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加权Čebyšev-Ostrowski 型不等式*

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摘要: 关于著名的Čebyšev 不等式,已有众多的研究成果. 通过建立积分不等式,来建立全新的加权Čebyšev 型积分不等式. 给予了独立的证明,并给出了此类不等式的新评价.

关 键 词: Čebyšev 不等式; 函数空间; 绝对连续函数; 权函数
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引 言

由Čebyšev[1]发现的一个经典不等式,是以积分不等式的形式给出(见文献[2] 207) :

$$T(f, g) \leqslant \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty, \tag{1}$$

式中$f, g: [a, b] \to \mathbf{R}$ 为 2 个绝对连续函数, 其一阶导数$f', g' \in L_\infty(a, b)$, 且满足

$$T(f, g) = \frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x - \left(\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right)\left(\frac{1}{b-a}\int_a^b g(x)\,\mathrm{d}x\right), \tag{2}$$

上式称为Čebyšev 函数, 规定所包含的积分存在.

近来年, 对上述不等式的研究引起了许多数学家的兴趣, 并产生了众多研究成果, 有通用型的、延拓型的、变分型的, 详见文献[2] 至文献[10] 及其后面的参考文献. 基于对(1) 式和(2) 式, 本文建立一种新的加权 Ostrowski 型不等式, 该式基于两个绝对连续函数之积, 其一阶导数属于 $L_\infty(a, b)$ 空间.

1 主 要 结 论

设权函数 $w: [a, b] \to [0, \infty)$ 为非负、可积, 并有

$$\int_a^b w(t)\,\mathrm{d}t < \infty.$$

w 域可以是有限的也可以是无限的, 定义零阶矩为:

$$m(a, b) = \int_a^b w(t)\,\mathrm{d}t.$$

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令$[a, b] \subset \mathbf{R}$, $a < b$; 使用如下记号简化细节的描述. 对合适的函数$f, g:[a, b] \to \mathbf{R}$, 取

$$F = m\left(a, \frac{5a+b}{6}\right) f(a) + m\left(\frac{5a+b}{6}, \frac{a+5b}{6}\right) f\left(\frac{a+b}{2}\right) + m\left(\frac{a+5b}{6}, b\right) f(b),$$

$$G = m\left(a, \frac{5a+b}{6}\right) g(a) + m\left(\frac{5a+b}{6}, \frac{a+5b}{6}\right) g\left(\frac{a+b}{2}\right) + m\left(\frac{a+5b}{6}, b\right) g(b),$$

$$S_w(f, g) = f(x) g(x) - \frac{f(x)}{m(a, b)} \int_a^b w(t) g(t) \mathrm{d}t - \frac{g(x)}{m(a, b)} \int_a^b w(t) f(t) \mathrm{d}t + \frac{1}{m^2(a, b)} \left(\int_a^b w(t) f(t) \mathrm{d}t\right) \left(\int_a^b w(t) g(t) \mathrm{d}t\right),$$

$$S_w(f, g) = FG - G \frac{1}{m(a, b)} \int_a^b w(x) f(x) \mathrm{d}x - F \frac{1}{m(a, b)} \int_a^b w(x) g(x) \mathrm{d}x + \left(\frac{1}{m(a, b)} \int_a^b w(x) f(x) \mathrm{d}x\right) \left(\frac{1}{m(a, b)} \int_a^b w(x) g(x) \mathrm{d}x\right),$$

$$T_w(f, g) = \frac{1}{m(a, b)} \int_a^b w(x) S_w(f, g) \mathrm{d}x = \frac{1}{m(a, b)} \int_a^b w(x) f(x) g(x) \mathrm{d}x - \left(\frac{1}{m(a, b)} \int_a^b w(x) f(x) \mathrm{d}x\right) \left(\frac{1}{m(a, b)} \int_a^b w(x) g(x) \mathrm{d}x\right),$$

$$H_w(f, g) = \frac{1}{m(a, b)} \int_a^b w(x) [F g(x) + G f(x)] \mathrm{d}x - 2\left(\frac{1}{m(a, b)} \int_a^b w(x) g(x) \mathrm{d}x\right) \left(\frac{1}{m(a, b)} \int_a^b w(x) f(x) \mathrm{d}x\right).$$

本文的主要结果由如下诸定理给出.

定理 1 设$f, g:[a, b] \to \mathbf{R}$为绝对连续函数, 其一阶导数$f'$、$g'$属于$L_\infty(a, b)$, 则对所有$x \in [a, b]$有如下不等式:

$$| T_w(f, g) | \leqslant \frac{\| f' \|_\infty \| g' \|_\infty}{m^3(a, b)} \int_a^b w(x) \left(\int_a^b \operatorname{sgn}(t - x)(t - x) w(t) \mathrm{d}t\right)^2 \mathrm{d}x, \tag{3}$$

$$| T_w(f, g) | \leqslant \frac{1}{2m^2(a, b)} \int_a^b w(x) (| g(x) | \| f' \|_\infty + | f(x) | \| g' \|_\infty) \times \left(\int_a^b \operatorname{sgn}(t - x)(t - x) w(t) \mathrm{d}t\right) \mathrm{d}x. \tag{4}$$

证明 假设, 对所有$x \in [a, b]$, 有如下等式成立(见文献[3] 319):

$$f(x) - \frac{1}{m(a, b)} \int_a^b w(t) f(t) \mathrm{d}t = \frac{1}{m(a, b)} \int_a^b p(x, t) f'(t) \mathrm{d}t \tag{5}$$

和

$$g(x) - \frac{1}{m(a, b)} \int_a^b w(t) g(t) \mathrm{d}t = \frac{1}{m(a, b)} \int_a^b p(x, t) g'(t) \mathrm{d}t, \tag{6}$$

对所有$x \in [a, b]$, 其中

$$p(x, t) = \begin{cases} \int_a^t w(u) \mathrm{d}u, & t \in [a, x], \\ \int_b^t w(u) \mathrm{d}u, & t \in [x, b]. \end{cases}$$

得到

$$\int_a^b | p(x, t) | \mathrm{d}t = \int_a^x \left| \int_a^t w(u) \mathrm{d}u \right| \mathrm{d}t + \int_x^b \left| \int_b^t w(u) \mathrm{d}u \right| \mathrm{d}t =$$

$$\int_a^x\int_a^t w(u)\,\mathrm{d}u\mathrm{d}t + \int_x^b\int_t^b w(u)\,\mathrm{d}u\mathrm{d}t =$$
$$-\int_a^x tw(t)\,\mathrm{d}t + \int_x^b tw(t)\,\mathrm{d}t + x\int_a^x w(t)\,\mathrm{d}t - x\int_x^b w(t)\,\mathrm{d}t =$$
$$\int_a^b \mathrm{sgn}(t-x)\,tw(t)\,\mathrm{d}t - x\int_a^b \mathrm{sgn}(t-x)w(t)\,\mathrm{d}t =$$
$$\int_a^b \mathrm{sgn}(t-x)(t-x)w(t)\,\mathrm{d}t. \tag{7}$$

将(5)式和(6)式的两边对应相乘, 可得

$$f(x)g(x) - \frac{f(x)}{m(a,b)}\int_a^b w(t)g(t)\,\mathrm{d}t - \frac{g(x)}{m(a,b)}\int_a^b w(t)f(t)\,\mathrm{d}t +$$
$$\frac{1}{m^2(a,b)}\left(\int_a^b w(t)f(t)\,\mathrm{d}t\right)\left(\int_a^b w(t)g(t)\,\mathrm{d}t\right) =$$
$$\frac{1}{m^2(a,b)}\left(\int_a^b p(x,t)f'(t)\,\mathrm{d}t\right)\left(\int_a^b p(x,t)g'(t)\,\mathrm{d}t\right).$$

上式两边乘以 $w(x)/(m(a,b))$, 并在 $[a,b]$ 上对 x 积分, 可得

$$T_w(f,g) = \frac{1}{m^3(a,b)}\int_a^b w(x)\left(\int_a^b p(x,t)f'(t)\,\mathrm{d}t\right)\left(\int_a^b p(x,t)g'(t)\,\mathrm{d}t\right)\mathrm{d}x.$$

对上式取模, 推得

$$|T_w(f,g)| \leqslant$$
$$\frac{1}{m^3(a,b)}\int_a^b w(x)\left(\int_a^b |p(x,t)||f'(t)|\,\mathrm{d}t\right)\times$$
$$\left(\int_a^b |p(x,t)||g'(t)|\,\mathrm{d}t\right)\mathrm{d}x \leqslant$$
$$\frac{1}{m^3(a,b)}\int_a^b w(x)\,\|f'\|_\infty\left(\int_a^b |p(x,t)|\,\mathrm{d}t\right)\|g'\|_\infty\left(\int_a^b |p(x,t)|\,\mathrm{d}t\right)\mathrm{d}x =$$
$$\frac{\|f'\|_\infty\|g'\|_\infty}{m^3(a,b)}\int_a^b w(x)\left(\int_a^b |p(x,t)|\,\mathrm{d}t\right)^2\mathrm{d}x. \tag{8}$$

利用(7)式和(8)式, 得到(3)式. 从而不等式(3)得证.

将(5)式和(6)式分别乘上 $g(x)$ 和 $f(x)$, 并将结果相加后乘上 $w(x)/(2m(a,b))$. 得到

$$\frac{w(x)f(x)g(x)}{m(a,b)} - \frac{w(x)f(x)}{2m^2(a,b)}\int_a^b w(t)g(t)\,\mathrm{d}t - \frac{w(x)g(x)}{2m^2(a,b)}\int_a^b w(t)f(t)\,\mathrm{d}t =$$
$$\frac{w(x)}{2m^2(a,b)}\left(g(x)\int_a^b p(x,t)f'(t)\,\mathrm{d}t + f(x)\int_a^b p(x,t)g'(t)\,\mathrm{d}t\right).$$

将上式两边, 在 $[a,b]$ 上对 x 积分, 可得

$$T_w(f,g) = \frac{1}{2m^2(a,b)}\int_a^b w(x)\left[g(x)\left(\int_a^b p(x,t)f'(t)\,\mathrm{d}t\right)+\right.$$
$$\left.f(x)\left(\int_a^b p(x,t)g'(t)\,\mathrm{d}t\right)\right]\mathrm{d}x,$$

对上式取模, 可导得

$$|T_w(f,g)| \leqslant \frac{1}{2m^2(a,b)}\int_a^b w(x)\left[|g(x)|\left(\int_a^b |p(x,t)||f'(t)|\,\mathrm{d}t\right)+\right.$$
$$\left.|f(x)|\left(\int_a^b |p(x,t)||g'(t)|\,\mathrm{d}t\right)\right]\mathrm{d}x \leqslant$$

$$\frac{1}{2m^2(a,b)}\int_a^b w(x)\left[|g(x)|\ \|f'\|_\infty\left(\int_a^b |p(x,t)|\,\mathrm{d}t\right)+\right.$$
$$\left.|f(x)|\ \|g'\|_\infty\left(\int_a^b |p(x,t)|\,\mathrm{d}t\right)\right]\mathrm{d}x=$$
$$\frac{1}{2m^2(a,b)}\int_a^b w(x)(|g(x)|\ \|f'\|_\infty+$$
$$|f(x)|\ \|g'\|_\infty)\left(\int_a^b |p(x,t)|\,\mathrm{d}t\right)\mathrm{d}x=$$
$$\frac{1}{2m^2(a,b)}\int_a^b w(x)(|g(x)|\ \|f'\|_\infty+|f(x)|\ \|g'\|_\infty)\times$$
$$\left(\int_a^b \mathrm{sgn}(t-x)(t-x)w(t)\mathrm{d}t\right)\mathrm{d}x.$$

上式便是我们期望的结果式(4). 因此定理 1 得证. □

定理 2 设$f, g:[a,b]\to\mathbf{R}$为绝对连续函数, 其一阶导数f'、g'属于$L_\infty(a,b)$, 则对所有$x\in[a,b]$, 有如下不等式:

$$|S_w(f,g)|\leqslant\frac{\|f'\|_\infty\|g'\|_\infty}{m^2(a,b)}N^2, \tag{9}$$

$$|H_w(f,g)|\leqslant$$
$$\frac{1}{m^2(a,b)}\int_a^b w(x)(|g(x)|\ \|f'\|_\infty+|f(x)|\ \|g'\|_\infty)N\mathrm{d}x, \tag{10}$$

对所有$x\in[a,b]$, 其中

$$N=\int_a^{(a+b)/2}\mathrm{sgn}\left(x-\frac{5a+b}{6}\right)m\left(\frac{5a+b}{6},x\right)\mathrm{d}x+$$
$$\int_{(a+b)/2}^b\mathrm{sgn}\left(x-\frac{a+5b}{6}\right)m\left(\frac{a+5b}{6},x\right)\mathrm{d}x.$$

证明 设有如下加权等式(见参考文献[11])

$$F-\frac{1}{m(a,b)}\int_a^b w(x)f(x)\mathrm{d}x=\frac{1}{m(a,b)}\int_a^b m(x)f'(t)\mathrm{d}x, \tag{11}$$

$$G-\frac{1}{m(a,b)}\int_a^b w(x)g(x)\mathrm{d}x=\frac{1}{m(a,b)}\int_a^b m(x)g'(t)\mathrm{d}x, \tag{12}$$

其中

$$m(x)=\begin{cases}\displaystyle\int_{(5a+b)/6}^x w(u)\mathrm{d}u, & x\in\left[a,\dfrac{a+b}{2}\right],\\ \displaystyle\int_{(a+5b)/6}^x w(u)\mathrm{d}u, & x\in\left(\dfrac{a+b}{2},b\right].\end{cases}$$

将式(11)和式(12)两边分别相乘, 得到

$$FG-G\frac{1}{m(a,b)}\int_a^b w(x)f(x)\mathrm{d}x-F\frac{1}{m(a,b)}\int_a^b w(x)g(x)\mathrm{d}x+$$
$$\left(\frac{1}{m(a,b)}\int_a^b w(x)f(x)\mathrm{d}x\right)\left(\frac{1}{m(a,b)}\int_a^b w(x)g(x)\mathrm{d}x\right)=$$
$$\frac{1}{m^2(a,b)}\left(\int_a^b m(x)f'(x)\mathrm{d}x\right)\left(\int_a^b m(x)g'(x)\mathrm{d}x\right),$$

这意味着

$$S_w(f,g)=\frac{1}{m^2(a,b)}\left(\int_a^b m(x)f'(x)\mathrm{d}x\right)\left(\int_a^b m(x)g'(x)\mathrm{d}x\right).$$

对上式取模, 导得

$$|S_w(f,g)| \leqslant \frac{\|f'\|_\infty \|g'\|_\infty}{m^2(a,b)} M^2,$$

其中

$$M = \int_a^b |m(x)| \, dx =$$
$$\int_a^{(a+b)/2} \left| \int_{(5a+b)/6}^x w(u) du \right| dx + \int_{(a+b)/2}^b \left| \int_{(a+5b)/6}^x w(u) du \right| dx =$$
$$\int_a^{(5a+b)/6} \left(\int_x^{(5a+b)/6} w(u) du \right) dx + \int_{(5a+b)/6}^{(a+b)/2} \left(\int_{(5a+b)/6}^x w(u) du \right) dx +$$
$$\int_{(a+b)/2}^{(a+5b)/6} \left(\int_x^{(a+5b)/6} w(u) du \right) dx + \int_{(a+5b)/6}^b \left(\int_{(a+5b)/6}^x w(u) du \right) dx =$$
$$\int_a^{(5a+b)/6} m\left(x, \frac{5a+b}{6}\right) dx + \int_{(5a+b)/6}^{(a+b)/2} m\left(\frac{5a+b}{6}, x\right) dx +$$
$$\int_{(a+b)/2}^{(a+5b)/6} m\left(x, \frac{a+5b}{6}\right) dx + \int_{(a+5b)/6}^b m\left(\frac{a+5b}{6}, x\right) dx +$$
$$\int_a^{(a+b)/2} \operatorname{sgn}\left(x - \frac{5a+b}{6}\right) m\left(\frac{5a+b}{6}, x\right) dx +$$
$$\int_{(a+b)/2}^b \operatorname{sgn}\left(x - \frac{a+5b}{6}\right) m\left(\frac{a+5b}{6}, x\right) dx = N.$$

因此不等式(9)得证.

将(11)式和(12)式分别乘上 $g(x)$ 和 $f(x)$, 将结果相加后得到

$$Fg(x) + Gf(x) - \frac{g(x)}{m(a,b)} \int_a^b w(x) f(x) dx - \frac{f(x)}{m(a,b)} \int_a^b w(x) g(x) dx =$$
$$\frac{g(x)}{m(a,b)} \int_a^b m(x) f'(x) dx + \frac{f(x)}{m(a,b)} \int_a^b m(x) g'(x) dx.$$

将上式乘上 $w(x)/(m(a,b))$, 并在 $[a,b]$ 上对 x 积分, 可得

$$H_w(f,g) = \frac{1}{m^2(a,b)} \int_a^b w(x) \left(g(x) \int_a^b m(x) f'(x) dx + f(x) \int_a^b m(x) g'(x) dx \right) dx.$$

对上式取模, 导得

$$|H_w(f,g)| \leqslant \frac{1}{m^2(a,b)} \int_a^b w(x) \left(|g(x)| \int_a^b |m(x)| |f'(x)| dx + \right.$$
$$\left. |f(x)| \int_a^b |m(x)| |g'(x)| dx \right) dx \leqslant$$
$$\frac{1}{m^2(a,b)} \int_a^b w(x) (|g(x)| \|f'\|_\infty + |f(x)| \|g'\|_\infty) \left(\int_a^b |m(x)| dx \right) dx =$$
$$\frac{1}{m^2(a,b)} \int_a^b w(x) (|g(x)| \|f'\|_\infty + |f(x)| \|g'\|_\infty) N dx,$$

其中 N 定义如上, 上面便是我们期望的结果(10)式. 因此定理 2 得证. □

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Weighted Čeby》ev-Ostrowski Type Inequalities Involving Functions Whose First Derivatives Belong to a Spaces of the Functions

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Abstract: On account of the famous Čeby》ev inequality, a rich theory has appeared in some literature. Some new weighted Čeby》ev type integral inequalities via certain integral inequalities for functions whose first derivatives belong to a space of the functions are established. The proofs are of independent interest and provide new estimates on these types of inequalities.

Key words: Čeby》ev type inequality; space of the function; absolutely continuous function; weight function