cdot\boldsymbol{n} = 0,\quad \langle\boldsymbol{k}, \boldsymbol{1}\rangle_\Gamma = 0,\quad \langle\boldsymbol{k}, \nabla\phi\rangle_\Gamma = 0,\quad \forall\phi \in H^2(\Omega), \tag{4}$$

其中 $\boldsymbol{1}$ 为 $\partial\Omega$ 上的单位切向量· 于是, 问题(Ⅰ)的混合变分问题为:

问题(I^*)　求 $(\boldsymbol{u}, \boldsymbol{B}) \in \mathscr{W}_{gq}(\Omega)$, $p \in L^2_0(\Omega)$ 满足

$$\begin{cases} a((\boldsymbol{u}, \boldsymbol{B}), (\boldsymbol{u}, \boldsymbol{B}), (\boldsymbol{v}, \Psi)) + b((\boldsymbol{v}, \Psi), p) = F(\boldsymbol{v}, \Psi), \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall(\boldsymbol{v}, \Psi) \in \mathscr{W}_{0\boldsymbol{n}}(\Omega), \\ b((\boldsymbol{u}, \boldsymbol{B}), \chi) = 0, \qquad \forall\chi \in L^2_0(\Omega), \end{cases} \tag{5}$$

其中

$$a((\boldsymbol{u}, \boldsymbol{B}), (\boldsymbol{v}, \Psi), (\boldsymbol{w}, \Phi)) :=$$

$$a_0((\boldsymbol{v},\ \Psi),(\boldsymbol{w},\ \Phi)) + a_1((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{v},\ \Psi),(\boldsymbol{w},\ \Phi)),$$

$$a_0((\boldsymbol{v},\ \Psi),(\boldsymbol{w},\ \Phi)) := \frac{1}{M^2}\int_\Omega \nabla\boldsymbol{v} : \nabla\boldsymbol{w}\,\mathrm{d}x +$$

$$\frac{1}{Re_{\mathrm{m}}^2}\int_\Omega\left\{(\nabla\times\Psi)\cdot(\nabla\times\Phi) + (\nabla\cdot\Psi)(\nabla\cdot\Phi)\right\}\mathrm{d}x,$$

$$a_1((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{v},\ \Psi),(\boldsymbol{w},\ \Phi)) := \frac{1}{N}\int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{w}\,\mathrm{d}x -$$

$$\frac{1}{Re_{\mathrm{m}}}\int_\Omega[(\nabla\times\Psi)\times\boldsymbol{B}\cdot\boldsymbol{w} - (\nabla\times\Phi)\times\boldsymbol{B}\cdot\boldsymbol{v}]\,\mathrm{d}x,$$

$$b((\boldsymbol{v},\ \Psi),\ \chi) := -\int_\Omega(\nabla\cdot\boldsymbol{v})\,\chi\,\mathrm{d}x,\quad F(\boldsymbol{v},\ \Psi) := \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x + \frac{1}{Re_{\mathrm{m}}}\int_{\partial\Omega}\boldsymbol{k}\cdot\Psi\,\mathrm{d}s.$$

令

$$b_1(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) = \frac{1}{2}\int_\Omega\sum_{i,j=1}^{3}\left[u_i\frac{\partial v_j}{\partial x_i}w_j - u_i\frac{\partial w_j}{\partial x_i}v_j\right]\mathrm{d}x,\qquad \boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in H^1(\Omega)^3.$$

三线性型 $b_1(\cdot;\cdot,\cdot)$ 有下面性质(可参见文献[7~10] 等): 对于任意的 $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in H^1(\Omega)^3$,

$$\begin{cases} b_1(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) = -b_1(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v});\ b_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0; \\ |b_1(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w})| \leqslant C\|\boldsymbol{u}\|_0^{1/2}|\boldsymbol{u}|_1^{1/2}(|\boldsymbol{v}|_1^{1/2}\|\boldsymbol{v}\|_0^{1/2}|\boldsymbol{w}|_1 + \\ \qquad |\boldsymbol{v}|_1\|\boldsymbol{w}\|_0^{1/2}|\boldsymbol{w}|_1^{1/2}); \\ |b_1(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w})| \leqslant C|\boldsymbol{u}|_1(|\boldsymbol{v}|_1\|\boldsymbol{v}\|_0\|\boldsymbol{w}\|_0|\boldsymbol{w}|_1)^{1/2} + \\ \qquad C|\boldsymbol{v}|_1(\|\boldsymbol{u}\|_0|\boldsymbol{u}|_1\|\boldsymbol{w}\|_0|\boldsymbol{w}|_1)^{1/2}, \end{cases} \tag{6}$$

其中 C 是与 $\boldsymbol{u}$、$\boldsymbol{v}$、$\boldsymbol{w}$ 无关的常数.

设 $\mathcal{T}_h$ 为 Ω 的拟一致三角形剖分(可参见文献[13] 或[14]), $X^h\subset H^1(\Omega)^3$ 和 $S_h^0\subset L_0^2(\Omega)$ 为 Bernardi_Raugel[14] 的一阶元, $Y^h\subset H^1(\Omega)^3$ 为分块一次多项式. 定义下列空间:

$$X_0^h(\Omega) := X^h(\Omega)\cap H_0^1(\Omega)^3,\quad Y_n^h(\Omega) := Y^h(\Omega)\cap H_n^1(\Omega)^3,$$

$$\mathscr{W}^h(\Omega) := X^h(\Omega)\times Y^h(\Omega),\quad \mathscr{W}_{0\boldsymbol{n}}^h(\Omega) := X_0^h(\Omega)\times Y_n^h(\Omega).$$

并定义 X_0^h 的子空间 Z^h 如下:

$$Z^h(\Omega) := \left\{\boldsymbol{w}_h\in X_0^h : \int_\Omega(\nabla\cdot\boldsymbol{w}_h)\,\chi_h\,\mathrm{d}x = 0,\ \chi_h\in S_0^h\right\}. \tag{7}$$

由文献[14] 和插值理论[13] 知 X^h、Y^h、S_h^0 满足下面的逼近性质: 对于任意的 $(\boldsymbol{u},\boldsymbol{B})\in\left\{\boldsymbol{w}\in H^1(\Omega)^3 : \boldsymbol{w}|_\Gamma = \boldsymbol{g}\right\}\times\left\{\Psi\in H^1(\Omega)^3 : \Psi\cdot\boldsymbol{n}|_\Gamma = q\right\}$ 及 $\forall\chi\in L_0^2(\Omega)$, 分别有

$$\begin{cases} \inf\limits_{(\boldsymbol{u}_h,\boldsymbol{B}_h)\in\mathscr{W}_{\boldsymbol{g}_h q_h}} \|\nabla((\boldsymbol{u},\boldsymbol{B}) - (\boldsymbol{u}_h,\boldsymbol{B}_h))\|_0 \leqslant Ch|(\boldsymbol{u},\boldsymbol{B})|_2; \\ \inf\limits_{\chi_h\in S_0^h}\|\chi - \chi_h\|_0 \leqslant Ch|\chi|_1, \end{cases} \tag{8}$$

并满足所谓的 inf_sup 条件, 即

$$\inf_{\chi_h\in S_0^h,\ \chi_h\neq 0}\ \sup_{(\boldsymbol{v}_h,\Psi_h)\in\mathscr{W}_{0\boldsymbol{n}}^h,\ (\boldsymbol{v}_h,\Psi_h)\neq 0}\frac{b((\boldsymbol{v}_h,\ \Psi_h),\ \chi_h)}{\|(\boldsymbol{v}_h,\ \Psi_h)\|_{\mathscr{W}}\|\chi_h\|_0}\geqslant\beta, \tag{9}$$

其中 β 是一个与 h 无关的常数.

一般 $Z^h \not\subset Z(\Omega)$, 因此我们必须稍微修改一下非线形项 a_1 使得满足离散函数的反对称条件. 我们定义

$$a_1((\boldsymbol{u}_h,\boldsymbol{B}_h),(\boldsymbol{v}_h,\ \Psi_h),(\boldsymbol{w}_h,\ \Phi_h)) := \frac{1}{2}a_1((\boldsymbol{u}_h,\boldsymbol{B}_h),(\boldsymbol{v}_h,\ \Psi_h),(\boldsymbol{w}_h,\ \Phi_h)) -$$

$$\frac{1}{2}a_1((\boldsymbol{u}_h,\boldsymbol{B}_h),(\boldsymbol{w}_h,\Phi_h)(\boldsymbol{v}_h,\Psi_h))\cdot$$

显然，当 $\boldsymbol{u}\in Z(\Omega)$ 时，$a_1=a_1$. 另外，我们有

$$a_1((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{v},\Psi),(\boldsymbol{w},\Phi))=-a_1((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{w},\Phi),(\boldsymbol{v},\Psi)),\qquad 在\ \mathscr{W}(\Omega)^3\ 内\cdot$$

将 a 的形式进行一下修改，定义为：对于 $(\boldsymbol{u}_h,\boldsymbol{B}_h),(\boldsymbol{v}_h,\Psi_h),(\boldsymbol{w}_h,\Phi_h)\in(\mathscr{W}^h)^3$,

$$a((\boldsymbol{u}_h,\boldsymbol{B}_h),(\boldsymbol{v}_h,\Psi_h),(\boldsymbol{w}_h,\Phi_h)):=a_0((\boldsymbol{v}_h,\Psi_h),(\boldsymbol{w}_h,\Phi_h))+$$
$$a_1((\boldsymbol{u}_h,\boldsymbol{B}_h),(\boldsymbol{v}_h,\Psi_h),(\boldsymbol{w}_h,\Phi_h))\cdot$$

注　为了得到误差估计，我们必须附加边界的正则条件，即

$$\boldsymbol{g}\in H^{3/2+\varepsilon}(\Omega)^3,\ q\in H^{3/2+\varepsilon}(\Omega)^3\cdot \tag{10}$$

对于离散的弱格式，用 $\boldsymbol{g}_h\in X^h|_\Gamma$ 和 $q_h\in\left\{(\Psi_h\cdot\boldsymbol{n})|_\Gamma:\Psi_h\in Y^h\right\}$ 分别逼近 $\boldsymbol{g}$ 和 q（$\mathscr{W}_{\boldsymbol{g}_hq_h}$ 与 $\mathscr{W}_{\boldsymbol{g}q}$ 类似）. 那么，问题（Ⅰ*）的经典 Galerkin 混合元格式叙述为：

问题（Ⅰ$_h$）　求 $(\boldsymbol{u}_h,\boldsymbol{B}_h)\in\mathscr{W}_{\boldsymbol{g}_hq_h}(\Omega)$, $p_h\in S_0^h(\Omega)$ 满足

$$\begin{cases}a((\boldsymbol{u}_h,\boldsymbol{B}_h),(\boldsymbol{u}_h,\boldsymbol{B}_h),(\boldsymbol{v}_h,\Psi_h))+b((\boldsymbol{v}_h,\Psi_h),p_h)=\\ \qquad F_h(\boldsymbol{v}_h,\Psi_h),\qquad \forall(\boldsymbol{v}_h,\Psi_h)\in\mathscr{W}_{0\boldsymbol{n}}^h(\Omega),\\ b((\boldsymbol{u}_h,\boldsymbol{B}_h),\chi_h)=0,\qquad \forall\chi\in S_0^h(\Omega)\cdot\end{cases} \tag{11}$$

下面的结果是经典的（可参见文献[12]，[15]）.

定理 1.1　如果 $\boldsymbol{f}\in H^{-1}(\Omega)^3$, $\boldsymbol{k}$ 满足(4)式，$q_h\in\left\{(\Psi_h\cdot\boldsymbol{n})|_{\partial\Omega}:\Psi_h\in Y^h(\Omega)\right\}$ 和 $\boldsymbol{g}_h\in X^h|_{\partial\Omega}$ 满足 $\|\boldsymbol{g}_h\|_{1/2,\partial\Omega}<\dfrac{1}{\gamma_1\gamma_3}\min\left\{\dfrac{Nk_1}{M^2},\dfrac{k_2}{\sqrt{2}Re_m}\right\}$，并且式(9)成立，又如果

$$\frac{1}{\alpha_h}\left[\|\boldsymbol{f}\|_{-1}+\frac{1}{Re_m}\|\boldsymbol{k}\|_{-1/2,\partial\Omega}+\frac{\gamma_1\gamma_2\gamma_3}{\sqrt{2}Re_m}\|q_h\|^2_{1/2,\partial\Omega}+\right.$$
$$\left.\gamma_1\gamma_3\left(\frac{\gamma_1}{N}+\frac{\gamma_2}{\sqrt{2}Re_m}\right)\|\boldsymbol{g}_h\|^2_{1/2,\partial\Omega}\right]+\gamma_2\left(1+\frac{5}{\alpha_hRe_m^2}\right)\|q_h\|_{1/2,\partial\Omega}+$$
$$\gamma_1\left(1+\frac{1}{\alpha_hM^2}\right)\|\boldsymbol{g}_h\|_{1/2,\partial\Omega}<\frac{1}{\sqrt{2}\gamma_3}\min\left\{\frac{k_1}{M^2},\frac{k_2}{Re_m^2}\right\}\Big\backslash\max\left\{\frac{1}{N},\frac{\sqrt{2}}{Re_m}\right\}, \tag{12}$$

那么存在一个常数 $h_0>0$ 使得对于所有的 $h\leqslant h_0$，问题（Ⅰ$_h$）存在唯一的解 $(\boldsymbol{u}_h,\boldsymbol{B}_h)\in\mathscr{W}^h(\Omega)$, $p_h\in S_0^h(\Omega)$，满足

$$\begin{cases}\|(\boldsymbol{u}_h,\boldsymbol{B}_h)\|_{\mathscr{W}}<\dfrac{1}{\sqrt{2}\gamma_3}\min\left\{\dfrac{k_1}{M^2},\dfrac{k_2}{Re_m^2}\right\}\Big\backslash\max\left\{\dfrac{1}{N},\dfrac{\sqrt{2}}{Re_m}\right\},\\ \|(\boldsymbol{u},\boldsymbol{B})-(\boldsymbol{u}_h,\boldsymbol{B}_h)\|_{\mathscr{W}}+\|p-p_h\|_0\leqslant Ch,\end{cases} \tag{13}$$

其中 $(\boldsymbol{u},\boldsymbol{B})\in\mathscr{W}_{\boldsymbol{g}q}(\Omega)$, $p\in L_0^2(\Omega)$ 是问题（Ⅰ）的精确解，C 是与 $\|(\boldsymbol{u},\boldsymbol{B})\|_{\mathscr{W}}$ 和 $\|p\|_1$ 有关的常数，γ_1、γ_2、γ_3、α_h 是与 h 无关的常数.

2　非线性 Galerkin 混合元解的存在性

设 h 和 H $(H\gg h>0)$ 是两个趋于 0 的参数，考虑有限元空间 X^h, X_H $(X_H\subset X^h)$, X_h^H $(X_h^H\subset X^h)$, Y^h, Y_H $(Y_H\subset Y^h)$, Y_h^H $(Y_h^H\subset Y^h)$ 和 S_0^h. 令

$$\begin{cases}X^h:=X_H+X_h^H,\ X_H:=R_HX^h,\ X_h^H:=(I-R_H)X^h;\\ Y^h:=Y_H+Y_h^H,\ Y_H:=R_HY^h,\ Y_h^H:=(I-R_H)Y^h,\end{cases} \tag{14}$$

其中 R_H 是从 X(或 Y) 到 X_H(或 Y_H) 的 Ritz 正交算子，即对所有的 $\boldsymbol{v} \in X$(或 Y) 满足

$$(\nabla(\boldsymbol{v} - R_H\boldsymbol{v}), \nabla \boldsymbol{v}_H) = 0, \qquad \forall \boldsymbol{v}_H \in X_H(\text{或 } Y_H), \tag{15}$$

X^h、Y^h 和 S_h^0 是对应于网格尺寸为 h 的细网格有限元空间，X_H 和 Y_H 是对应于网格尺寸为 H 的粗网格有限元空间•

下面关于空间 $X_H(Y_H)$、$X_h^H(Y_h^H)$ 和算子 R_H 的性质将经常被使用，这些性质可由(15)式及对偶原理导出(可参见文献[13]和文献[16])•

引理 2.1 空间 $X_H(Y_H)$、$X_h^H(Y_h^H)$ 和算子 R_H 满足 $\forall \boldsymbol{v} \in H^l(\Omega)^3 \cap X$(或 Y)，

$$|\boldsymbol{v} - R_H\boldsymbol{v}|_s \leqslant CH^{l-s}|\boldsymbol{v}|_l, \qquad s = 0, 1, s \leqslant l \leqslant m+1; \tag{16}$$

$$\begin{cases} |R_H\boldsymbol{v}|_1 \leqslant |\boldsymbol{v}|_1, & \boldsymbol{v} \in X(\text{或 } Y); \\ \|\tau\|_0 \leqslant CH|\tau|_1, & \forall \tau \in X_h^H(\text{或 } Y_h^H); \end{cases} \tag{17}$$

$$(\nabla\phi, \nabla\tau) = 0, \ |\phi|_1^2 + |\tau|_1^2 = |\phi + \tau|_1^2,$$
$$\forall \phi \in X_H(\text{或 } Y_H), \ \tau \in X_h^H(\text{或 } Y_h^H)• \tag{18}$$

现在引进非线性 Galerkin 混合有限元格式:

问题(Ⅰ ^{h}) 求 $\boldsymbol{u}^h := \boldsymbol{u}^H + \boldsymbol{w}^h$, $\boldsymbol{B}^h := \boldsymbol{B}^H + \boldsymbol{A}^h$ 和 $p^h \in S_0^h$，其中 $\boldsymbol{u}^h \in X^h$, $\boldsymbol{u}^H \in X_H$, $\boldsymbol{w}^h \in X_h^H$; $\boldsymbol{B}^h \in Y^h$, $\boldsymbol{B}^H \in Y_H$, $\boldsymbol{A}^h \in Y_h^H$，满足

$$\begin{aligned} & a_0((\boldsymbol{u}^H, \boldsymbol{B}^H), (\phi, \tau)) + a_1((\boldsymbol{u}^H, \boldsymbol{B}^H), (\boldsymbol{u}^H, \boldsymbol{B}^H), (\phi, \tau)) + \\ & \quad a_1((\boldsymbol{u}^H, \boldsymbol{B}^H), (\boldsymbol{w}^h, \boldsymbol{A}^h), (\phi, \tau)) + a_1((\boldsymbol{w}^h, \boldsymbol{A}^h), (\boldsymbol{u}^H, \boldsymbol{B}^H), (\phi, \tau)) + \\ & \quad b((\phi, \tau), p^h) + b((\boldsymbol{u}^H, \boldsymbol{B}^H), \chi) + \frac{1}{Re_{\mathrm{m}}^2}\int_\Omega (\nabla \times \boldsymbol{A}^h)\cdot(\nabla \times \tau)\,\mathrm{d}x + \\ & \quad \frac{1}{Re_{\mathrm{m}}}\int_\Omega (\nabla \times \tau) \times \boldsymbol{A}^h \cdot \boldsymbol{w}^h\,\mathrm{d}x = F(\phi, \tau), \\ & \qquad\qquad \forall (\phi, \tau) \in X_H \times Y_H, \ \chi \in S_0^h, \end{aligned} \tag{19}$$

$$\begin{aligned} & a_0((\boldsymbol{w}^h, \boldsymbol{A}^h), (\mu, \sigma)) + a_1((\boldsymbol{u}^H, \boldsymbol{B}^H), (\boldsymbol{u}^H, \boldsymbol{B}^H), (\mu, \sigma)) + \\ & \quad b((\mu, \sigma), p^h) + b((\boldsymbol{w}^h, \boldsymbol{A}^h), \chi) + \frac{1}{Re_{\mathrm{m}}^2}\int_\Omega (\nabla \times \boldsymbol{B}^H)\cdot(\nabla \times \sigma)\,\mathrm{d}x - \\ & \quad \frac{1}{Re_{\mathrm{m}}}\int_\Omega [(\nabla \times \boldsymbol{A}^h) \times \boldsymbol{B}^H \cdot \mu - (\nabla \times \sigma) \times \boldsymbol{B}^H \cdot \boldsymbol{w}^h]\,\mathrm{d}x - \\ & \quad \frac{1}{Re_{\mathrm{m}}}\int_\Omega (\nabla \times \boldsymbol{B}^H) \times \boldsymbol{A}^h \cdot \mu\,\mathrm{d}x = F(\mu, \sigma), \\ & \qquad\qquad \forall (\mu, \sigma) \in X_h^H \times Y_h^H, \ \chi \in S_0^h• \end{aligned} \tag{20}$$

从文献[12]中得知下面的结果：对于任意的 $\varepsilon > 0$ 和足够小的 h 有

$$\boldsymbol{u}_0^h \in X^h(\Omega), \ \boldsymbol{u}_0^h|_{\partial\Omega} = \boldsymbol{g}^h, \ \|\boldsymbol{u}_0^h\|_1 \leqslant \gamma_1\|\boldsymbol{g}^h\|_{1/2,\partial\Omega}, \tag{21}$$

$$\boldsymbol{B}_0^h \in Y^h(\Omega), (\boldsymbol{B}_0^h\cdot\boldsymbol{n})|_{\partial\Omega} = q^h, \ \|\boldsymbol{B}_0^h\|_1 \leqslant \gamma_2\|q^h\|_{1/2,\partial\Omega}, \tag{22}$$

$$\|\nabla \times \boldsymbol{B}_0^h\|_0 \leqslant \varepsilon, \ \|\nabla\cdot\boldsymbol{B}_0^h\|_0 \leqslant \varepsilon•$$

令 $\boldsymbol{u}^h := \boldsymbol{u}_0^h + \hat{\boldsymbol{u}}^h$, $\boldsymbol{B}^h := \boldsymbol{B}_0^h + \boldsymbol{B}^h$, $\hat{\boldsymbol{u}}^h \in X_0^h(\Omega)$, $\boldsymbol{B}^h \in Y_n^h(\Omega)$，则

$$\inf_{\chi_h \in S_0^h, \chi_h \neq 0} \ \sup_{(\hat{\boldsymbol{v}}_h, \Psi_h) \in \mathscr{W}_{0\boldsymbol{n}}^h(\Omega), (\hat{\boldsymbol{v}}_h, \Psi_h) \neq 0} \frac{b((\hat{\boldsymbol{v}}_h, \Psi_h), \chi_h)}{\|(\hat{\boldsymbol{v}}_h, \Psi_h)\|_{\mathscr{W}}\|\chi_h\|_{0,\Omega}} \geqslant \beta, \tag{23}$$

其中 $\beta > 0$ 是一个与 h 无关的常数• 从而问题(Ⅰ ^{h}) 变为

问题(Ⅰ ^{h*}) 求 $\hat{\boldsymbol{u}}^h := \boldsymbol{u}^H + \boldsymbol{w}^h$, $\boldsymbol{B}^h := \boldsymbol{B}^H + \boldsymbol{A}^h$ 和 $p^h \in S_0^h$，其中 $\hat{\boldsymbol{u}}^h \in X_0^h$, $\boldsymbol{u}^H \in X_H$, $\boldsymbol{w}^h \in X_h^H$; $\boldsymbol{B}^h \in Y_n^h$, $\boldsymbol{B}^H \in Y_H$, $\boldsymbol{A}^h \in Y_h^H$，满足

$$
\begin{aligned}
&a_0((\boldsymbol{u}^H, \boldsymbol{B}^H), (\phi, \tau)) + a_1((\boldsymbol{u}^H, \boldsymbol{B}^H), (\boldsymbol{u}^H, \boldsymbol{B}^H), (\phi, \tau)) + \\
&\quad a_1((\boldsymbol{u}^H, \boldsymbol{B}^H), (\boldsymbol{w}^h, \boldsymbol{A}^h), (\phi, \tau)) + a_1((\boldsymbol{w}^h, \boldsymbol{A}^h), (\boldsymbol{u}^H, \boldsymbol{B}^H), (\phi, \tau)) + \\
&\quad a_1((\boldsymbol{u}^H, \boldsymbol{B}^H), (\boldsymbol{u}_0^h, \boldsymbol{B}_0^h), (\phi, \tau)) + a_1((\boldsymbol{w}^h, \boldsymbol{A}^h), (\boldsymbol{u}_0^h, \boldsymbol{B}_0^h), (\phi, \tau)) + \\
&\quad a_1((\boldsymbol{u}_0^h, \boldsymbol{B}_0^h), (\boldsymbol{u}^H, \boldsymbol{B}^H), (\phi, \tau)) + a_1((\boldsymbol{u}_0^h, \boldsymbol{B}_0^h), (\boldsymbol{w}^h, \boldsymbol{A}^h), (\phi, \tau)) + \\
&\quad b((\phi, \tau), p^h) + b((\boldsymbol{u}^H, \boldsymbol{B}^H), \chi) + \frac{1}{Re_{\mathrm{m}}^2}\int_\Omega (\nabla \times \boldsymbol{A}^h)\cdot(\nabla \times \tau)\,\mathrm{d}x + \\
&\quad \frac{1}{Re_{\mathrm{m}}}\int_\Omega (\nabla \times \tau) \times \boldsymbol{A}^h \cdot \boldsymbol{w}^h\,\mathrm{d}x = \tilde{F}_h(\phi, \tau), \\
&\qquad\qquad\qquad \forall (\phi, \tau) \in X_H \times Y_H,\ \chi \in S_0^h,
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
&a_0((\boldsymbol{w}^h, \boldsymbol{A}^h), (\mu, \sigma)) + a_1((\boldsymbol{u}^H, \boldsymbol{B}^H), (\boldsymbol{u}^H, \boldsymbol{B}^H), (\mu, \sigma)) + \\
&\quad a_1((\boldsymbol{u}^H, \boldsymbol{B}^H), (\boldsymbol{u}_0^h, \boldsymbol{B}_0^h), (\mu, \sigma)) + a_1((\boldsymbol{u}_0^h, \boldsymbol{B}_0^h), (\boldsymbol{u}^H, \boldsymbol{B}^H), (\mu, \sigma)) + \\
&\quad b((\mu, \sigma), p^h) + b((\boldsymbol{w}^h, \boldsymbol{A}^h), \chi) + \frac{1}{Re_{\mathrm{m}}^2}\int_\Omega (\nabla \times \boldsymbol{B}^H)\cdot(\nabla \times \sigma)\,\mathrm{d}x - \\
&\quad \frac{1}{Re_{\mathrm{m}}}\int_\Omega [(\nabla \times \boldsymbol{A}^h) \times \boldsymbol{B}^H \cdot \mu - (\nabla \times \sigma) \times \boldsymbol{B}^H \cdot \boldsymbol{w}^h]\,\mathrm{d}x - \\
&\quad \frac{1}{Re_{\mathrm{m}}}\int_\Omega (\nabla \times \boldsymbol{B}^H) \times \boldsymbol{A}^h \cdot \mu\,\mathrm{d}x - \frac{1}{Re_{\mathrm{m}}}\int_\Omega (\nabla \times \boldsymbol{B}_0^h) \times \boldsymbol{A}^h \cdot \mu\,\mathrm{d}x - \\
&\quad \frac{1}{Re_{\mathrm{m}}}\int_\Omega (\nabla \times \boldsymbol{A}^h) \times \boldsymbol{B}_0^h \cdot \mu\,\mathrm{d}x + \frac{1}{Re_{\mathrm{m}}}\int_\Omega (\nabla \times \sigma) \times \boldsymbol{B}_0^h \cdot \boldsymbol{w}^h\,\mathrm{d}x = \\
&\quad \tilde{F}_h(\mu, \sigma), \qquad \forall (\mu, \sigma) \in X_h^H \times Y_h^H,\ \chi \in S_0^h,
\end{aligned} \tag{25}
$$

其中　$\tilde{F}_h(\mu, \sigma) = F_h(\mu, \sigma) + a((\boldsymbol{u}_0^h, \boldsymbol{B}_0^h), (\boldsymbol{u}_0^h, \boldsymbol{B}_0^h), (\mu, \sigma))$•

与文献[18]中的定理2.2同理可证得下一定理•

定理2.2　在定理1.1的条件下,如果

$$
\frac{\beta_1\gamma_1}{N}\|\boldsymbol{g}^h\|_{1/2,\partial\Omega} + \frac{\beta_2\gamma_2}{Re_{\mathrm{m}}}\|q^h\|_{1/2,\partial\Omega} < \min\left\{\frac{k_1}{M^2}, \frac{k_2}{Re_{\mathrm{m}}^2}\right\},
$$

则存在一个常数H^* 使得对于任意的$H \leqslant H^*$ 问题(I^h)都存在唯一的解$(\boldsymbol{u}^H, \boldsymbol{B}^H) := ((\boldsymbol{u}^H + \boldsymbol{w}^h), (\boldsymbol{B}^H + \boldsymbol{A}^h)) \in \mathscr{W}^h(\Omega), p^h \in S_0^h$满足

$$
\|(\boldsymbol{u}^h, \boldsymbol{B}^h)\|_{\mathscr{W}^h} < \frac{1}{\sqrt{2}\gamma_3}\min\left\{\frac{k_1}{M^2}, \frac{k_2}{Re_{\mathrm{m}}^2}\right\}\Big/\max\left\{\frac{1}{N}, \frac{\sqrt{2}}{Re_{\mathrm{m}}}\right\} := R• \tag{26}
$$

3　非线性Galerkin混合元解的误差估计

本节的目的是导出问题(I^h)的非线性Galerkin混合元解关于参数H和$h(h \ll H)$的误差估计•

定理3.1　在定理2.2的条件下,如果$\boldsymbol{f} \in L^2(\Omega)$而且(5)式的精确解为$\boldsymbol{u} = ((\boldsymbol{u}, \boldsymbol{B}), p) \in \mathscr{W}_{gq}(\Omega)$,那么存在一个正常数$h^*$使得对于$h \ll H \leqslant h^*$都有

$$
\|(\boldsymbol{u}_h, \boldsymbol{B}_h) - (\boldsymbol{u}^h, \boldsymbol{B}^h)\|_{\mathscr{W}} + \|p_h - p^h\|_0 \leqslant CH^2, \tag{27}
$$

其中$((\boldsymbol{u}_h, \boldsymbol{B}_h), p_h)$是问题$(\mathrm{I}_h)$的解,$((\boldsymbol{u}^h, \boldsymbol{B}^h), p^h)$是问题$(\mathrm{I}^h)$的解•

证明　定理3.1的证明将分为下面两步:

(a) 预备论述

对于问题(I_h)的解的$(\boldsymbol{u}_h, \boldsymbol{B}_h)$令

$$\begin{cases} \boldsymbol{u}_h = \boldsymbol{u}_H + \boldsymbol{w}_h,\ \boldsymbol{u}_H = R_H\boldsymbol{u}_h,\ \boldsymbol{w}_h = (I - R_H)\boldsymbol{u}_h, \\ \boldsymbol{B}_h = \boldsymbol{B}_H + \boldsymbol{A}_h,\ \boldsymbol{B}_H = R_H\boldsymbol{B}_h,\ \boldsymbol{A}_h = (I - R_H)\boldsymbol{B}_h, \end{cases} \tag{28}$$

并记

$$\boldsymbol{e} = \boldsymbol{u}_H - \boldsymbol{u}^H,\ \boldsymbol{E} = \boldsymbol{w}_h - \boldsymbol{w}^h,\ \boldsymbol{d} = \boldsymbol{B}_H - \boldsymbol{B}^H,\ \boldsymbol{D} = \boldsymbol{A}_h - \boldsymbol{A}^h,\ \delta = p_h - p^h. \tag{29}$$

引理 3.2 在定理 3.1 的条件下, 问题 (I_h) 的解 $(\boldsymbol{u}^h, \boldsymbol{B}^h)$ 使得 $\boldsymbol{w}_h = (I - R_H)\boldsymbol{u}_h$, $\boldsymbol{A}_h = (I - R_H)\boldsymbol{B}_h$ 满足

$$\|(\boldsymbol{w}_h, \boldsymbol{A}_h)\|_0 + H|(\boldsymbol{w}_h, \boldsymbol{A}_h)|_1 \leqslant CH^2. \tag{30}$$

证明 $\boldsymbol{w}_h$、$\boldsymbol{A}_h$ 可分别分解为

$$\begin{cases} \boldsymbol{w}_h = (I - R_H)\boldsymbol{u}_h = (I - R_H)\boldsymbol{u} + (I - R_H)(\boldsymbol{u}_h - \boldsymbol{u}), \\ \boldsymbol{A}_h = (I - R_H)\boldsymbol{B}_h = (I - R_H)\boldsymbol{B} + (I - R_H)(\boldsymbol{B}_h - \boldsymbol{B}). \end{cases} \tag{31}$$

由定理 1.1 和引理 2.1 可得

$$\begin{aligned} &\|(\boldsymbol{w}_h, \boldsymbol{A}_h)\|_0 + H|(\boldsymbol{w}_h, \boldsymbol{A}_h)|_1 \leqslant CH|(\boldsymbol{w}_h, \boldsymbol{A}_h)|_1 \leqslant \\ &\quad CH(|(\boldsymbol{u}, \boldsymbol{B}) - R_H(\boldsymbol{u}, \boldsymbol{B})|_1 + |(\boldsymbol{u}, \boldsymbol{B}) - (\boldsymbol{u}_h, \boldsymbol{B}_h)|_1) \leqslant \\ &\quad C(H^2 + Hh) \leqslant CH^2. \end{aligned}$$

定义离散的 Laplace 算子 $\mathscr{A}_h \in L(X_H, X_H)$ 为

$$(\mathscr{A}_h\boldsymbol{v}, \boldsymbol{w}) = \frac{1}{M^2}\int_\Omega \nabla\boldsymbol{v}\cdot\nabla\boldsymbol{w}\,\mathrm{d}x, \qquad \forall\, \boldsymbol{v}, \boldsymbol{w} \in X_H. \tag{32}$$

那么, 下面估计[7],[18~20]成立: 对于任意的 $\boldsymbol{u}_h \in Z_h$ 和任意的 $\boldsymbol{v}_h, \boldsymbol{w}_h \in X_H$, 有

$$\begin{aligned} &|b_1(\boldsymbol{v}_h; \boldsymbol{u}_h, \boldsymbol{w}_h)| + |b_1(\boldsymbol{u}_h; \boldsymbol{v}_h, \boldsymbol{w}_h)| \leqslant \\ &\quad |\boldsymbol{u}_h|_1^{1/2}\|\mathscr{A}_h\boldsymbol{u}_h\|_0^{1/2}\|\boldsymbol{v}_h\|_0|\boldsymbol{w}_h|_1. \end{aligned} \tag{33}$$

而且对问题 (I_h) 的解 $\boldsymbol{u}_h$ 有

$$\|\mathscr{A}_h\boldsymbol{u}_h\|_0 \leqslant C. \tag{34}$$

问题 (I^h) 的线性化问题是

$$\begin{aligned} &a_0((\boldsymbol{u}^H, \boldsymbol{B}^H), (\phi, \tau)) + a_1((\boldsymbol{u}_H, \boldsymbol{B}_H), (\boldsymbol{u}^H, \boldsymbol{B}^H), (\phi, \tau)) + \\ &\quad a_1((\boldsymbol{u}_H, \boldsymbol{B}_H), (\boldsymbol{w}^h, \boldsymbol{A}^h), (\phi, \tau)) + a_1((\boldsymbol{w}_h, \boldsymbol{A}_h), (\boldsymbol{u}^H, \boldsymbol{B}^H), (\phi, \tau)) + \\ &\quad b((\phi, \tau), p^h) + b((\boldsymbol{u}^H, \boldsymbol{B}^H), \chi) + \frac{1}{Re_{\mathrm{m}}^2}\int_\Omega(\nabla\times\boldsymbol{A}^h)\cdot(\nabla\times\tau)\,\mathrm{d}x + \\ &\quad \frac{1}{Re_{\mathrm{m}}}\int_\Omega(\nabla\times\tau)\times\boldsymbol{A}^h\cdot\boldsymbol{w}^h\,\mathrm{d}x = \\ &\quad F(\phi, \tau), \qquad \forall(\phi, \tau) \in X_H\times Y_H,\ \chi \in S_0^h. \end{aligned} \tag{35}$$

$$\begin{aligned} &a_0((\boldsymbol{w}^h, \boldsymbol{A}^h), (\mu, \sigma)) + a_1((\boldsymbol{u}_H, \boldsymbol{B}_H), (\boldsymbol{u}^H, \boldsymbol{B}^H), (\mu, \sigma)) + \\ &\quad b((\mu, \sigma), p^h) + b((\boldsymbol{w}^h, \boldsymbol{A}^h), \chi) + \frac{1}{Re_{\mathrm{m}}^2}\int_\Omega(\nabla\times\boldsymbol{B}^H)\cdot(\nabla\times\sigma)\,\mathrm{d}x - \\ &\quad \frac{1}{Re_{\mathrm{m}}}\int_\Omega[(\nabla\times\boldsymbol{A}^h)\times\boldsymbol{B}^H\cdot\mu - (\nabla\times\sigma)\times\boldsymbol{B}^H\cdot\boldsymbol{w}^h]\,\mathrm{d}x - \\ &\quad \frac{1}{Re_{\mathrm{m}}}\int_\Omega(\nabla\times\boldsymbol{B}^H)\times\boldsymbol{A}^h\cdot\mu\,\mathrm{d}x + \frac{1}{Re_{\mathrm{m}}}\int_\Omega(\nabla\times\sigma)\times\boldsymbol{B}_0^h\cdot\boldsymbol{w}^h\,\mathrm{d}x = \\ &\quad F(\mu, \sigma), \qquad \forall(\mu, \sigma) \in X_h^H\times Y_h^H,\ \chi \in S_0^h. \end{aligned} \tag{36}$$

为了方便我们下面将 a_1 简记为 a_1.

(b) 定理 3.1 的证明

利用(28) 式和(19) 式, 问题 (I^h) 可写为

$$
\begin{aligned}
&a_0((\boldsymbol{u}_H, \boldsymbol{B}_H), (\phi, \tau)) + a_1((\boldsymbol{u}_h, \boldsymbol{B}_h), (\boldsymbol{u}_h, \boldsymbol{B}_h), (\phi + \mu, \tau + \sigma)) + \\
&\quad a_0((\boldsymbol{w}_h, \boldsymbol{A}_h), (\mu, \sigma)) + b((\phi + \mu, \tau + \sigma), p_h) - b((\boldsymbol{u}_H + \boldsymbol{w}_h, \boldsymbol{B}_H + \boldsymbol{A}_h), \chi) + \\
&\quad \frac{1}{Re_{\mathrm{m}}^2}\int_\Omega (\nabla \times \boldsymbol{B}_H)\cdot(\nabla \times \sigma)\mathrm{d}x + \frac{1}{Re_{\mathrm{m}}^2}\int_\Omega (\nabla \times \boldsymbol{A}_h)\cdot(\nabla \times \tau)\mathrm{d}x = \\
&\quad F(\phi + \mu, \tau + \sigma), \qquad \forall \phi \in X_H, \mu \in X_h^H, \tau \in Y_H, \sigma \in Y_h^H, \chi \in S_0^h. \qquad (37)
\end{aligned}
$$

在(37)式中取 $\phi = \boldsymbol{e} = \boldsymbol{u}_H - \boldsymbol{u}^H$, $\mu = \boldsymbol{E} = \boldsymbol{w}_h - \boldsymbol{w}^h$, $\tau = \boldsymbol{d} = \boldsymbol{B}_H - \boldsymbol{B}^H$, $\sigma = \boldsymbol{A}_h - \boldsymbol{A}^h$, $\chi = \delta = p_h - p^h$. 并与(35)、(36)式相减得

$$
\begin{aligned}
&a_0((\boldsymbol{e} + \boldsymbol{E}, \boldsymbol{d} + \boldsymbol{D}), (\boldsymbol{e} + \boldsymbol{E}, \boldsymbol{d} + \boldsymbol{D})) = a_1((\boldsymbol{u}^h, \boldsymbol{B}^h), (\boldsymbol{u}^h, \boldsymbol{B}^h), (\boldsymbol{e}, \boldsymbol{d})) - \\
&\quad a_1((\boldsymbol{w}^h, \boldsymbol{A}^h), (\boldsymbol{w}^h, \boldsymbol{A}^h), (\boldsymbol{e}, \boldsymbol{d})) + a_1((\boldsymbol{u}^H, \boldsymbol{B}^H), (\boldsymbol{u}^H, \boldsymbol{B}^H), (\boldsymbol{E}, \boldsymbol{D})) - \\
&\quad a_1((\boldsymbol{u}_h, \boldsymbol{B}_h), (\boldsymbol{u}_h, \boldsymbol{B}_h), (\boldsymbol{e}, \boldsymbol{d})) - a_1((\boldsymbol{u}_h, \boldsymbol{B}_h), (\boldsymbol{u}_h, \boldsymbol{B}_h), (\boldsymbol{E}, \boldsymbol{D})) + \\
&\quad \frac{1}{Re_{\mathrm{m}}}\int_\Omega (\nabla \times \boldsymbol{d}) \times \boldsymbol{A}^h\cdot \boldsymbol{w}^h\mathrm{d}x - \frac{1}{Re_{\mathrm{m}}}\int_\Omega (\nabla \times \boldsymbol{B}^H) \times \boldsymbol{A}^h\cdot \boldsymbol{E}\mathrm{d}x - \\
&\quad \frac{1}{Re_{\mathrm{m}}}\int_\Omega (\nabla \times \boldsymbol{A}^h) \times \boldsymbol{B}^H\cdot \boldsymbol{E}\mathrm{d}x + \frac{1}{Re_{\mathrm{m}}}\int_\Omega (\nabla \times \boldsymbol{D}) \times \boldsymbol{B}^H\cdot \boldsymbol{w}^h\mathrm{d}x = \\
&\quad a_1((\boldsymbol{u}_h, \boldsymbol{B}_h), (\boldsymbol{w}_h, \boldsymbol{A}_h), (\boldsymbol{E}, \boldsymbol{D})) + a_1((\boldsymbol{e} + \boldsymbol{E}, \boldsymbol{d} + \boldsymbol{D}), (\boldsymbol{u}_h, \boldsymbol{B}_h), (\boldsymbol{e}, \boldsymbol{d})) + \\
&\quad a_1((\boldsymbol{e} + \boldsymbol{w}_h, \boldsymbol{d} + \boldsymbol{A}_h), (\boldsymbol{u}_h, \boldsymbol{B}_h), (\boldsymbol{E}, \boldsymbol{D})) - a_1((\boldsymbol{e}, \boldsymbol{d}), (\boldsymbol{w}_h, \boldsymbol{A}_h), (\boldsymbol{E}, \boldsymbol{D})) - \\
&\quad a_1((\boldsymbol{E}, \boldsymbol{D}), (\boldsymbol{w}_h, \boldsymbol{A}_h), (\boldsymbol{e}, \boldsymbol{d})) - a_1((\boldsymbol{w}_h, \boldsymbol{A}_h), (\boldsymbol{w}_h, \boldsymbol{A}_h), (\boldsymbol{E}, \boldsymbol{D})) + \\
&\quad a_1((\boldsymbol{w}_h, \boldsymbol{A}_h), (\boldsymbol{w}_h, \boldsymbol{A}_h), (\boldsymbol{e}, \boldsymbol{d})) + \frac{1}{Re_{\mathrm{m}}}\int_\Omega (\nabla \times \boldsymbol{d}) \times \boldsymbol{A}^h\cdot \boldsymbol{w}^h\mathrm{d}x - \\
&\quad \frac{1}{Re_{\mathrm{m}}}\int_\Omega (\nabla \times \boldsymbol{B}^H) \times \boldsymbol{A}^h\cdot \boldsymbol{E}\mathrm{d}x - \frac{1}{Re_{\mathrm{m}}}\int_\Omega (\nabla \times \boldsymbol{A}^h) \times \boldsymbol{B}^H\cdot \boldsymbol{E}\mathrm{d}x + \\
&\quad \frac{1}{Re_{\mathrm{m}}}\int_\Omega (\nabla \times \boldsymbol{D}) \times \boldsymbol{B}^H\cdot \boldsymbol{w}^h\mathrm{d}x. \qquad (38)
\end{aligned}
$$

由 $a_0(\cdot,\cdot)$ 的正定性(参见文献[12])知,(38)式的左端项有

$$
a_0((\boldsymbol{e} + \boldsymbol{E}, \boldsymbol{d} + \boldsymbol{D})(\boldsymbol{e} + \boldsymbol{E}, \boldsymbol{d} + \boldsymbol{D})) \geqslant C_0(\|(\boldsymbol{e}, \boldsymbol{d})\|_{\mathscr{W}}^2 + \|(\boldsymbol{E}, \boldsymbol{D})\|_{\mathscr{W}}^2), \qquad (39)
$$

其中 $C_0 = \min\left\{k_1/M^2, k_2/Re_{\mathrm{m}}^2\right\}$. 由(33)、(34)式,引理 2.1、引理 4.2 和定理 1.1,利用 Hölder 和嵌入定理有,(38)式的右端第 1 项

$$
\leqslant (CH^2 + CH)(\|\boldsymbol{E}\|_1 + \|\boldsymbol{D}\|_1). \qquad (40)
$$

(38)式的右端第 2 项

$$
\begin{aligned}
&a_1((\boldsymbol{e} + \boldsymbol{E}, \boldsymbol{d} + \boldsymbol{D}), (\boldsymbol{u}_h, \boldsymbol{B}_h), (\boldsymbol{e}, \boldsymbol{d})) = \\
&\qquad a_1((\boldsymbol{e}, \boldsymbol{d}), (\boldsymbol{u}_h, \boldsymbol{B}_h), (\boldsymbol{e}, \boldsymbol{d}))s + a_1((\boldsymbol{E}, \boldsymbol{D}), (\boldsymbol{u}_h, \boldsymbol{B}_h), (\boldsymbol{e}, \boldsymbol{d})). \\
&a_1((\boldsymbol{e}, \boldsymbol{d}), (\boldsymbol{u}_h, \boldsymbol{B}_h), (\boldsymbol{e}, \boldsymbol{d})) \leqslant CR(\|\boldsymbol{e}\|_1^2 + \|\boldsymbol{d}\|_1^2). \qquad (41) \\
&a_1((\boldsymbol{E}, \boldsymbol{D}), (\boldsymbol{u}_h, \boldsymbol{B}_h), (\boldsymbol{e}, \boldsymbol{d})) \leqslant \\
&\qquad CR(\|\boldsymbol{e}\|_1^2 + \|\boldsymbol{d}\|_1^2 + \|\boldsymbol{E}\|_1^2 + \|\boldsymbol{D}\|_1^2). \qquad (42)
\end{aligned}
$$

(38)式的右端第 3 项

$$
\begin{aligned}
&a_1((\boldsymbol{e} + \boldsymbol{w}_h, \boldsymbol{d} + \boldsymbol{A}_h), (\boldsymbol{u}_h, \boldsymbol{B}_h), (\boldsymbol{E}, \boldsymbol{D})) = \\
&\qquad a_1((\boldsymbol{e}, \boldsymbol{d}), (\boldsymbol{u}_h, \boldsymbol{B}_h), (\boldsymbol{E}, \boldsymbol{D})) + a_1((\boldsymbol{w}_h, \boldsymbol{A}_h), (\boldsymbol{u}_h, \boldsymbol{B}_h), (\boldsymbol{E}, \boldsymbol{D})). \\
&a_1((\boldsymbol{e}, \boldsymbol{d}), (\boldsymbol{u}_h, \boldsymbol{B}_h), (\boldsymbol{E}, \boldsymbol{D})) \leqslant \\
&\qquad CR(\|\boldsymbol{e}\|_1^2 + \|\boldsymbol{d}\|_1^2 + \|\boldsymbol{E}\|_1^2 + \|\boldsymbol{D}\|_1^2). \qquad (43) \\
&a_1((\boldsymbol{w}_h, \boldsymbol{A}_h), (\boldsymbol{u}_h, \boldsymbol{B}_h), (\boldsymbol{E}, \boldsymbol{D})) \leqslant (CH^2 + CH)(\|\boldsymbol{E}\|_1 + \|\boldsymbol{D}\|_1). \qquad (44)
\end{aligned}
$$

(38) 式的右端第 4 项~ 第 11 项

$$\leqslant CH(\|\boldsymbol{e}\|_1^2+\|\boldsymbol{d}\|_1^2+\|\boldsymbol{E}\|_1^2+\|\boldsymbol{D}\|_1^2)+CH^2(\|\boldsymbol{d}\|_1^2+\|\boldsymbol{D}\|_1^2). \tag{45}$$

由(39)~(45)式得

$$\begin{aligned}&\min\left\{\frac{k_1}{M^2},\frac{k_2}{Re_{\mathrm{m}}^2}\right\}(\|(\boldsymbol{e},\boldsymbol{d})\|_{\mathscr{W}}^2+\|(\boldsymbol{E},\boldsymbol{D})\|_{\mathscr{W}}^2)\leqslant\\&\quad(C_1R+CH)(\|\boldsymbol{e}\|_1^2+\|\boldsymbol{d}\|_1^2+\|\boldsymbol{E}\|_1^2+\|\boldsymbol{D}\|_1^2)+\\&\quad CH^2(\|\boldsymbol{e}\|_1^2+\|\boldsymbol{d}\|_1^2+\|\boldsymbol{E}\|_1^2+\|\boldsymbol{D}\|_1^2)^{1/2},\end{aligned} \tag{46}$$

即

$$\begin{aligned}&\left[\min\left\{\frac{k_1}{M^2},\frac{k_2}{Re_{\mathrm{m}}^2}\right\}-C_1R-CH\right](\|(\boldsymbol{e},\boldsymbol{d})\|_{\mathscr{W}}^2+\|(\boldsymbol{E},\boldsymbol{D})\|_{\mathscr{W}}^2)\leqslant\\&\quad CH^2(\|(\boldsymbol{e},\boldsymbol{d})\|_{\mathscr{W}}^2+\|(\boldsymbol{E},\boldsymbol{D})\|_{\mathscr{W}}^2)^{1/2}.\end{aligned} \tag{47}$$

由于 $\gamma=\min\left\{k_1/M^2,k_2/Re_{\mathrm{m}}^2\right\}-C_1R>0$, 从而取 $h^*=\gamma/(2C)$, 其中 C 是常数, 则对于所有的 $H\leqslant h^*$ 有

$$(\|(\boldsymbol{e},\boldsymbol{d})\|_{\mathscr{W}}^2+\|(\boldsymbol{E},\boldsymbol{D})\|_{\mathscr{W}}^2)^{1/2}\leqslant CH^2. \tag{48}$$

由 inf_sup 不等式(9)有

$$\beta\|p_h-p^h\|_0\leqslant\sup_{(\phi+\mu,\tau+\sigma)\in\mathscr{W}^h}\frac{b((\phi+\mu,\tau+\sigma),p_h-p^h)}{\|(\phi+\mu,\tau+\sigma)\|_{\mathscr{W}}}. \tag{49}$$

由(37)式与(35)、(36)式相减可得

$$\begin{aligned}&b((\phi+\mu,\tau+\sigma),\delta)=a_0((\boldsymbol{e}+\boldsymbol{E},\boldsymbol{d}+\boldsymbol{D}),(\phi+\mu,\tau+\sigma))+\\&\quad a_1((\boldsymbol{u}_h,\boldsymbol{B}_h),(\boldsymbol{u}_h,\boldsymbol{B}_h),(\phi,\tau))-a_1((\boldsymbol{u}^h,\boldsymbol{B}^h),(\boldsymbol{u}^h,\boldsymbol{B}^h),(\phi,\tau))+\\&\quad a_1((\boldsymbol{w}^h,\boldsymbol{A}^h),(\boldsymbol{w}^h,\boldsymbol{A}^h),(\phi,\tau))+a_1((\boldsymbol{u}_h,\boldsymbol{B}_h),(\boldsymbol{u}_h,\boldsymbol{B}_h),(\mu,\sigma))-\\&\quad a_1((\boldsymbol{u}^H,\boldsymbol{B}^H),(\boldsymbol{u}^H,\boldsymbol{B}^H),(\mu,\sigma))-\frac{1}{Re_{\mathrm{m}}}\int_\Omega(\nabla\times\tau)\times\boldsymbol{A}^h\cdot\boldsymbol{w}^h\mathrm{d}x+\\&\quad\frac{1}{Re_{\mathrm{m}}}\int_\Omega(\nabla\times\boldsymbol{B}^H)\times\boldsymbol{A}^h\cdot\mu\mathrm{d}x+\frac{1}{Re_{\mathrm{m}}}\int_\Omega(\nabla\times\boldsymbol{A}^h)\times\boldsymbol{B}^H\cdot\mu\mathrm{d}x-\\&\quad\frac{1}{Re_{\mathrm{m}}}\int_\Omega(\nabla\times\sigma)\times\boldsymbol{B}^H\cdot\boldsymbol{w}^h\mathrm{d}x,\\&\qquad\qquad\forall\phi\in X_H,\ \mu\in X_h^H,\ \tau\in Y_H,\ \sigma\in Y_h^H.\end{aligned} \tag{50}$$

由于

$$\begin{aligned}&a_1((\boldsymbol{u}_h,\boldsymbol{B}_h),(\boldsymbol{u}_h,\boldsymbol{B}_h),(\phi,\tau))-a_1((\boldsymbol{u}^h,\boldsymbol{B}^h),(\boldsymbol{u}^h,\boldsymbol{B}^h),(\phi,\tau))+\\&\quad a_1((\boldsymbol{w}^h,\boldsymbol{A}^h),(\boldsymbol{w}^h,\boldsymbol{A}^h),(\phi,\tau))=\\&\quad a_1((\boldsymbol{u}_h,\boldsymbol{B}_h),(\boldsymbol{e}+\boldsymbol{E},\boldsymbol{d}+\boldsymbol{D}),(\phi,\tau))+\\&\quad a_1((\boldsymbol{e}+\boldsymbol{E},\boldsymbol{d}+\boldsymbol{D}),(\boldsymbol{u}_h,\boldsymbol{B}_h),(\phi,\tau))+\\&\quad a_1((\boldsymbol{w}_h,\boldsymbol{A}_h),(\boldsymbol{w}_h,\boldsymbol{A}_h),(\phi,\tau))-\\&\quad a_1((\boldsymbol{e}+\boldsymbol{E},\boldsymbol{d}+\boldsymbol{D}),(\boldsymbol{e}+\boldsymbol{E},\boldsymbol{d}+\boldsymbol{D}),(\phi,\tau)),\end{aligned} \tag{51}$$

所以由(48)式、引理 2.1 和引理 3.2 有

$$\begin{aligned}&|a_1((\boldsymbol{u}_h,\boldsymbol{B}_h),(\boldsymbol{u}_h,\boldsymbol{B}_h),(\phi,\tau))-a_1((\boldsymbol{u}^h,\boldsymbol{B}^h),(\boldsymbol{u}^h,\boldsymbol{B}^h),(\phi,\tau))+\\&\quad a_1((\boldsymbol{w}^h,\boldsymbol{A}^h),(\boldsymbol{w}^h,\boldsymbol{A}^h),(\phi,\tau))|\leqslant\\&\quad CH^2(\|\phi+\mu\|_1^2+\|\tau+\sigma\|_1^2)^{1/2}=CH^2\|(\phi+\mu,\tau+\sigma)\|_{\mathscr{W}}.\end{aligned} \tag{52}$$

再由式(48)、(6)和引理 3.2 可得

$$|a_1((\boldsymbol{u}_h, \boldsymbol{B}_h), (\boldsymbol{u}_h, \boldsymbol{B}_h), (\mu, \sigma)) - a_1((\boldsymbol{u}^H, \boldsymbol{B}^H), (\boldsymbol{u}^H, \boldsymbol{B}^H), (\mu, \sigma))| \leqslant CH^2(\|\phi+\mu\|_1^2 + \|\tau+\sigma\|_1^2)^{1/2} = CH^2\|(\phi+\mu, \tau+\sigma)\|_{\mathscr{W}}. \tag{53}$$

$$\frac{1}{Re_{\mathrm{m}}}\int_\Omega(\nabla\times\boldsymbol{B}^H)\times\boldsymbol{A}^h\cdot\mu\,\mathrm{d}x \leqslant CH^2\|\mu\|_1 \leqslant CH^2\|\phi+\mu\|_1. \tag{54}$$

与(54)式同理有

$$\left|\frac{1}{Re_{\mathrm{m}}}\int_\Omega(\nabla\times\tau)\times\boldsymbol{A}^h\cdot\boldsymbol{w}^h\,\mathrm{d}x + \frac{1}{Re_{\mathrm{m}}}\int_\Omega(\nabla\times\boldsymbol{A}^h)\times\boldsymbol{B}^H\cdot\mu\,\mathrm{d}x + \frac{1}{Re_{\mathrm{m}}}\int_\Omega(\nabla\times\sigma)\times\boldsymbol{B}^H\cdot\boldsymbol{w}^h\,\mathrm{d}x\right| \leqslant CH^2(\|\tau+\sigma\|_1 + \|\phi+\mu\|_1). \tag{55}$$

又

$$\begin{aligned} &a_0((\boldsymbol{e}+\boldsymbol{E}, \boldsymbol{d}+\boldsymbol{D}), (\boldsymbol{e}+\boldsymbol{E}, \boldsymbol{d}+\boldsymbol{D})) \leqslant \\ &\quad \frac{1}{M^2}|\boldsymbol{e}+\boldsymbol{E}|_1|\phi+\mu|_1 + \frac{5}{Re_{\mathrm{m}}}\|\boldsymbol{d}+\boldsymbol{D}\|_1\|\tau+\sigma\|_1 \leqslant \\ &\quad CH^2(\|\phi+\mu\|_1 + \|\tau+\sigma\|_1) \leqslant \\ &\quad CH^2(\|\phi+\mu\|_1^2 + \|\tau+\sigma\|_1^2)^{1/2} = CH^2\|(\phi+\mu, \tau+\sigma)\|_{\mathscr{W}}. \end{aligned} \tag{56}$$

结合(49)～(56)式即得

$$\|p_h - p^h\|_0 \leqslant CH^2. \tag{57}$$

结合(48)和(57)式得(27)式. 定理 3.1 证毕.

结合定理 1.1 和定理 3.1 得到下面的推论:

推论 3.3 在定理 3.1 的条件下, 下面的估计成立:

$$\|(\boldsymbol{u}, \boldsymbol{B}) - (\boldsymbol{u}^h, \boldsymbol{B}^h)\|_{\mathscr{W}} + \|p - p^h\|_0 \leqslant C(h + H^2). \tag{58}$$

注 如果取 $H = O(h^{1/2})$, 那么我们的非线性 Galerkin 混合元解与经典的混合有限元解(参见文献[12]和文献[15])具有同阶的逼近的误差估计. 等价地, 如果取 $h = O(H^2)$, 非线性 Galerkin 混合元法相对于 H 具有超收敛.

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Nonlinear Galerkin Mixed Element Methods for the Stationary Incompressible Magnetohydrodynamics

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Abstract: A nonlinear Galerkin mixed element (NGME) method for the stationary incompressible magnetohydrodynamics equations was presented. And the existence and error estimates of the NGME solution were derived.

Key words: equation of magnetohydrodynamics; nonlinear Galerkin mixed element method; error estimate